Optimal Camera Orientation from Points and Straight Lines

Marc Luxen and Wolfgang Förstner

Institut für Photogrammetrie, Universität Bonn
Nußallee 15, D-53115 Bonn
Germany
luxen/wf@ipb.uni-bonn.de,
WWW home page: http://www.ipb.uni-bonn.de

Abstract. This paper presents an optimal estimate for the projection matrix for points of a camera from an arbitrary mixture of six or more observed points and straight lines in object space. It gives expressions for determining the corresponding projection matrix for straight lines together with its covariance matrix. Examples on synthetic and real images demonstrate the feasibility of the approach.

1 Introduction

Determining the orientation of a camera is a basic task in computer vision and photogrammetry. Direct as well as iterative solutions for calibrated and uncalibrated cameras are well known, in case only points are used as basic entities. Hartley and Zisserman (cf. [4], pp. 166-169) indicate how to use point and line observations simultaneously for estimating the projection matrix for points. However, they give no optimal solution to this problem.

We present a procedure for optimal estimation of the projection matrix and for determining the covariance matrix of its entries. This can be used to derive the covariance matrix of the projection matrix for lines, which allows to infer the uncertainty of projecting lines and planes from observed points and lines.

The paper is organized as follows: Section 2 presents some basic tools from algebraic projective geometry, gives expressions for the projection of 3D points and lines from object space into the image plane of a camera and explains the determination of the matrix Q for line projection from the matrix P for point projection. Section 3 describes the procedure to estimate P and to derive the covariance matrices of the orientation parameters. Finally we give examples to demonstrate the feasibility of the approach.

2 Basics

We first give the necessary tools from algebraic projective geometry easing the computation of statistically uncertain geometric entities.

2.1 Points, lines and planes and their images

Points $\mathbf{x}^\mathsf{T} = (\boldsymbol{x}_0^\mathsf{T}, x_h)$ and lines $\mathbf{l} = (\boldsymbol{l}_h^\mathsf{T}, l_0)$ in the plane are represented with 3-vectors, splitted into their homogeneous part, indexed with h, and non-homogeneous part indexed with 0. Points $\mathbf{X} = (\boldsymbol{X}_0^\mathsf{T}, X_h)$ and planes $\mathbf{A} = (\boldsymbol{A}_h^\mathsf{T}, A_0)$ in 3D space are represented similarly. Lines $\mathbf{L}$ in 3D are represented with their *Plücker coordinates* $\mathbf{L}^\mathsf{T} = (\boldsymbol{L}_h^\mathsf{T}, \boldsymbol{L}_0^\mathsf{T})$. It can be derived from two points $\mathbf{X}$ and $\mathbf{Y}$ by the direction $\boldsymbol{L}_h = Y_h \boldsymbol{X}_0 - X_h \boldsymbol{Y}_0$ of the line and the normal $\boldsymbol{L}_0 = \boldsymbol{X}_0 \times \boldsymbol{Y}_0$ of the plane through the line and the origin. We will need the dual 3D-line $\overline{\mathbf{L}}^\mathsf{T} = (\boldsymbol{L}_0^\mathsf{T}, \boldsymbol{L}_h^\mathsf{T})$ which has homogeneous and non-homogeneous part interchanged. The line parameters have to fulfill the *Plücker constraint* $L_1L_4 + L_2L_5 + L_3L_6 = \boldsymbol{L}_h^\mathsf{T}\boldsymbol{L}_0 = \frac{1}{2}\mathbf{L}^\mathsf{T}\overline{\mathbf{L}} = 0$ which is clear, as $\boldsymbol{L} = Y_h\boldsymbol{X}_0 - X_h\boldsymbol{Y}_0$ is orthogonal to $\boldsymbol{L}_0 = \boldsymbol{X}_0 \times \boldsymbol{Y}_0$. All 6-vectors $\mathbf{L} \neq 0$ fulfilling the Plücker constraint represent a 3D line.

