To appear at the IEEE Workshop on Perceptual Organization in Computer Vision

Santa Barbara, June 26 1998

A Dual, Scalable and Hierarchical Representation for Perceptual Organization of Binary Images

Stephan Heuel, Wolfgang Förstner
{Stephan.Heuel|Wolfgang.Foerstner}@ipb.uni-bonn.de
Institute for Photogrammetry, University Bonn
Nußallee 15, 53115 Bonn – Germany
Phone: +49 228 732713, Fax: +49 228 732712

Abstract

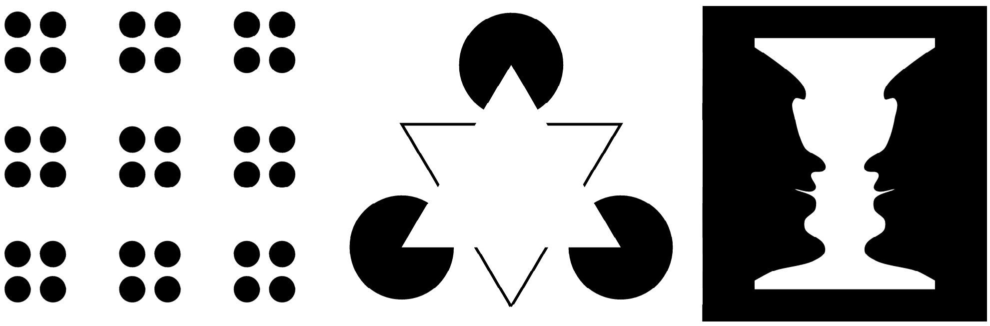
We propose a new representation for segmented images useful for Perceptual Organization. The representation has four distinct properties: (1) It is topologically consistent, i.e. the image plane is completely described; (2) the representation treats fore- and background symmetrically, a change of fore- and background has a well-defined and transparent impact on the representation; (3) the hierarchical structure of the representation explicitly reflects the aggregation of parts and objects; (4) finally the representation has an associated scale, which refers to the significance of image parts and of their relationships.

We present an example for such a representation, where the images consist of area type features and the significance of the relationships of the blobs are based on their proximity.

1 Introduction

1.1 Motivation

In their overview paper, Sarkar and Boyer 1993b define the term perceptual organization "[...] as the ability to impose structural organization on sensory data, so as to group sensory primitives arising from a common underlying cause". Numerous examples of this ability have been demonstrated by artificial imagery following the widely known Gestalt laws of proximity, similarity, closure, continuation and symmetry, see Lowe 1985 and figures 1(a) and (b). Researchers have proposed computational models for one of these aspects, like the detection of illusory or subjective contours, see Heitger and von der Heydt 1993 or Guy and Medioni 1996, but also for a number of different grouping principles like McCafferty 1990. Besides the theoretical and psychological interest in perceptual grouping,



(a) (b) (c)

Figure 1: Proximity criterion (a), Kanizsa triangle, see Kanizsa 1979 (b) and faces–vase illusion (c)

the work on this topic is motivated by the fact, that every feature extraction method like edge detection produces cluttered image descriptions with qualitative and quantitative errors. The goal of most computational methods of perceptual organization is to overcome some of these problems in order to provide an improved description to subsequent vision modules. But the structure and the quality of the description depends on the application: for example, if we know that the object in the image is an industrial building, we probably only want to extract straight lines and group them to a polygon, e.g. see Lin *et al.* 1994; if the object is a coffee mug, we need to find and group curved lines. In both cases the grouping process uses this specific knowledge in order to find an appropriate group of lines. The decision whether we look for curved or linear lines belongs to a top-down process as opposed to the bottom-up process of edge detection and grouping. Sarkar and Boyer 1993a introduced a (modified) Bayesian network to guide the grouping of straight and curved lines to parallelograms, triangles, circles, etc. They restrict the complexity of a Bayesian network by using the local neighborhood dependencies of the features and structural similarities of features in different locations. The advantage of this approach is the flexibility with respect to other applications since they are able to explicitly model specific scene knowledge. The construction of the network and the choice of algorithms however,may not be obvious and is (currently) the task of the system user. Castaño and Hutchinson 1996 also use a probabilistic approach to find a most likely set of groupings which are ordered by their associated probability. Therefore a user can choose between several possibilities to find the grouping which fits best to his or her application. They implemented straight line and symmetry grouping as examples of geometric structures and computed probabilities that a set of features is to be grouped together for a given image.

1.2 The Task

A module for perceptual organization in general requires the following decisions made by the calling routine:

1. the selection of the basic features and the type of tokens to be organized.

2. the organization criteria, evaluation or optimization functions or constraints.

In this way it is possible to specify the needs of the user. The complexity of the organization process results from the variety of links between features and the evaluation criteria for these links which depend not only on the attributes of the features but also on the density of the

Figure 2: Examples of blob extraction. On the left blobs are extracted by thresholding, the right shows homogeneous blobs of an aerial image, labeled by grey value.

linking network. As perceptual organization is a repeated aggregation process the ease of specification and the efficiency of the realization heavily depends on the representation of the features and their links.

This paper presents a new representation for segmented images useful for perceptual organization. It starts from

1. spatial features which may be the result of some segmentation procedure, in general yielding attributed point type, line type and area type features together with their relations. We assume the figure ground separation has been made, see for example figure 2.

