Distances for Uncertain Topological Relations

Stephan Winter

Institute of Photogrammetry, University of Bonn
Stephan.Winter@ipb.uni-bonn.de

Erstmals veröffentlicht in: *ESF-NSF Summer Institute in Geographic Information*, Berlin, 24. Juli bis 1. August 1996. Taylor & Francis, London, in Vorbereitung.

Abstract

Considering uncertainty of spatial data in any GIS analysis is a theme of actual research, but far from practicability. One prerequisite is the availability of descriptions concerning uncertainty (*meta-data*), and another is the attachment of uncertainty to spatial analysis.

In our contribution we propose a method to determine the topological relation between two positional uncertain regions. The decision is based on morphological distances along a skeleton through uncertain intersection sets. These measures are equivalent to the known representation by intersection sets, but yield additionally metric information. These distances are applied to a statistical model of the observation process. A Bayesian classification yields distance classes which allow to deduce the topological relationship.

We discuss also importance and perspectives of the method.

1 Introduction

Uncertainty is an inherent property of observations. Abstracting the real world to conceptional objects is a step of generalization, and the measurement, taking place on abstracted objects, propagates this uncertainty, adding systematic, gross and random errors. Each spatial analysis is infected by these sources of uncertainty. Therefore it is necessary to introduce propagation of uncertainty in spatial analysis to allow an assessment of the results. The scope of this paper is to combine the process of observation with a mathematical model of qualitative spatial relations, modeling the randomness of the observations. A methodology is presented for probability-based decisions about spatial relations.

When determining spatial relations from positional uncertain objects, one has to distinguish between quantitative relations, which become *imprecise*, and qualitative relations, which become *uncertain*. Topological relations, being of qualitative nature, may or may not be true in presence of positional uncertainty. E. g., assuming the overlay of two independent objects indicates a very small overlap, the question arises whether the two objects could be neighbored in reality. Comparing the degree of overlap to the size of uncertainty will allow to make a decision, and to assess this decision.

We describe the inference from positional uncertain objects to observations characterizing the topological relations. As observations we introduce a morphological distance function, based on the skeleton. These observations allow an equivalent representation of topological relations, but yield additionally metric information. The inference from this description to the uncertainty of derived topological relations is treated with a statistical classification approach. Probabilities of single relations are determined, and the relation with maximum probability, given the evidence from observations, is chosen.

It is shown that it is sufficient to use the minimum and maximum distance, and to classify the relation, depending on the signs of these two values. For the first time the imprecision of measurement and the uncertainty of abstraction are described using probability densities. They are used to determine a vector of probabilities the sign of the distance values have, which is the basis for a Bayesian classification. From these distance classes the decision about the topological relation of the two objects is derived.

Other approaches for handling uncertainty are discrete, using error bands, or fuzzy, with the problem of weaker results. With the strong connection to an observation process we hope to give a more valuable decision method, with probabilities as interpretable results, which should be useful for the assessment and propagation in spatial reasoning processes.

2 Related and Fundamental Work

2.1 Uncertainty of Objects and Relations

There are a few ideas to handle positional uncertainty of spatial objects in GIS, mainly by bands or by fuzzy sets. Also the modeling of spatial relations was based up to now on these models.

Statistical models are neglected in GIS because of their complexity. While the positional uncertainty of a point can be described by a 2×2-covariance matrix, objects of higher dimensionality — curves, or bounded regions — need additional efforts in modeling correlations and superpositions. For these reasons existing models remain discrete or fuzzy.

Bands, replacing linear boundaries of regions, are a discrete two-dimensional representation of positional uncertainty. There are bands in use with a constant width (ε-bands: Perkal 1956, Chrisman 1982), and so-called error-bands, which may have variable widths. The error band for linear segments can be calculated stochastically as of hyperbolic shape (Wolf 1975, Caspary and Scheuring 1992).