All links between two geometric elements are bilinear in their homogeneous coordinates, an example being the line joining two points in 3D . Thus the coordinates of new entities can be written in the form

$$\boldsymbol{\gamma} = \mathsf{A}(\boldsymbol{\alpha})\boldsymbol{\beta} = \mathsf{B}(\boldsymbol{\beta})\boldsymbol{\alpha} \qquad \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\beta}} = \mathsf{A}(\boldsymbol{\alpha}) \qquad \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\alpha}} = \mathsf{B}(\boldsymbol{\beta})$$

Thus the matrices $\mathsf{A}(\boldsymbol{\alpha})$ and $\mathsf{B}(\boldsymbol{\beta})$ have entries being linear in the coordinates $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. At the same time they are the Jacobians of $\boldsymbol{\gamma}$.

We then may use error propagation or the propagation of covariances of linear $\boldsymbol{y} = \mathsf{A}\boldsymbol{x}$ functions of $\boldsymbol{x}$ with covariance matrix $\boldsymbol{\Sigma}_{xx}$ leading to $\boldsymbol{\Sigma}_{yy} = \mathsf{A}\boldsymbol{\Sigma}_{xx}\mathsf{A}^\mathsf{T}$ to obtain rigorous expressions for the covariance matrices of constructed entities $\boldsymbol{\gamma}$,

$$\boldsymbol{\Sigma}_{\gamma\gamma} = \mathsf{A}(\boldsymbol{\alpha})\boldsymbol{\Sigma}_{\beta\beta}\mathsf{A}^\mathsf{T}(\boldsymbol{\alpha}) + \mathsf{B}(\boldsymbol{\beta})\boldsymbol{\Sigma}_{\alpha\alpha}\mathsf{B}^\mathsf{T}(\boldsymbol{\beta})$$

in case of stochastic independence.

We need the line $\mathbf{L}$ as intersection of two planes $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{L} = \mathbf{A} \cap \mathbf{B} = \begin{pmatrix} X_h \mathsf{I} & -\boldsymbol{X}_0 \\ \mathsf{S}_{X_0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_h \\ B_0 \end{pmatrix} = \mathsf{\Pi}(\mathbf{A})\mathbf{B} = -\mathsf{\Pi}(\mathbf{B})\mathbf{A}$$

inducing the matrix $\mathsf{\Pi}(\mathbf{A})$ depending on the entries of the plane $\mathbf{A}$. We also need the incidence constraints for points $\mathbf{x}$ and lines $\mathbf{l}$ in 2D, for points $\mathbf{X}$ and planes $\mathbf{A}$ in 3D and for two lines $\mathbf{L}$ and $\mathbf{M}$ in 3D:

$$\mathbf{x}^\mathsf{T}\mathbf{l} = 0 \qquad \mathbf{X}^\mathsf{T}\mathbf{A} = 0 \qquad \mathbf{L}^\mathsf{T}\overline{\mathbf{M}} = 0 \tag{1}$$

2.2 Projection of points and lines in homogeneous coordinates

Point projection. As explained in [4], the relation between a 3D point in object space and its image point in the image plane can be written as

$$\mathbf{x}' = \mathsf{P}\mathbf{X} \quad \text{with} \quad \mathsf{P}_{3\times4} = \begin{pmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \\ \mathbf{C}^\mathsf{T} \end{pmatrix} \tag{2}$$

where $\mathbf{X}$ is the homogeneous coordinate vector of a point in object space, $\mathbf{x}'$ the homogeneous coordinate vector representing its image point and P the 3×4-matrix for point projection. Due to the homogeneity of P it only contains 11 independent elements. Its rows $\mathbf{A}^\mathsf{T}$, $\mathbf{B}^\mathsf{T}$ and $\mathbf{C}^\mathsf{T}$ can be interpreted as homogeneous coordinates of the three coordinate planes $\mathrm{x}'_1 = 0$, $\mathrm{x}'_2 = 0$ and $\mathrm{x}'_3 = 0$ intersecting in the projection center $\mathbf{X}_c$ of the camera (cf. [4]). Analogously, the columns of P can be interpreted as the images of the three points at infinity of the three coordinate axes and of the origin.