   This covers a large class of classical tasks. In this paper we concentrate on binary images only. However, generalization of the type of imagery is straight forward.

2. a significance measure for the features themselves as well as for their relations.

   This gives enough freedom for specification. In this paper we restrict the significance measure to a size and a proximity measure, establishing close links to published approaches. However, taking other attributes of the features and their relations into account is easy.

The envisaged representation should fulfill the following properties:

- The representation should be topologically consistent. The complete image plane should be represented. Features to be grouped may be typed and are assumed to not overlap. Neighborhood relations should not lead to inconsistencies as e.g. in case one uses the N4-neighborhood for fore- and background of a binary image. We achieve this by explicitly representing the background, especially holes.

- The representation should treat fore- and background symmetrically. Changing fore- and background should change the representation in a transparent way. This goal follows from the observation, that both fore- and background contain important information, e.g. see the faces-vase illusion in figure 1(c).

  The proposed representation consists of two interleaved attributed trees, one for the foreground and one for the background which are exchanged. Especially grouping and partitioning and generation and deletion of features are explicitly represented as dual operations.

- The representation should be hierarchical. The tree structure explicitly represents the aggregation hierarchy identifying significant parts of the foreground and the background.

- The hierarchical representation should be scalable, i. e. depending on—a user specified—scaling measure. This measure is defined by the significance of the individual features as well as by the significance of the relations between the features, The significance depend on the attributes of the features and their relations. This gives enough freedom for specification (cf. the approach of McCafferty 1990).

In this work the used features and tokens are just area type features, also called blobs; our goal is to correct for spatial uncertainty of the blobs. We are only interested in topological errors (sec. 3) and do not model geometric errors. In the next sections we will see that we can choose the principle of *proximity* between blobs as the grouping criteria for our task.

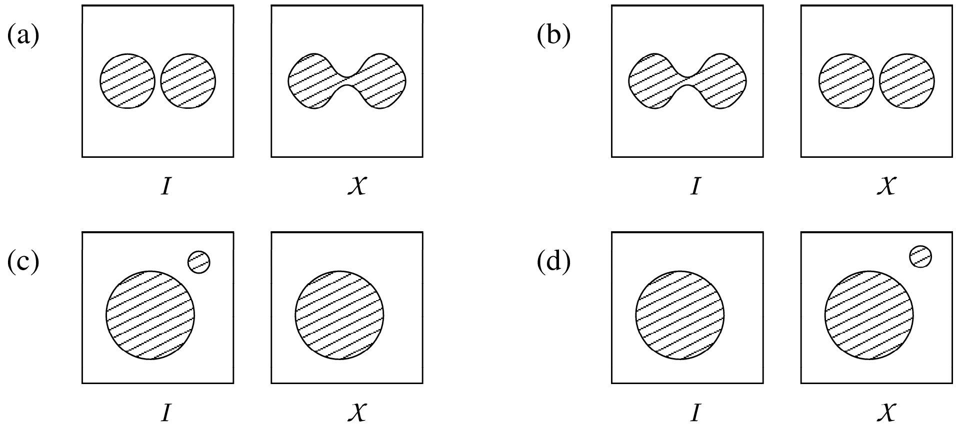


Figure 3: Four types of qualitative or topological errors for blob extractors. $\mathcal{I}$ denotes the ideal image and $\mathcal{X}$ the observed image.

1.3 Related Work

The idea of representing a blob process in a tree was already described by LINDEBERG 1993: he investigated grey scale blobs defined by specific properties in the grey value surface of the image and depend on a scale parameter. He observed that four events can happen: creation, deletion, splitting and merging. The "scale of the events" were defined similar to our critical points. KOENDERINK AND VAN DOORN 1997 also noticed an hierarchy structure when analyzing the qualitative structure of images. He referred to the so-called Morse-points, which are singularities in a landscape namely summits, immits and saddle points. Looking at the level curves of the landscape one can define a nested qualitative description. The close connection between our representation and this nesting will become clear in the subsequent sections.

Our grouping and partitioning method uses the proximity of blobs to merge features. This is similar to a neighborhood grouping as in FUCHS 1998 (cf. FUCHS AND FÖRSTNER 1995): she uses the neighborhood relations to correct topological inconsistent situations of blobs, lines and point features. These neighborhood relations are obtained using a skeleton of the non-feature regions, which is similar to AHUJA AND TUCERYAN 1989, who computed regions and curves from a dot pattern using the properties of the Voronoi-neighborhoods of the dots.

KIMIA *et al.* 1995 developed a method for describing two-dimensional shapes by analyzing the evolution of shape boundaries under the process of constant and curvature deformation. They categorized four singular events called shocks. In a subsequent work SIDDIQI *et al.* 1998 use the shocks to define a graph describing a shape.

In this work we use the following notations: given the set $\mathcal{A} \in \mathbb{R}^2$, we denote the interior of $\mathcal{A}$ as $\overset{\circ}{\mathcal{A}}$ and the boundary as $\partial\mathcal{A}$. The closure of a set is the unification of boundary and interior: $\overline{\mathcal{A}} = \overset{\circ}{\mathcal{A}} \cup \partial\mathcal{A}$. The complement is given by $\mathcal{A}^c$.