Bands disturb Euclidean topology, but they allow in a first instance to differentiate topological relations for their uncertainty (Clementini and Di Felice 1996). The interpretability of a discrete sub-relation is poor, of course. An assessment or a numerical scale cannot be given.

Kraus and Haussteiner 1993 calculate a map of the probabilities of points in $\mathbb{R}^2$ being inside of a polygon. In principle their map is also characterized by hyperbolic isolines. Propagation of positional uncertainty to derived metric parameters — line intersections, surface areas — is treated e. g. by Kraus and Kager 1994.

Another way of modeling uncertainty is the interpretation of regions as two-dimensional fuzzy sets (Zadeh 1965, and e. g. Molenaar 1994). Then the problem arises how to determine the fuzzy membership values, and how to interpret the results of fuzzy reasoning. In contrast to fuzzy sets a probability distribution can be interpreted by specifying an experiment, which follows the distribution.

Wazinski 1993 used an ε-band to derive graded topological relations between very restricted objects, comparable to fuzzy measures.

2.2 Representation of Topological Relations

In this paper we will make use of the *9-intersection*, a specific representation model for binary topological relations. Therefore, we will recall the principle of this model, with emphasis on details of special interest here.

The nine intersection sets between the interior, the boundaries and the exterior of two spatial objects are used to characterize sets of topological relations. For simple and regular closed regions (Worboys and Bofakos 1993) in $\mathbb{R}^2$ a set of eight relations can be distinguished[1]. Spatial objects with other topological properties differ in the number of topological relations, but this number is always greater than eight

[1] In this special case a subset of the nine intersection sets would be sufficient to differentiate, but we keep the general form.

(Egenhofer and Herring 1991). We limit ourselves to simple, regular closed regions, being interested more in developing a method than in being complete.

Then the set of topological relations Ω_R consists of:

$$\begin{aligned}\Omega_R \;=\; & \{\textsc{Disjunct}, \textsc{Touch}, \textsc{Overlap}, \\ & \textsc{Covers}, \textsc{CoveredBy}, \textsc{Equal}, \textsc{Contains}, \textsc{ContainedBy}\}\end{aligned} \tag{1}$$

This set can be ordered by a planar graph, the *conceptual neighborhood graph* (CNG), based on concepts like topological neighborhood or topological distance (Egenhofer and Al-Taha 1992). Additionally it is possible to direct the edges of the graph, using the concept of dominance (Galton 1994, Winter 1994).

Definition (dominance): We call a relation *dominating* against its neighboring relations, if it holds in some translation or deformation only at a point.

We define the edge directions from the dominating relation to the dominated relation. Fig. 1 shows the CNG, extended by some ideas from section 3.

In the next section we develop a distance function to replace the non-emptiness of intersection sets in the determination of topological relations. For these observable values we apply a statistical model of observation in section 4, which allow to classify distances, and further, the topological relationship.

3 Observations for Topological Relations

In this section we develop an alternative representation to the 9-intersection with full compatibility, but with additional properties, yielding metric information of the distance of two regions.

3.1 Relation Clusters

We restrict the considered regions by two conditions:

1. The uncertainty about the position of each region has to be small against the size of the region.
2. The question whether a point is inside a region must be related (in practice) at most to one segment of the boundary.

In geodetic contexts these restrictions usually are fulfilled. In cadastral surveying object dimensions are in decameters while precision is in centimeters, and in topographic mapping object dimensions are in hectometers while precision is in meters.

The geometric restrictions give justification to partition the CNG into two connected subgraphs, or to partition Ω_R into two relation clusters C^1, C^2.