Line projection. For line projection, it holds (c. f. [2])

$$\mathbf{l}' = \mathsf{Q}\overline{\mathbf{L}} \text{ with } \underset{3\times 6}{\mathsf{Q}} = \begin{pmatrix} \mathbf{U}^\mathsf{T} \\ \mathbf{V}^\mathsf{T} \\ \mathbf{W}^\mathsf{T} \end{pmatrix} = \begin{pmatrix} (\mathbf{B} \cap \mathbf{C})^\mathsf{T} \\ (\mathbf{C} \cap \mathbf{A})^\mathsf{T} \\ (\mathbf{A} \cap \mathbf{B})^\mathsf{T} \end{pmatrix} = \begin{pmatrix} \Pi(\mathbf{B})\mathbf{C} \\ \Pi(\mathbf{C})\mathbf{A} \\ \Pi(\mathbf{A})\mathbf{B} \end{pmatrix} = \begin{pmatrix} -\Pi(\mathbf{C})\mathbf{B} \\ -\Pi(\mathbf{A})\mathbf{C} \\ -\Pi(\mathbf{B})\mathbf{A} \end{pmatrix} \tag{3}$$

where the 6×1 vector $\overline{\mathbf{L}}$ contains the Plücker coordinates of the straight line dual to the line $\mathbf{L}$ and $\mathbf{l}'$ denotes the coordinates of the image of $\mathbf{L}$. As shown in [2], the rows of the 3×6 projection matrix Q for line projection represent the intersections of the planes mentioned above. Therefore they can be interpreted as the Plücker coordinates of the three coordinate axes $\mathrm{x}'_2 = \mathrm{x}'_3 = 0$, $\mathrm{x}'_3 = \mathrm{x}'_1 = 0$ and $\mathrm{x}'_1 = \mathrm{x}'_2 = 0$.

Inversion. Inversion of the projection leads to projection rays $\mathbf{L}'$ for image points $\mathbf{x}'$ and for projection planes $\mathbf{A}'$ for image lines $\mathbf{l}'$

$$\mathbf{L}' = \mathsf{Q}^\mathsf{T}\mathbf{x}' \qquad \mathbf{A}' = \mathsf{P}^\mathsf{T}\mathbf{l}' \tag{4}$$

The expression for $\mathbf{L}'$ results from the incidence relation $\mathbf{x}'^\mathsf{T}\mathbf{l}' = 0$ (cf. (1a)), for all lines $\mathbf{l}' = \mathsf{Q}\overline{\mathbf{L}}$ passing through $\mathbf{x}'$, leading to $(\mathbf{x}'^\mathsf{T}\mathsf{Q})\,\overline{\mathbf{L}} = \mathbf{L}'^\mathsf{T}\overline{\mathbf{L}} = 0$ using (1c). The expression for $\mathbf{A}'$ results from the incidence relation $\mathbf{l}'^\mathsf{T}\mathbf{x}' = 0$ for all points $\mathbf{x}' = \mathsf{P}\mathbf{X}$ on the line $\mathbf{l}'$, leading to $(\mathbf{l}'^\mathsf{T}\mathsf{P})\,\mathbf{X} = \mathbf{A}'^\mathsf{T}\mathbf{X} = 0$ using (1b). Especially each point $\mathbf{X}$ on the line $\mathbf{L}$ lies in the projecting plane $\mathbf{A}'$, therefore it holds

$$\mathbf{l}'^\mathsf{T}\mathsf{P}\mathbf{X} = 0. \tag{5}$$

2.3 Determination of Q from P.

For a given matrix P together with the covariance matrix $\boldsymbol{\Sigma}_{\beta\beta}$ of its elements $\beta = (\mathbf{A}^\mathsf{T}, \mathbf{B}^\mathsf{T}, \mathbf{C}^\mathsf{T})^\mathsf{T}$ we easily can derive the matrix $\mathsf{Q} = (\mathbf{U}, \mathbf{V}, \mathbf{W})^\mathsf{T}$ using (3). By error propagation we see that the covariance matrix $\boldsymbol{\Sigma}_{\gamma\gamma}$ of $\gamma = (\mathbf{U}, \mathbf{V}, \mathbf{W})^\mathsf{T}$ is given by

$$\boldsymbol{\Sigma}_{\gamma\gamma} = \mathbf{M}\boldsymbol{\Sigma}_{\beta\beta}\mathbf{M}^\mathsf{T}$$

with the 18×12 matrix

$$\mathbf{M} = \begin{pmatrix} 0 & -\Pi(\mathbf{C}) & \Pi(\mathbf{B}) \\ \Pi(\mathbf{C}) & 0 & -\Pi(\mathbf{A}) \\ -\Pi(\mathbf{B}) & \Pi(\mathbf{A}) & 0 \end{pmatrix}.$$