2 Complete Partitioning and Tide Partitioning

2.1 Complete Partitioning

We define a partitioning as the splitting of regions into two or more disjunct subregions, based on SIDDIQI AND KIMIA 1995: they define a partitioning with part–lines, which are curves that connect two boundary points, lie within the interior and divide the region in two connected components. Because the definition can not deal with holes, we will extend this notion of partitioning.

First we give a set-oriented definition of a region image. Let $\mathcal{W}_1, \ldots, \mathcal{W}_n \subset \mathbb{R}^2$ be a set of connected regions and $\overline{\mathcal{W}_i}$ disjunct. Then we call $\mathcal{X} = \bigcup_{i=1}^{n} \mathcal{W}_i$ a binary figure with n components $\mathcal{W}_i$. A binary image consists of two elements, the binary figure $\mathcal{X}$ (foreground) and its dual figure, the complement $\mathcal{X}^c$ (background) of $\mathcal{X}$. Components of the foreground will be denoted by letters $\mathcal{W}$ and background components with $\mathcal{V}$. Now we can define a *part–line* similar to SIDDIQI AND KIMIA 1995 as a regular curve $\alpha : [0, 1] \to \mathcal{X}$ with $\alpha(]0, 1[) \subset \overset{\circ}{\mathcal{X}}$, which divides a component $\mathcal{W}_i$ of $\mathcal{X}$ in two disjunct subsets $\mathcal{W}_{i_1}$ and $\mathcal{W}_{i_2}$. Let $L = \{\alpha_1, \ldots, \alpha_n\}$ be a set of part-lines L for a figure $\mathcal{X}$. We denote a *partitioning* of a figure as the difference $\mathcal{X} \setminus L$.

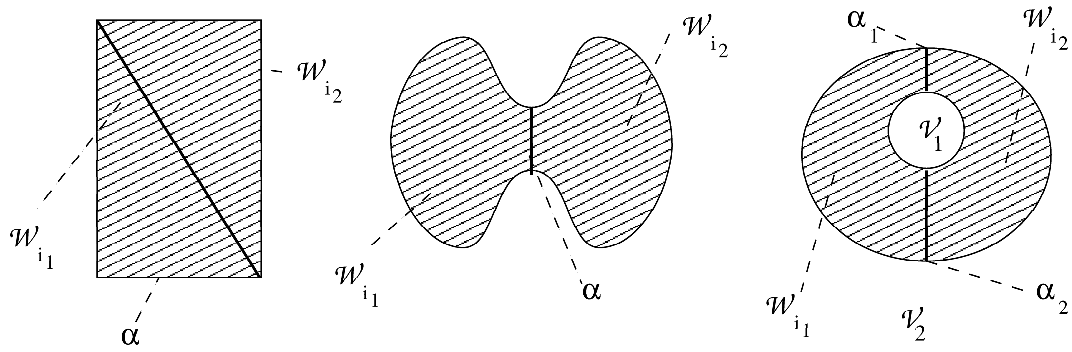


Figure 4: Examples of part-lines

But this scheme is not general enough, if we a allow "holes" in the components, the component is not simply connected. One example can be seen in figure 4: the curves α_1 and α_2 are not part-lines on their own, but together they divide the component in two parts. If we look only at one of the above part-lines, we see that this line actually merges two background components, namely $\mathcal{V}_1$ and $\mathcal{V}_2$. This observation suggests a more general definition of part–lines: either the number of components increases (first case) or the number of holes decrease (new case). We can use the genus (also called Euler number $\epsilon(\mathcal{X})$) of a binary figure $\mathcal{X}$, which denotes the difference between the number of components $\kappa(\mathcal{X})$ and the number of holes $\lambda(\mathcal{X})$: a *general part–line* is a regular curve $\alpha : [0, 1] \to \mathcal{X}$, which increases the Euler number $\epsilon(\mathcal{X})$ by one, i.e. $\epsilon(\mathcal{X} \setminus \alpha) = \epsilon(\mathcal{X}) + 1$. A partitioning is now the difference between $\mathcal{X}$ and a set of general part–lines. We call this partitioning *complete*, if the Euler number equals the number of components, i.e. no holes exist.

2.2 The Geometry of Part-lines

Up to now we have not described how the part-lines are computed. Since our task is to correct for the spatial uncertainty of blobs, we look again at figure 3(a). If one component in the observed image refers to two components in the ideal image, one apparently divides the one component at an "obvious" location. What does "obvious" mean in this context? Apparently two ideal components can not have an arbitrary distance such that they are observed as one component in the real image. The smaller the distance the larger the probability that they are mistakenly merged. WINTER 1995 distinguishes between a strong and a weak overlap of two sets. If one has two disjunct sets, an uncertain observation results only in a weak overlap. This means in our case that the connection between the ideal components only cover a small area compared to the real objects. This argumentation corresponds to the proximity criterion when grouping features.