Definition (clusters of relations): We call the set of relations which consists of Touch and its neighbored relations the cluster C^1:

$$C^1 \;=\; \{\textsc{Disjunct}, \textsc{Touch}, \textsc{WeakOverlap}\}$$

We call the set of relations which consists of Equal and its neighbored relations the cluster C^2:

$$\begin{aligned}C^2 \;=\; & \{\textsc{StrongOverlap}, \textsc{Covers}, \textsc{CoveredBy}, \\ & \textsc{Contains}, \textsc{ContainedBy}, \textsc{Equal}\}\end{aligned}$$

This partitioning cuts the central relation Overlap into a WeakOverlap and a StrongOverlap, so that both clusters are centered to a dominant relation (Equal resp. Touch) and contain all relations which are directly neighbored (Fig. 1). The weight of the relation Overlap is based on an overlap factor OF:

$$OF = \frac{|\mathcal{A}^\circ \cap \mathcal{B}^\circ|}{min(|\mathcal{A}^\circ|, |\mathcal{B}^\circ|)} \tag{2}$$

With that it can be defined simply:

$$\begin{aligned} \text{Rel} = \text{WeakOverlap} &: \quad OF \leq 0.5 \\ \text{Rel} = \text{StrongOverlap} &: \quad OF > 0.5 \end{aligned}$$

The idea behind splitting Overlap is the observation, that a situation with an overlap factor near to 0.5 is insensitive to imprecision and always to be classified as Overlap.

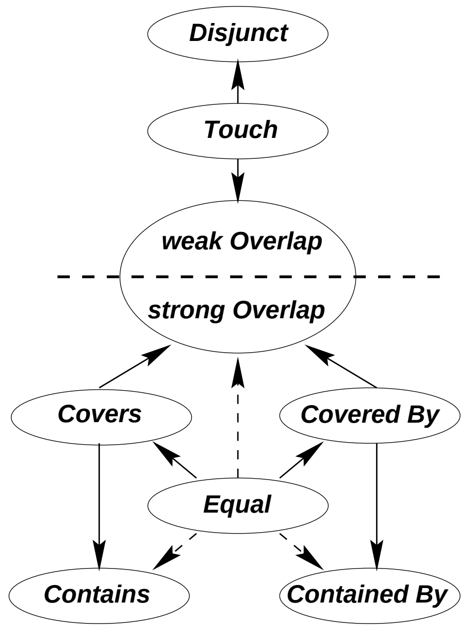

Figure 1: The conceptual neighborhood graph, partitioned into clusters of the neighborhood of Equal and the neighborhood of Touch.

3.2 Ternary Skeletons

If an observed relation between two regions $\mathcal{A}$ and $\mathcal{B}$ is out of C^1, then positional uncertainty is linked with the uncertainty of the relation about the intersection sets $\mathcal{A}^\circ \cap \mathcal{B}^\circ$ and $\partial\mathcal{A} \cap \partial\mathcal{B}$. And if an observed relation is out of C^2, then the positional uncertainty is linked with the uncertainty of the relation about the intersection sets $\mathcal{A}^\circ \cap \mathcal{B}^c$, $\mathcal{A}^c \cap \mathcal{B}^\circ$, and $\partial\mathcal{A} \cap \partial\mathcal{B}$. The link between positional uncertainty and other intersection sets is then redundant.

On the way to develop a metric measure of the uncertainty of intersection sets we label three zones $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$, each consisting of the following intersection sets:

$$\text{Rel} \in C^1 : \quad \begin{cases} \mathcal{O} = \mathbb{R}^2 \setminus \mathcal{P}, \mathcal{Q} \\ \mathcal{P} = \mathcal{A}^\circ \cap \mathcal{B}^c \\ \mathcal{Q} = \mathcal{A}^c \cap \mathcal{B}^\circ \end{cases} \tag{3}$$

$$\text{Rel} \in C^2 : \quad \begin{cases} \mathcal{O} = \mathbb{R}^2 \setminus \mathcal{P}, \mathcal{Q} \\ \mathcal{P} = \mathcal{A}^c \cap \mathcal{B}^c \\ \mathcal{Q} = \mathcal{A}^\circ \cap \mathcal{B}^\circ \end{cases} \tag{4}$$

All zones are open sets. — With that we recall the concept of a zonal skeleton (Lantuejoul 1978). For skeletonization we consider the closure of the zones $\mathcal{P}$ and $\mathcal{Q}$ as foreground $\mathcal{X}$, and $\mathcal{O}$ as background $\overline{\mathcal{X}^c}$. A skeleton $\mathcal{S}(\mathcal{X})$ is the set of centers of the maximal discs which are in a closed set $\mathcal{X}$ (Serra 1982). Then an exoskeleton is the skeleton $\mathcal{S}(\overline{\mathcal{X}^c})$.