In a similar manner we may now determine the uncertainty of the projecting lines and planes in (4).

3 Estimation of the projection matrix P

There are several possibilities to determine the projection matrix P for points from observations of points or straight lines in an image.

For the case that only points are observed, eq. (2) allows us to determine P using the well known DLT algorithm (cf. [4], p. 71). If no points but only straight lines are observed, we might use a similar algorithm based on eq. (3) to determine the matrix Q for lines. Then we need to derive P from Q, which is possible but of course not a very elegant method for determining P. Integrating point and line observations would require a separate determination of P independently estimated from observed points and lines and a fusion of the two projection matrices, which, moreover, would require a minimum of 6 points and 6 lines.

We therefore follow the proposal in [4] and use (2) for points and (5) for lines, as they both are linear in the unknown parameters of the projection matrix P. It only requires at least six pints and lines, however, in an arbitrary mixture. We here give an explicit derivation of the estimation process which not only gives an algebraic solution but a statistically optimal one.

Observations of points. For each observed point $\mathbf{x}'_i = (u'_i, v'_i, w'_i)^\mathsf{T}$, $i = 1, \ldots, I$ it holds (cf. (2))

$$\frac{u'_i}{w'_i} = \frac{\mathbf{A}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i} \qquad \text{and} \qquad \frac{v'_i}{w'_i} = \frac{\mathbf{B}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i}$$

which leads to the two constraints $u'_i\mathbf{C}^\mathsf{T}\mathbf{X}_i - w'_i\mathbf{A}^\mathsf{T}\mathbf{X}_i = 0$ and $v'_i\mathbf{C}^\mathsf{T}\mathbf{X}_i - w'_i\mathbf{B}^\mathsf{T}\mathbf{X}_i = 0$ which are dedicated to estimate the elements of P. In matrix representation, these two constraints can be formulated as bilinear forms

$$\mathsf{A}_i(\mathbf{x}'_i)\boldsymbol{\beta} = \begin{pmatrix} -w'_i\mathbf{X}_i^\mathsf{T} & 0 & u'_i\mathbf{X}_i^\mathsf{T} \\ 0 & -w'_i\mathbf{X}_i^\mathsf{T} & v'_i\mathbf{X}_i^\mathsf{T} \end{pmatrix} \boldsymbol{\beta} = \mathbf{e}_i \qquad \text{resp.} \tag{6}$$

$$\mathsf{B}_i(\boldsymbol{\beta})\mathbf{x}'_i = \begin{pmatrix} \mathbf{C}^\mathsf{T}\mathbf{X}_i & 0 & -\mathbf{A}^\mathsf{T}\mathbf{X}_i \\ 0 & \mathbf{C}^\mathsf{T}\mathbf{X}_i & -\mathbf{B}^\mathsf{T}\mathbf{X}_i \end{pmatrix} \mathbf{x}_i = \mathbf{e}_i \tag{7}$$

where the vector $\boldsymbol{\beta}^\mathsf{T} = (\mathbf{A}^\mathsf{T}, \mathbf{B}^\mathsf{T}, \mathbf{C}^\mathsf{T})$ contains all unknown elements of P. Eq. (6) will be used to estimate the $\boldsymbol{\beta}$, (7) will be used for determining the uncertainty of the residuals $\mathbf{e}_i$ by error propagation.

As each observed point $\mathbf{x}'_i$ yields 2 constraints to determine the 11 unknown elements $\boldsymbol{\beta}$, $I \geq 6$ observed points would be needed if only points are observed in the image.