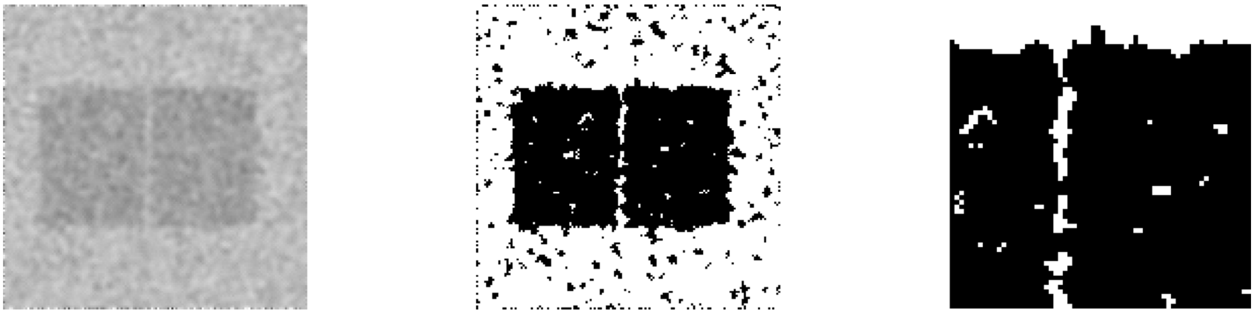


Figure 5: Artificial example of the impact of spatial uncertainty of blobs. The left shows the grey scale image from figure 2 with additional blurred Gaussian noise. The middle binary image is computed by thresholding the grey scale image. The right shows a magnified patch of the binary image.

An artificial example stresses the last argumentation: we add noise to the grey-scale image with the two boxes from figure 2 and smooth it thereafter, see figure 5. If we threshold the processed image, we get a noisy binary image with cluttered boundaries of the rectangles, which are now merged to one connected component. But the connections on which the two rectangles merge, are relatively thin depending on the strength of the noise. For our purposes, it seems to be sufficient to draw a part-line at the thinnest location of the connection.

For a mathematical definition of part-lines we need the distance transformation $d_{\mathcal{X}}(x) : \mathcal{X} \rightarrow \mathbb{R}^2$ of a binary figure $\mathcal{X}$, which assigns every point of $\mathcal{X}$ the minimal distance to the boundary $\partial\mathcal{X}$ of $\mathcal{X}$, see also SERRA 1982. As one can see in figure 6(a) and (b), a saddle point p of $d_{\mathcal{X}}$ defines a part-line $\alpha(p)$ by connecting p with its nearest boundary points. TARI *et al.* 1997 used this criterion for segmenting shapes and defines a limb part-line in the sense of SIDDIQI

AND KIMIA 1995. Unfortunately the distance transformation of the shape in figure 6(c) does not have a saddle point, but for our problem we still want to draw a partitioning line.

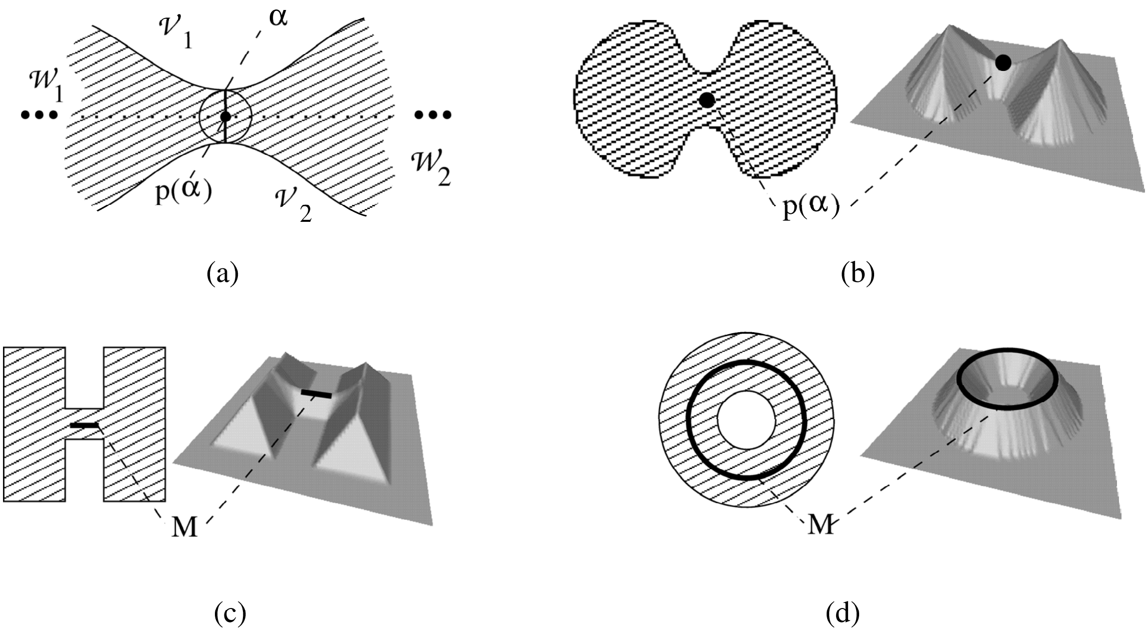


Figure 6: Tide-partitioning. Figure (a) shows a schematic drawing of a thin connection between two regions $p(\alpha)$ is a part-point, which defines the part-line α. In (b) we have a unique splitting-point, whereas in (c) we have a set M of possible part-points. The part-line can be defined with any of the points in M. In (d) we have again a set of splitting-points, the part-line can be anywhere within the torus.