Definition (zonal skeleton): A (ternary) zonal skeleton is the subset of an exoskeleton on disjunct particles $\mathcal{P}$ and $\mathcal{Q}$, where the maximal discs touch both $\mathcal{P}$ and $\mathcal{Q}$.

With two different initializations of $\mathcal{P}$ and $\mathcal{Q}$ (Eq. 3, 4) we distinguish also $\mathcal{S}^1$ (REL $\in C^1$) and $\mathcal{S}^2$ (REL $\in C^2$). A zonal skeleton is a finite union of simple lines (Lantuejoul 1978), but neither $\mathcal{S}^1$ nor $\mathcal{S}^2$ must be connected.

3.3 Morphological Distance Functions

Now we can use the zonal skeleton to introduce a distance function between two particles, i. e. a diameter function in uncertain intersection sets. The idea is based on the involvation of the diameter d of the maximal discs at each point s of the skeleton $\mathcal{S}^i$, $i \in \{1, 2\}$.

Definition (morphological distance): We call a function $\vartheta_{\mathcal{AB}}(s)$ between two regions $\mathcal{A}$ and $\mathcal{B}$ with the following properties:

$$\begin{aligned} \vartheta_{\mathcal{AB}}(s) &= d && \text{if} \quad s \in \mathcal{A}^c \\ \vartheta_{\mathcal{AB}}(s) &= -d && \text{if} \quad s \in \mathcal{A}^\circ \\ \vartheta_{\mathcal{AB}}(s) &= 0 && \text{if} \quad s \in \partial\mathcal{A} \end{aligned}$$

for $s \in \mathcal{S}^i$ and $d \in \mathbb{R}^+$ the *morphological distance.*

The name and sign of the morphological distance follows from the morphological operations of dilation ($\oplus$) or erosion ($\ominus$) of $\mathcal{A}$ to reach the skeleton. — In the following we will denote the morphological distance shortly by $\vartheta_{\mathcal{AB}}$.

It is now easy to show that the morphological distance is symmetric along $\mathcal{S}^1$ ($\vartheta_{\mathcal{AB}} = \vartheta_{\mathcal{BA}}$), and anti-symmetric along $\mathcal{S}^2$ ($\vartheta_{\mathcal{AB}} = -\vartheta_{\mathcal{BA}}$).

In the next step we concentrate on the range of morphological distances between $\mathcal{A}$ and $\mathcal{B}$, and reduce the actual values further to distance classes based on their sign:

Definition (distance classes): We define the following classes for morphological distances:

$$\Omega_\vartheta = \{\omega_1, \omega_0, \omega_2\} \qquad \text{with} \begin{cases} \vartheta_{\mathcal{AB}} \in \omega_1 & \text{if } \vartheta_{\mathcal{AB}} < 0 \\ \vartheta_{\mathcal{AB}} \in \omega_0 & \text{if } \vartheta_{\mathcal{AB}} = 0 \\ \vartheta_{\mathcal{AB}} \in \omega_2 & \text{if } \vartheta_{\mathcal{AB}} > 0 \end{cases}$$

With the triple consisting of the relation cluster C^i, the class $\omega_{\min}$ of the minimum distance $\vartheta_{\min} = \min(\vartheta_{\mathcal{AB}})$, and the class $\omega_{\max}$ of the maximum distance $\vartheta_{\max} = \max(\vartheta_{\mathcal{AB}})$ along $\mathcal{S}^i$ we have found an equivalent representation of the 9-intersection, cf. Tab. 1:

$$\{C^i, \omega_{\min}, \omega_{\max}\} \equiv \mathbf{I}^9_{\mathcal{AB}} \tag{5}$$

We can show that the variations of the triple in Tab. 1 are complete.