Observations of straight lines. For each observed straight line $\mathbf{l}'_j = (a'_j, b'_j, c'_j)^{\mathsf{T}}$, $j = 1, \ldots, J$ eq. (5) is valid.

Thus, if $\mathbf{X}_{A,j}$ and $\mathbf{X}_{E,j}$ are two points lying on the 3D line with image $\mathbf{l}'$, one yields the two constraints

$$\mathbf{l}'^{\mathsf{T}}\mathsf{P}\mathbf{X}_{A,j} = 0 \qquad \text{resp.} \qquad a_j\mathbf{X}_{A,j}^{\mathsf{T}}\mathbf{A} + b_j\mathbf{X}_{A,j}^{\mathsf{T}}\mathbf{B} + c_j\mathbf{X}_{A,j}^{\mathsf{T}}\mathbf{C} = 0 \qquad \text{and}$$

$$\mathbf{l}'^{\mathsf{T}}\mathsf{P}\mathbf{X}_{E,j} = 0 \qquad \text{resp.} \qquad a_j\mathbf{X}_{E,j}^{\mathsf{T}}\mathbf{A} + b_j\mathbf{X}_{E,j}^{\mathsf{T}}\mathbf{B} + c_j\mathbf{X}_{E,j}^{\mathsf{T}}\mathbf{C} = 0$$

or in matrix representation as bilinear constraints

$$\mathsf{C}_j(\mathbf{l}'_j)\boldsymbol{\beta} = \begin{pmatrix} a_j\mathbf{X}_{A,j}^{\mathsf{T}} & b_j\mathbf{X}_{A,j}^{\mathsf{T}} & c_j\mathbf{X}_{A,j}^{\mathsf{T}} \\ a_j\mathbf{X}_{E,j}^{\mathsf{T}} & b_j\mathbf{X}_{E,j}^{\mathsf{T}} & c_j\mathbf{X}_{E,j}^{\mathsf{T}} \end{pmatrix} \boldsymbol{\beta} = \mathbf{e}_j \qquad \text{resp.} \tag{8}$$

$$\mathsf{D}_j(\boldsymbol{\beta})\mathbf{l}'_j = \begin{pmatrix} \mathbf{X}_{A,j}^{\mathsf{T}}\mathbf{A} & \mathbf{X}_{A,j}^{\mathsf{T}}\mathbf{B} & \mathbf{X}_{A,j}^{\mathsf{T}}\mathbf{C} \\ \mathbf{X}_{E,j}^{\mathsf{T}}\mathbf{A} & \mathbf{X}_{E,j}^{\mathsf{T}}\mathbf{B} & \mathbf{X}_{E,j}^{\mathsf{T}}\mathbf{C} \end{pmatrix} \mathbf{l}'_j = \mathbf{e}_j \tag{9}$$

Again, (8) will be used to determine $\boldsymbol{\beta}$, while (9) will be used for determining the uncertainty of the residuals $\mathbf{e}_j$ by error propagation.

We see that each observation of a straight line in an image yields 2 constraints to determine P. Thus again, if only straight lines are observed, $J \geq 6$ lines are needed.

Note that in the following entities concerning observations of points are indexed with i and entities concerning observations of straight lines are indexed with j.

Parameter estimation. Now we seek an optimal estimate for $\boldsymbol{\beta}$. Due to the homogeneity of P we search for an optimal estimate just under the constraint $|\boldsymbol{\beta}| = 1$. Thus, we optimize

$$\Omega = \sum_i \mathbf{e}_i^{\mathsf{T}}(\boldsymbol{\beta})\ \boldsymbol{\Sigma}_{e_ie_i}^{-1}(\boldsymbol{\beta})\ \mathbf{e}_i(\boldsymbol{\beta}) + \sum_j \mathbf{e}_j^{\mathsf{T}}(\boldsymbol{\beta})\ \boldsymbol{\Sigma}_{e_je_j}^{-1}(\boldsymbol{\beta})\ \mathbf{e}_j(\boldsymbol{\beta})$$

under the constraint $|\boldsymbol{\beta}| = 1$. In case the observations are normally distributed this is the ML-estimate (under this model).