2.3 Tide Partitioning

We therefore introduce another definition which is based on the morphological operation *erosion* $\mathcal{X} \ominus \rho_0 B$, where B is the unit disc and $\rho_0 \in \mathbb{R}^+$. It can be shown that $\mathcal{X} \ominus \rho_0 B = \{x \in \mathbb{R}^2 \mid d_{\mathcal{X}}(x) \geq \rho_0\}$ (for dilation, we have $\mathcal{X} \oplus \rho_0 B = \{x \in \mathbb{R}^2 \mid d_{\mathcal{X}^c}(x) \geq \rho_0\}$). We can clarify an erosion (and later on a dilation) by the following analogy: imagine that the binary image is a simple cartography map where the foreground corresponds to land areas and the background to water areas. An erosion $\mathcal{X} \ominus \rho_0 B$ can now be expressed by an increase of the water-level by ρ_0. If we regard the increase as a process, we are interested in those events at which an island is split into two parts or a lake merges with another lake or sea[1]. Such a critical event defines a point or a set of points of the original land areas at which the splitting of an island or merging of two seas happen. If seas merge, we also regard this event as a *splitting event* because it splits an area of an island. Therefore we call the points *splitting points*. Note that in figure 6(d) the set of splitting points has the shape of a circle; a part-line $\alpha(p_s)$ is be defined by an arbitrary splitting point $p_s \in M$.

Mathematically, a splitting event ρ_0 and the set of splitting points M_{ρ_0} can be defined as follows:

$$\rho_0 \text{ is a splitting event} \quad :\Leftrightarrow \quad \exists \epsilon\, \forall \rho \in [\rho_0 - \epsilon, \rho_0) : \quad \begin{array}{l} \kappa(\mathcal{X} \ominus \rho B) = \kappa(\mathcal{X} \ominus \rho_0 B) - 1 \vee \\ \lambda(\mathcal{X} \ominus \rho B) = \lambda(\mathcal{X} \ominus \rho_0 B) + 1 \end{array}$$

$$M_{\rho_0} := \bigcap_{\rho > 0} (\mathring{\mathcal{X}} \ominus \rho_0 \mathring{B}) - (\mathring{\mathcal{X}} \ominus \rho_0 \mathring{B}) \circ \rho \overline{B})$$

where M_{ρ_0} is defined as in Serra 1982, page 376 as the subset $s_{\rho_0}(\mathcal{X})$ of the skeleton $S(\mathcal{X})$, $\kappa(\mathcal{X})$ denotes the number of components of a binary figure $\mathcal{X}$ and $\lambda(\mathcal{X})$ the number of holes. Note that the splitting events are shape shocks of second and third order in the sense of Kimia *et al.* 1995.

We can now introduce a new partitioning scheme: let $M_{\rho_1}, \ldots, M_{\rho_n}$ be the sets of splitting points for all possible splitting events $\rho_1, \ldots, \rho_n$. Assuming that M_{ρ_i} are connected[2], we choose for every M_{ρ_i} an arbitrary point $p_{\rho_i} \in M_{\rho_i}$ which defines a general part-line $\alpha(p_{\rho_i})$ by connecting its nearest boundary points of $\mathcal{X}$. Because of the above analogy we call the partitioning $\mathcal{X} \setminus L_t$ with $L_t = \bigcup_i \alpha(p_{\rho_i})$ the *tide-partitioning* of a binary figure $\mathcal{X}$. It can be shown that the tide-partitioning is complete, i.e. there are no holes in $\mathcal{X} \setminus L_t$ (cf. Heuel 1997). Note that the tide-partitioning is invariant under translation, rotation and scaling. An example for a tide-partitioning is shown in figure 7.

The arbitrariness if the partitioning point $p_{\rho_i} \in M_{\rho_i}$ does not effect our partitioning scheme, as all partitionings with $p_{\rho_i} \in M_{\rho_i}$ are seen to be equivalent. In this case one would indicate p_{ρ_i} to be uncertain to some degree.

[1]We assume that the ground-water-level is at any time equal to the sea-level.
[2]i.e. the critical events are unique

Figure 7: Tide-partitioning for the artificial example in figure 5. On the left is the original binary image. The middle shows the same image where blobs with size smaller than 9 pixels are removed. The right shows the connected components of the tide-partitioned image.

3 Scaled Partitioning

3.1 Significance as Scale

Like many other segmentation methods the tide-partitioning may result in an over-segmentation. The decision whether a segmentation is over-, under- or correctly segmented depends on the vision task; only with high-level knowledge one is able to make this decision. For this reason we want to provide a number of partitionings from which one can choose the best related to the task. We can obtain the partitionings by assigning every part-line α_i with a *significance measure* $\sigma(\alpha_i)$: the larger the similarity the more important is the part-line; the significance measures the so-called *Prägnanz* of the split. With this measure we not only obtain an order of the part-lines but we can also determine the relative significance of two part-lines by their difference of significance. Applying the part-lines subsequently according to their order, one get an list of partitionings with size n, if n is the number of part-lines.