C	$\vartheta_{\min}$	$\vartheta_{\max}$	Relation
C^1	ω_2	ω_2	DISJUNCT
	ω_0	ω_2	TOUCH
	ω_1	ω_2	WEAKOVERLAP
C^2	ω_1	ω_2	STRONGOVERLAP
	ω_1	ω_0	COVERS
	ω_0	ω_2	COVEREDBY
	ω_1	ω_1	CONTAINS
	ω_2	ω_2	CONTAINEDBY
	ω_0	ω_0	EQUAL

Table 1: Equivalence in the relations of the 9-intersection and of the triple of the relation cluster C and the classes of extremal morphological distances.

4 Classification of Topological Relations

4.1 Uncertainty of Abstraction

Up to now we used distance classes with a mathematical definition. ω_0 stands for $\vartheta_{\mathcal{AB}} = 0$, which is for a continuous random variable like ϑ impossible ($P(\vartheta_{\mathcal{AB}} = 0) = 0$). This argument coincides with the hint, that human concepts of ISZERO always have a natural width. The width depends merely on the context of an observation.

Consider e. g. a surveyor, who will always avoid to mark a new point in a distance of, let us say, $5cm$ of an already captured point; instead he will use that one. Therefore, if we find in cadastral datasets two points nearer than $5cm$ we have strong support for the assumption that the same point was meant.

Modeling the width of a concept can be done by asking experts. Let us collect a high number of answers from independent experts. Then the function which describes the number of agreements with the concept for all values of $\mathbb{R}$ is a probability density function. Let us further describe this function by two parameters: a mean width β, and a smoothness of the concept, given with σ_β. Then a density function can be given by the convolution ('$*$') of a constant distribution D_β and a Gaussian[2] G_{σ_β}:

$$p(\underline{\mu} = \mu_\vartheta \mid \omega_0) = D_\beta \, * \, G_{\sigma_\beta} \quad (6)$$

Such a model is consistent with the mathematical concept of '=0', because in the case of $\beta = 0$ and $\sigma_\beta = 0$ the density function degenerates to a δ-function. — Density functions with ω_1 and ω_2 can be set up analogously, because both intervals are in practice limited, either by the region diameters or by a possibly minimum bounding rectangle containing $\mathcal{A}$ and $\mathcal{B}$.

4.2 Uncertainty of Measurement

The width of a concept is (in a first step) independent from measurement, which introduces additionally gross, systematic and random errors. Here we only cope with random errors, because gross and systematic errors are avoidable in principle.

Measured variables are the boundaries of the regions. In a simple model we assume that the random errors may be described by $\sigma_\mathcal{A}$ and $\sigma_\mathcal{B}$ for two regions $\mathcal{A}$ and $\mathcal{B}$. Further we simplify the morphological distance to a functional difference between $\partial\mathcal{A}$ and $\partial\mathcal{B}$. Then the uncertainty of observing $\vartheta_{\mathcal{AB}}$ can be described with σ_ϑ:

$$\sigma_\vartheta^2 = \sigma_\mathcal{A}^2 + \sigma_\mathcal{B}^2 \quad (7)$$

[2]The Gaussian G_σ is defined as $G_\sigma = 1/\sqrt{2\pi\sigma^2}\ \exp(-\frac{1}{2}(x/\sigma)^2)$

by the density function of an observation error ε:

$$p_{\varepsilon} = G_{\sigma_\vartheta} \tag{8}$$

4.3 Combined Observation Uncertainty

With the assumed independence between abstraction and measurement we can write:

$$\underline{\varepsilon} \mid \underline{\mu}_\vartheta, \underline{\omega}_i = \underline{\varepsilon} \tag{9}$$