The solution can be achieved by iteratively solving

$$\boldsymbol{\beta}^{(\nu+1)\mathsf{T}}\Big[\sum_i \mathsf{A}_i{}^{\mathsf{T}}(\widehat{\mathbf{x}}_i'^{(\nu)}) \left(\mathsf{B}_i(\boldsymbol{\beta}^{(\nu)})\boldsymbol{\Sigma}_{x'_ix'_i}\mathsf{B}_i{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})\right)^{-1} \mathsf{A}_i(\mathbf{x}'_i)$$
$$+ \sum_j \mathsf{C}_j{}^{\mathsf{T}}(\widehat{\mathbf{l}}_j'^{(\nu)}) \left(\mathsf{D}_j(\boldsymbol{\beta}^{(\nu)})\boldsymbol{\Sigma}_{l_jl_j}\mathsf{D}_j{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})\right)^{-1} \mathsf{C}_j(\mathbf{l}'_j)\Big]\boldsymbol{\beta}^{(\nu+1)} \underset{|\boldsymbol{\beta}^{(\nu+1)}|=1}{\rightarrow} \min \tag{10}$$

(cf. [6]). Observe, this is the solution of an ordinary eigenvalue problem with a non-symmetric matrix, as the left factor, e. g. $\mathsf{A}_i{}^{\mathsf{T}}(\widehat{\mathbf{x}}_i'^{(\nu)})$ uses the fitted observations $\widehat{\mathbf{x}}_i'^{(\nu)}$, whereas the right factor $\mathsf{A}_i(\mathbf{x}'_i)$ uses the observed values.

The covariance matrices of $\mathbf{e}_i$ and $\mathbf{e}_j$ are given by $\mathsf{B}_i(\boldsymbol{\beta}^{(\nu)})\boldsymbol{\Sigma}_{x'_ix'_i}\mathsf{B}_i{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})$ and $\mathsf{D}_i(\boldsymbol{\beta}^{(\nu)})\boldsymbol{\Sigma}_{l'_il'_i}\mathsf{D}_i{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})$, so that they have to use the previous estimate $\boldsymbol{\beta}^{(\nu)}$.

The procedure is initiated with $\boldsymbol{\Sigma}_{e_ie_i} = \boldsymbol{\Sigma}_{e_je_j} = \mathbf{I}$ and using the observed values $\widehat{\mathbf{x}}'_i$ and $\widehat{\mathbf{l}}'_j$ as initial values $\widehat{\mathbf{x}}'^{(0)}_i$ and $\widehat{\mathbf{l}}'^{(0)}_j$ for the fitted observations. This leads in the first step to the classical algebraic solution (cf. [1], p. 332, p. 377) and no approximate values are needed.

The estimated observations $\widehat{\mathbf{x}}'^{(\nu)}_i$ and $\widehat{\mathbf{l}}'^{(\nu)}_j$ needed in (10) can be calculated by (cf. [6])

$$\widehat{\mathbf{x}}'^{(\nu)}_i = \left(\mathsf{I} - \boldsymbol{\Sigma}_{x'_ix'_i}\mathsf{B}_i{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})\left(\mathsf{B}_i(\boldsymbol{\beta}^{(\nu)})\boldsymbol{\Sigma}_{x'_ix'_i}\mathsf{B}_i{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})\right)^{-1}\mathsf{B}_i(\boldsymbol{\beta}^{(\nu)})\right)\mathbf{x}'_i \quad (11)$$

$$\widehat{\mathbf{l}}'^{(\nu)}_j = \left(\mathsf{I} - \boldsymbol{\Sigma}_{l'_jl'_j}\mathsf{D}_j{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})\left(\mathsf{D}_j(\boldsymbol{\beta}^{(\nu)})\boldsymbol{\Sigma}_{l'_jl'_j}\mathsf{D}_j{}^{\mathsf{T}}(\boldsymbol{\beta}^{(\nu)})\right)^{-1}\mathsf{D}_j(\boldsymbol{\beta}^{(\nu)})\right)\mathbf{l}'_j \quad (12)$$