For our special tide-partitioning we suggest that the significance of a part-line $\alpha_i(p)$ should be anti-proportional to a function of its length. In the analogy the length corresponds to the water-level of the critical event or the height of the splitting point and we can define: $\sigma(\alpha_i(p)) \sim \frac{1}{f(d_{\mathcal{X}}(p))}$ where p is the splitting point and f is some monotonous increasing function, the simplest case would be $f \equiv id$, or $f \equiv \exp(d_{\mathcal{X}}(p))$. Observe that any similarity measure can be used for scaling, which gives freedom to the user. Also combined measures, e.g. depending on distance and spectral similarity of blobs can be used.

3.2 Scaled Deletion

In our opinion a partitioning should not only break objects into parts, but also delete those parts that are not important to the task. Looking back to the analogy: when raising the water-level we not only have splitting and merging as topological events, also islands can disappear if the water-level is increased sufficiently. In this sense we now include the deletion of parts in the tide-partitioning scheme: we define a significance measure of the parts which can again be based on the distance transformation $d_{\mathcal{X}}$: let $\mathcal{W}_i$ be a component in the complete partitioned figure $\mathcal{X} \setminus L_t$, then: $\sigma(\mathcal{W}_i) \sim \frac{1}{f(d_{\mathcal{X}}(p(\mathcal{W}_i)))}$ where $p \in \mathcal{W}_i$ has the largest distance transformation value in the component, which refers to the top of the mountain of the island $\mathcal{W}_i$. The function f is not necessarily the same function as above. This significance can be related to a topological event called the deletion event, which is similar to a shock of fourth or third order in Kimia *et al.* 1995. Using the deletion of parts, the ordered list of partitionings grows by the number of these parts in the complete tide-partitioning.

3.3 Formal Definition of Scaled Partitioning

For a mathematical formulation of a scaled partitioning, we need three requirements for the significance measures $\sigma(\alpha_i)$ and $\sigma(\mathcal{W}_j)$. First they have to be *consistent*: a part $\mathcal{W}_j$ should not be deleted if not previously all its neighbor parts were split from $\mathcal{W}_j$, or expressed with part-lines: for all α_i is neighbored to $\mathcal{W}_j$: $\sigma(\alpha_i) > \sigma(\mathcal{W}_j)$. This means that for an island with two mountains, first the mountains are split before they disappear. The second requirement is that the significance measures of the part-lines are *completely ordered*[3]. This ensures a unique identification of splitting events and will be needed later on for the hierarchy. Third, the significance measures must not be zero, which would mean that a split or a deletion would

[3]You can transform every semi–order to a complete order by randomly sort equal elements.

never occur, and they must not be equal to infinity, which would mean that the events would always appear.

Definition 1 (Scaled Partitioning)

Let $\mathcal{X} \setminus L$ be a complete partitioning of a binary figure $\mathcal{X}$ with $L = \{\alpha_1, \dots, \alpha_n\}$ a set of general part-lines and $K = \{\mathcal{W}_1, \dots, \mathcal{W}_m\}$ the connected components of $\mathcal{X} \setminus L$. For every part-line α_i we have a significance measure $\sigma(\alpha_i)$ and for every part $\mathcal{W}_i$ we have $\sigma(\mathcal{W}_i)$, where $\sigma(\bullet)$ are consistent, completely ordered and positive real numbers. Then we call the mapping $\mathcal{X} : \mathbb{R}_0^- \to \mathcal{X}$,

$$\mathcal{X}(-\sigma) = (\mathcal{X} \setminus L^*(\sigma)) \setminus K^*(\sigma) \quad \text{with } \sigma \in \mathbb{R}_0^+ \text{ and}$$
$$L^*(\sigma) = \{\alpha_i |\ \alpha_i \in L \text{ and } \sigma(\alpha_i) \geq \sigma\}\ ,\ K^*(\sigma) = \{\mathcal{W}_i |\ \mathcal{W}_i \in K \text{ and } \sigma(\mathcal{W}_i) \geq \sigma\}$$

a *scaled partitioning of the binary figure* $\mathcal{X}$. ■

A scaled partitioning results in an ordered list of images $\mathcal{X}(I_0), \dots, \mathcal{X}(I_n)$, where $I_i = [\sigma_i, \sigma_{i+1})$ with $\sigma_0 = 0, \sigma_{n+1} = \infty$ and $n = |L| + |K|$.[4]