It follows that:

$$\begin{aligned} \underline{\vartheta} \mid \underline{\omega}_i &= (\underline{\vartheta} - \underline{\mu}_\vartheta) \mid \underline{\omega}_i + \underline{\mu}_\vartheta \mid \underline{\omega}_i \\ &= \underline{\varepsilon} + \underline{\mu}_\vartheta \mid \underline{\omega}_i \end{aligned} \tag{10}$$

Probabilities for continuous variables need a non-zero interval. With a small $\Delta\vartheta$ we define:

$$\begin{aligned} P(\underline{\vartheta} = \vartheta) &\stackrel{!}{=} P(\vartheta \leq \underline{\vartheta} < \vartheta + \Delta\vartheta) = F(\vartheta + \Delta\vartheta) - F(\vartheta) \\ &\stackrel{!}{=} p_\vartheta(\vartheta)\, \Delta\vartheta \quad \text{for small } \Delta\vartheta \end{aligned}$$

This definition leaves a dependency of $P(\underline{\vartheta} = \vartheta)$ and $\Delta\vartheta$, which will be cancelled later. — With such a convention, and referring to Eq. 10, we can write for the probability of the evidence ϑ:

$$\begin{aligned} P(\underline{\vartheta} = \vartheta \mid \omega_i) &= p_{\vartheta|\omega}(\vartheta)\, \Delta\vartheta \\ &= (p_{\vartheta-\mu|\omega_i} * p_{\mu|\omega_i})(\underline{\vartheta})\, \Delta\vartheta \\ &= (p_\varepsilon * p_{\mu|\omega_i})(\underline{\vartheta})\, \Delta\vartheta \end{aligned} \tag{11}$$

Setting in our proposed density functions, we can show by some transformations that this probability can be calculated simply by two error functions (Papoulis 1965, Winter 1996):

$$P(\underline{\vartheta} \mid \omega_0) = (\mathrm{erf}_\sigma(\vartheta - \beta_l) - \mathrm{erf}_\sigma(\vartheta - \beta_r))\, \Delta\vartheta \tag{12}$$

and analogously for ω_1 and ω_2, with $\mathrm{erf}_\sigma(x) = \int_{-\infty}^{x} G_\sigma(t)\, dt$.

4.4 Classification

Under the given uncertainties we have to classify an observed (extremal) morphological distance into one of the classes of Ω_R. With a maximum likelihood classification:

$$\underline{\vartheta} \mapsto \hat{\underline{\omega}}_i \quad \text{if } P(\underline{\omega}_i \mid \underline{\vartheta}) \geq P(\underline{\omega}_j \mid \underline{\vartheta}) \quad \text{for all } j \neq i \tag{13}$$

we have to calculate the vector of three conditional probabilities, $i \in \{1, 0, 2\}$, which we do using the theorem of Bayes (Koch 1987):

$$P(\underline{\omega}_i \mid \underline{\vartheta}) = \frac{P(\underline{\vartheta} \mid \underline{\omega}_i)\, P(\underline{\omega}_i)}{\sum\limits_{\omega_j \in \Omega_\vartheta} P(\underline{\vartheta} \mid \underline{\omega}_j)\, P(\underline{\omega}_j)} \tag{14}$$

The probability $P(\underline{\vartheta} \mid \underline{\omega}_i)$ can be calculated with Eq. 11, with the benefit of eliminating the factor $\Delta\vartheta$, which appears in each product of numerator and denominator.

Because of finite intervals in Ω_ϑ, the probability of a class ω_i can given with the length of the interval i, which leads to a low probability of ω_0, and high probabilities for ω_1 and ω_2.