With the final weight matrix $\boldsymbol{\Sigma}^{+}_{ee} = \mathrm{Diag}(\mathrm{Diag}(\boldsymbol{\Sigma}^{+}_{e_ie_i}), \mathrm{Diag}(\boldsymbol{\Sigma}^{+}_{e_je_j}))$ of the contradictions $\mathbf{e}_i$ and $\mathbf{e}_j$ the covariance matrix

$$\boldsymbol{\Sigma}_{ll} = \mathrm{Diag}(\mathrm{Diag}(\boldsymbol{\Sigma}_{x'_ix'_i}), \mathrm{Diag}(\boldsymbol{\Sigma}_{l'_jl'_j}))$$

of the observations, the covariance matrix of the estimated values is given by

$$\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}\widehat{\boldsymbol{\beta}}} = \mathsf{M}^{+} \quad \text{with} \quad \mathsf{M} = \sum_i \mathsf{A}_i{}^{\mathsf{T}}(\widehat{\mathbf{x}}'_i)\left(\mathsf{B}_i(\widehat{\boldsymbol{\beta}})\boldsymbol{\Sigma}_{x'_ix'_i}\mathsf{B}_i{}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}})\right)^{-1}\mathsf{A}_i(\widehat{\mathbf{x}}'_i)$$
$$+ \sum_j \mathsf{C}_j{}^{\mathsf{T}}(\widehat{\mathbf{l}}'_j)\left(\mathsf{D}_j(\widehat{\boldsymbol{\beta}})\boldsymbol{\Sigma}_{l'_jl'_j}\mathsf{D}_j{}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}})\right)^{-1}\mathsf{C}_j(\widehat{\mathbf{l}}'_j)$$

The pseudo inverse M^{+} can be determined exploiting the fact that $\widehat{\boldsymbol{\beta}}$ should be the null-space of M^{+}. We obtain an estimate for the unknown variance factor $\hat{\sigma}^2 = \Omega/R$ with $R = \mathrm{rk}(\boldsymbol{\Sigma}_{\mathbf{ee}}) - 11$ with the redundancy R and the weighted sum of squared residuals of the constraints Ω. The redundancy results form the fact that we effectively have $\mathrm{rk}(\boldsymbol{\Sigma}_{\mathbf{ee}})$ constraints which have to determine 11 unknown independent parameters of P. Therefore the estimated covariance matrix of the estimated parameters is given by

$$\hat{\boldsymbol{\Sigma}}_{\widehat{\boldsymbol{\beta}}\widehat{\boldsymbol{\beta}}} = \hat{\sigma}^2\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\beta}}\widehat{\boldsymbol{\beta}}}$$

4 Examples

4.1 Synthetic data

The presented method was applied on a synthetical image, only using straight lines as observations. The straight lines used connect all point pairs of a synthetic cube. The cube has size 2 [m] and is centered at the origin, the projection center is $(10, -3, 4)^{\mathsf{T}}$ [m] and therefore has a distance of approx. 11.2 [m]. The principal distance is 500 [pel], the assumed accuracy for the observed lines is 1 [pel], which

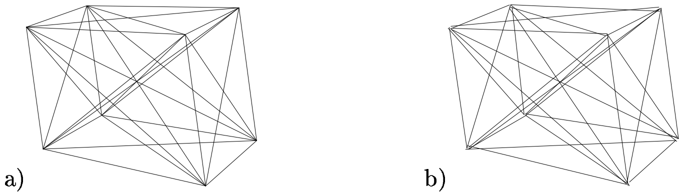

Fig. 1. Synthetic test data. (**a**) Ideal observations of lines in an image of a synthetic Cube. (**b**) Erroneous observations, assuming an accuracy of 1[pel] referring to the end points.

refers to the two end points. The cube appears with a diameter of approx. 200 [pel], thus the viewing angle is approx. 23^o and the measuring accuracy is approx. 1 : 200. The figure shows the image of the ideal cube and the image of the erroneous lines. The estimated projection matrix, determined from 28 observed lines, differs from the ideal one by 0.6 %, an accuracy which is to be expected.