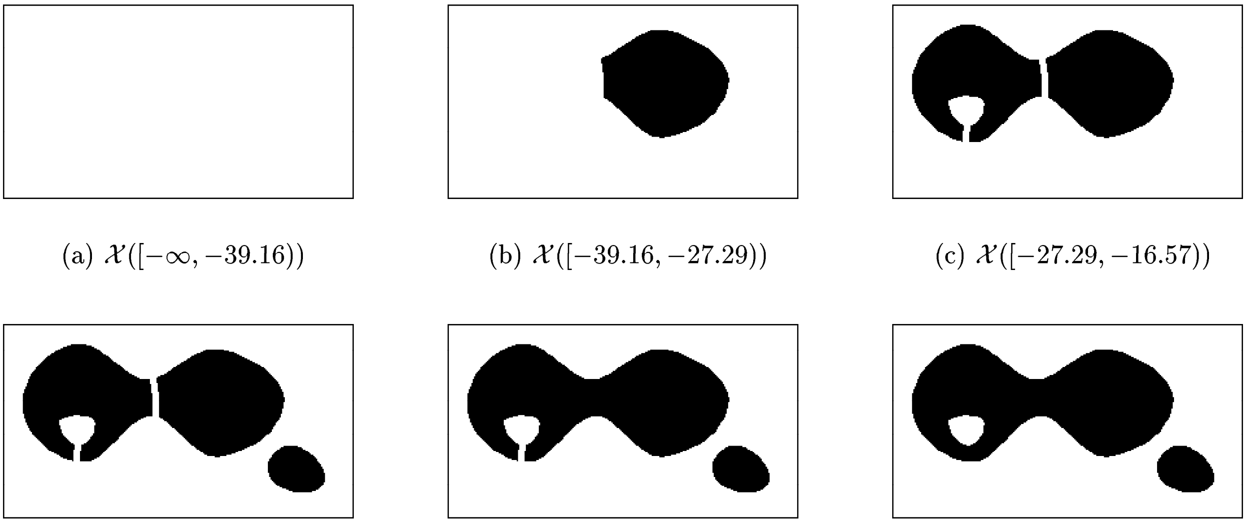
Note that the definition holds for any algorithm which produces complete partitionings, not only for the tide-partitioning scheme. Extremely scaled partitionings are $\mathcal{X}(-\infty) = \mathcal{X}$ and $\mathcal{X}(0) = \emptyset$. An example of a scaled partitioning is shown in figure 8. One can think of part-lines as channels which are built through an island and the significance measures as cost-functions for building a channel or flattening parts of the land areas. The higher the costs the less will be built resp. flattened.

4 Scaled Grouping

4.1 Merging Events

In the previous section we have concentrated on the foreground $\mathcal{X}$ of the binary image and used the erosion to define splitting and deletion operations. Now we are interested in merging

[4] The lower index of σ and I indicates that the significance and the intervals refer to partitioning. Later on in section 4 we will use upper indices for significance measures and intervals to denote grouping.



(a) $\mathcal{X}([-\infty, -39.16))$ (b) $\mathcal{X}([-39.16, -27.29))$ (c) $\mathcal{X}([-27.29, -16.57))$

(d) $\mathcal{X}([-16.57, -14.61))$ (e) $\mathcal{X}([-14.61, -6.02))$ (f) $\mathcal{X}([-6.02, 6.74))$

Figure 8: A binary figure $\mathcal{X}$ (f) and its scaled partitionings using the tide-partitioning. The intervals are given for distance d, not for significance σ. For an explanation of the interval of figure (f), see section 4.4

and creating blobs. Since we want to correct for the spatial uncertainty of blobs we can argue that the smaller the minimal distance between two feature blobs in the observed image, the larger is the probability that they are connected in the ideal image, cf. figure 3. Note that we do not include knowledge about the shape of blobs. For example for the Kanizsa triangle in figure 1(b), it would be better to incorporate linear boundary information for merging blobs.

So again we concentrate on the proximity criterion, but this time for merging instead of splitting blobs. For computing the merge operation we extend the definition of the distance transformation for the whole image:

$$D_{\mathcal{X}}(x) = \begin{cases} d_{\mathcal{X}}(x) & \text{if } x \in \mathcal{X}, \\ -d_{\mathcal{X}^c}(x) & \text{otherwise} \end{cases}$$

In our analogy this means that we can decrease the water-level which corresponds to the dilation of the figure (see page 10). If we have two separated islands a constant decrease of the water-level will eventually result in the connection of the two islands—we call this event *merging event.* Another possible merging event would be the connection of two coast lines so that a bay becomes a lake. We can define a merging event similar to a splitting event:

$$\rho_0 \text{ is a merging event} \quad :\Leftrightarrow \quad \exists\epsilon\ \forall\rho \in [\rho_0 - \epsilon, \rho_0) : \kappa(\mathcal{X} \oplus \rho B) = \kappa(\mathcal{X} \oplus \rho_0 B) + 1 \ \vee$$
$$\lambda(\mathcal{X} \oplus \rho B) = \lambda(\mathcal{X} \oplus \rho_0 B) - 1$$

Using the duality of erosion and dilation $\mathcal{X} \oplus \rho B = (\mathcal{X}^c \ominus \rho B)^c$ we obtain:

$$\rho_0 \text{ is a merging event} \quad :\Leftrightarrow \quad \exists\epsilon\ \forall\rho \in [\rho_0 - \epsilon, \rho_0) : \lambda(\mathcal{X}^c \ominus \rho B) = \lambda(\mathcal{X}^c \ominus \rho_0 B) + 1 \ \vee$$
$$\kappa(\mathcal{X}^c \ominus \rho B) = \kappa(\mathcal{X}^c \ominus \rho_0 B) - 1$$

Comparing the above to the definition of a splitting event in the previous section it can be seen that merging two blobs means splitting a background blob! So we can apply the methods for partitioning a binary figure to group components: let β_i be a part-line for the background $\mathcal{X}^c$; its significance measure can be related to the distance transformation with $\sigma(\beta_i(p)) \sim \frac{1}{f(d_{\mathcal{X}^c}(p))}$.