With that Eq. 14 can be solved with given β, σ_β, and ϑ. Now we have to show the transition from classifying distances to a classification of relations. An observed triple $\{C^i, \vartheta_{\min}, \vartheta_{\max}\}$ can be classified by Eqs. 13, 14 to $\{C^i, \omega_{\min}, \omega_{\max}\}$. With $P(C = C^i) = 1$, a consequence of the discussion in Section 3.1, we find:

$$P(C^i,\, \omega_{\min},\, \omega_{\max} \mid C^i,\, \vartheta_{\min},\, \vartheta_{\max}) = P(\omega_{\min} \mid \vartheta_{\min})\, P(\omega_{\max} \mid \vartheta_{\max}) \tag{15}$$

But this probability is the probability of the topological relation between $\mathcal{A}$ and $\mathcal{B}$ (cf. Tab. 1). Probabilities of alternative classifications can be calculated additionally, which allows to assess a decision.

5 Discussion

We present a morphological distance function which we use to determine the topological relation between two regions. We show the equivalence of the extremal distances and the known representation by intersection sets. Considering the two regions as (positional) uncertain, the observation of the distances can be modeled statistically. Within this model we propagate the uncertainty of the observations to the uncertainty of topological relations. Applying a statistical decision rule, the decision yields probabilities of the classification result, and also of alternative relations, which allow an assessment of the decision.

Our approach has some new aspects:

- a combined statistical model of observation uncertainty and relation uncertainty;
- a statistical model of the lack of definition in spatial abstraction;
- a new view on the conceptual-neighborhood-graph, which earlier was used in the context of motion or deformation, and now is adapted to positional uncertainty;
- a partitioning of the conceptual-neighborhood-graph, based on weighting the central relation OVERLAP;
- the use of morphological distances, which keep metric information about the magnitude of intersection sets, instead of empty or non-empty intersection sets.

The proposed method is relevant for all aspects in GIS:

Input. Single data layers are less involved in our question, because they follow semantical constraints in their topological structure. But data homogenization between layers of different thematic classes, as e. g. after data import, requires techniques to support decisions for eliminating slivers *etc.* The problem of a geometric determination of common boundaries is not touched.

Management. The topological structure usually is the basis of a data model. In a first step data homogenization may lead to a topological structure keeping the relations certain, e. g. by maximum likelihood decisions, cutting alternatives and probabilities. Then storing positional uncertainty of objects has to be solved elsewhere.

But for keeping alternatives and probabilities further, data models must be developed, and the propagation of uncertain topological structures in spatial analysis must be investigated.

Analysis. Up to now it is a logical problem to reason from a set of known topological relations to additional ones. Now the propagation of probabilities could be introduced in reasoning. Also the existence and the probability of alternative relations to the known uncertain relations has to be investigated. Possibly additional rules for reasoning, or combined probabilities in reasoning are to be handled with.

Another problem is the consistency in a network of topological relations, which includes alternative relations.

Presentation. Visualizing uncertainty is an actual theme of research, but in the context of topological relations, or more general of qualitative relations it is, for our knowledge, a completely new question.

The presented ideas are worked out for simple regions and conditions for a small positional uncertainty. Further work should extend these ideas for complex objects — an intermediate step could be generalized regions (Egenhofer *et al.* 1994) —, or for objects of other dimensions. It is to proof that the ideas hold for simple 1D-objects in $\mathbb{R}$, or for simple 3D-objects in $\mathbb{R}^3$. Also it is to investigate whether the method is to transfer to other qualitative spatial relationships.

Another aspect of further research is the extension of sources of uncertainty, or the combination with other techniques of handling uncertainty, e. g. knowledge from thematic properties, or cartographic generalization rules.

References

Caspary, Wilhelm; Scheuring, Robert (1992): Error-Bands as Measures of Geometrical Accuracy. In: *EGIS 92*, pages 226–233, Utrecht, 1992.