4.2 Real data

The presented method to determine P was also applied on real data, again using only observations of straight lines. The test object has the shape of an unsymmetric T with a height of 12 cm and a width of 14.5 cm (cf. Fig. 4.2). It is not very precise, so that object edges are not very sharp. Object information was gained manually by measuring the lengths of the object edges, supposing that the object edges are orthogonal. An image of the test object was taken (cf. Fig. 4.2a) using a digital camera SONY XC77 CE with a CCD chip of size 750 $\times$ 560 pixel. From the image straight lines were extracted automatically using the feature extraction software FEX (cf. [3]), leading to the results shown in Fig. 4.2b.

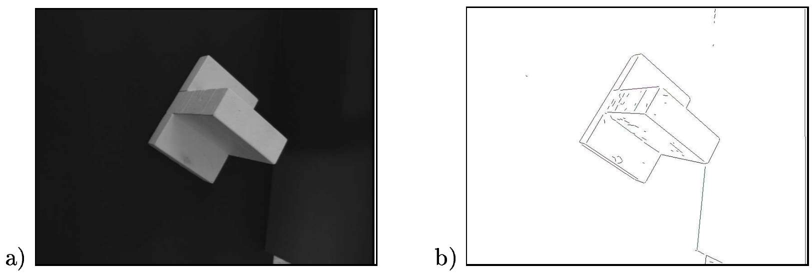

Fig. 2. Real test data. (**a**) Image of the test object. (**b**) Results of the feature extraction with FEX.

With revised data (short and spurious lines were thrown out) the parameter estimation delivered the estimate

$$\widehat{\mathsf{P}} = \begin{pmatrix} 0.00295953 & -0.00320425 & -0.00106391 & 0.564433 \\ 0.00397216 & 0.00257314 & 0.0003446 & 0.743515 \\ 0.00000259 & 0.00000141 & -0.0000050 & 0.00246196 \end{pmatrix}$$

with the relative accuracies $\widehat{\sigma}_{P_{ij}}/\widehat{P}_{ij}$ given in the following table:

(i,j)	1	2	3	4
1	0.883 %	0.598 %	1.711 %	0.116 %
2	0.964 %	1.492 %	9.504 %	0.068 %
3	3.013 %	8.218 %	1.393 %	0.049 %

The relative accuracies indicate that most elements of P are determined with good precision, particularly in view of the fact that the object information we used is not very precise and that the observed object edges are not very sharp. Obviously, the relative accuracy of some of the elements of P is inferior to the others. This is caused by the special geometry of the view in this example and not by the estimation method applied.

5 Conclusion

The examples demonstrate the feasibility of the presented method to estimate the projection matrix P for points. The method is practical, as it needs no approximate values for the unknown parameters and statistically optimal, as it leads to the maximum likelihood estimate. Therefore we think that it could be frequently used in computer vision and photogrammetry.

Acknowledgement

This work was supported by the German Research Council (DFG).

References

1. Duda, R. O., Hart, P. E.: Pattern Classification and Scene Analysis. Wiley (1973)
2. Faugeras, O, Papadopoulo, T.: Grassmann-Caley Algebra for Modeling Systems of Cameras and the Algebraic Equations of the Manifold of Trifocal Tensors, Trans. of the ROYAL SOCIETY A 356, (1998) 1123 – 1152
3. Fuchs, C.: Extraktion polymorpher Bildstrukturen und ihre topologische und geometrische Gruppierung, DGK, Bayer. Akademie der Wissenschaften, Reihe C, Heft 502
4. Hartley, R. Zisserman, A.: Multiple View Geometry in Computer Vision, Cambridge University Press (2000)

5. Förstner, W., Brunn, A., Heuel, S.: Statistically Testing Uncertain Geometric Relations, In G. Sommer, N. Krüger and Ch. Perwass, editors, *Mustererkennung 2000*, pp. 17-26. DAGM, Springer, September 2000.
6. Förstner , W.: Algebraic Projective Geometry and Direct Optimal Estimation of Geometric Entities, In Stefan Scherer, editor, *Computer Vision, Computer Graphics and Photogrammetry - a Common Viewpoint*, pp. 67-86. ÖAGM/AAPR, Österreichische Computer Gesellschaft (2001)