Chrisman, Nicholas R. (1982): A theory of cartographic error and its measurement in digital databases. In: *Auto-Carto 5*, pages 159–168, Crystal City, 1982.

Clementini, Eliseo; Di Felice, Paolino (1996): An Algebraic Model for Spatial Objects with Indeterminate Boundaries. In: Burrough, Peter A.; Frank, Andrew U. (Eds.), *Geographic Objects with Indeterminate Boundaries*, Band 2 der Reihe GISDATA, Chapter 11, pages 155–169. Taylor & Francis, 1996.

Egenhofer, Max J.; Al-Taha, Khaled K. (1992): Reasoning about Gradual Changes of Topological Relationships. In: Frank, A. U.; Campari, I.; Formentini, U. (Eds.), *Theories and Models of Spatio-Temporal Reasoning in Geographic Space*, pages 196–219, New York, 1992. Springer LNCS 639.

Egenhofer, Max J.; Herring, John R. (1991): Categorizing Binary Topological Relationships Between Regions, Lines, and Points in Geographic Databases. Technical report, Department of Surveying Engineering, University of Maine, Orono, ME, 1991.

Egenhofer, Max J.; Clementini, Eliseo; di Felice, Paolino (1994): Topological relations between regions with holes. *International Journal of Geographical Information Systems*, 8(2):129–142, 1994.

Galton, Anthony (1994): Perturbation and Dominance in the Qualitative Representation of Continous State-Spaces. Technical Report 270, Department of Computer Science, University of Exeter, Exeter, 1994.

Koch, Karl Rudolf (1987): *Parameterschätzung und Hypothesentests*. Dümmler, Bonn, 2. edition, 1987.

Kraus, Karl; Haussteiner, Karl (1993): Visualisierung der Genauigkeit geometrischer Daten. *GIS*, 6(3):7–12, 1993.

Kraus, Karl; Kager, H. (1994): Accuracy of Derived Data in a Geographic Information System. *Computer, Environment and Urban Systems*, 18(2):87–94, 1994.

Lantuejoul, Christian (1978): *La squelettisation et son application aux mesures topologiques des mosaiques polycristallines*. PhD thesis, Ecole Nationale Superieure des Mines de Paris, 1978.

Molenaar, Martien (1994): A Syntax for the Representation of Fuzzy Spatial Objects. In: Molenaar, Martien; Hoop, Sylvia de (Eds.), *Advanced Geographic Data Modelling*, pages 155–169, Delft, 1994. Proc. AGDM '94, Netherlands Geodetic Commission.

Papoulis, Athanasios (1965): *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, 1965.

Perkal, J. (1956): On Epsilon Length. *Bulletin de l'Academie Polonaise des Sciences*, 4:399–403, 1956.

Serra, Jean (Ed.) (1982): *Image Analysis and Mathematical Morphology*, Band 1. Academic Press, 1982.

Wazinski, Peter (1993): Graduated Topological Relations. Technical Report 54, Universität des Saarlandes, 1993.

Winter, Stephan (1994): Uncertainty of Topological Relations in GIS. In: Ebner, H.; Heipke, C.; Eder, K. (Eds.), *Proc. of ISPRS Comm. III Symposium* Spatial Information from Digital Photogrammetry and Computer Vision, pages 924–930, München, 1994.

Winter, Stephan (1996): *Unsichere topologische Beziehungen zwischen ungenauen Flächen*. PhD thesis, Landwirtschaftliche Fakultät der Universität Bonn, 1996.

Wolf, Helmut (1975): *Ausgleichungsrechnung*. Dümmler, Bonn, 1975.

Worboys, Michael F.; Bofakos, Petros (1993): A Canonical Model for a Class of Areal Spatial Objects. In: Abel, David; Ooi, Beng Chin (Eds.), *Advances in Spatial Databases*, pages 36–52. Springer (LNCS 692), 1993.

Zadeh, L. A. (1965): Fuzzy Sets. *Information and Control*, 8:338–353, 1965.