4.2 Creation of Blobs

As we stated in the beginning of the section we also look for an event of creating blobs. Complementary to the blob deletion this means to fill holes of a binary figure; using the analogy we dry out lakes as opposed to flood a part of an island. Again we assign a significance measure for this event: let $\mathcal{V}_i$ be the background component, then $\sigma(\mathcal{V}_i) \sim \frac{1}{f(d_{\mathcal{X}^c}(p(\mathcal{V}_i)))}$ where p is the point with the lowest distance transformation value in $\mathcal{V}_i$.

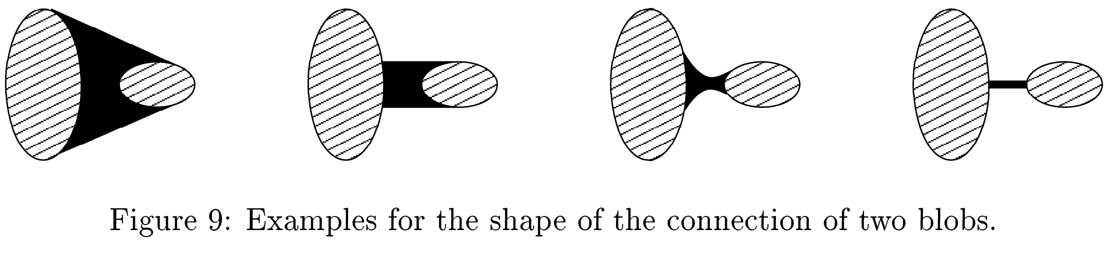


Figure 9: Examples for the shape of the connection of two blobs.

4.3 Shape of the Connections

The geometric shape of a connection of two blobs has not been discussed yet; it is not obvious, examples for possible connections can be seen in figure 9. One can interpret the gap between the blobs as a lack of information in the observed image. To connect the blobs one needs to extrapolate the blob shapes. Without any knowledge about the shape of the blobs we can not infer a specific shape of the connection. Therefore we suggest to include as low information to the image as possible. If we use the perimeter as a measure for the information of the figure, the minimal connection is the shortest line between the two components. In this way we can justify that the part-line of the background is the connection of two blobs.

4.4 Formal Definition of Scaled Grouping

The three requirements for significance measures also hold for merging and creation so we define a scaled grouping of blobs:

Definition 2 (Scaled Grouping)

Let $\mathcal{X} \setminus L_c$ be a complete partitioning of the complement $\mathcal{X}^c$ of a binary figure with $L_c = \{\alpha_1, \ldots, \alpha_n\}$ a set of general part-lines and $K_c = \{\mathcal{V}_1, \ldots, \mathcal{V}_m\}$ the connected components of $\mathcal{X}^c \setminus L$. For every part-line β_i we have a significance measure $\sigma(\beta_i)$ and for every part $\mathcal{V}_i$ we have $\sigma(\mathcal{V}_i)$, where $\sigma(\bullet)$ are consistent, completely ordered and positive real numbers. Then we call the mapping $\mathcal{X} : \mathbb{R}_0^+ \to \mathcal{X}$,

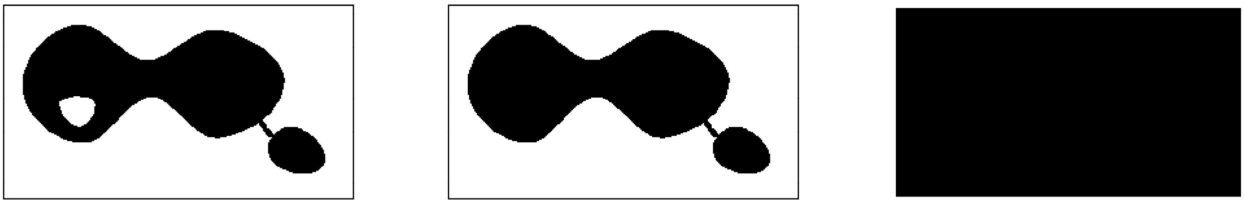
$$\mathcal{X}(\sigma) = \mathcal{X} \cup L_c^*(\sigma) \cup K_c^*(\sigma) \text{with } \sigma \in \mathbb{R}_0^+ \text{ and}$$

$$L_c^*(\sigma) = \{\beta_i | \ \beta_i \in L_c \text{ and } \sigma(\beta_i) \geq \sigma\} \ , \ K_c^*(\sigma) = \{\mathcal{V}_i | \ \mathcal{V}_i \in K_c \text{ and } \sigma(\mathcal{V}_i) \geq \sigma\}$$

a *scaled grouping of the binary figure* $\mathcal{X}$. ■

Like the partitioning, the scaled grouping results in an ordered list of grouped images $\mathcal{X}(I^0), \ldots, \mathcal{X}(I^m)$, where $I^i = [\sigma^i, \sigma^{i+1})$ with $\sigma^0 = \infty$, $\sigma^{m+1} = 0$ and $m = |L_c| + |K_c|$.

The completeness of the background partitioning ensures that all components of $\mathcal{X}$ are eventually connected to each other, because $(\mathcal{X} \setminus L_c)^c \supset \mathcal{X}$ is connected. For every scaled grouping we have $\mathcal{X}(\infty) = \mathcal{X}$ and $\mathcal{X}(0) = \mathbb{R}^2$. A connection line between two blobs can be regarded as a dam between two islands or two coast lines; closing a hole is similar to drying out a lake. Again the significance measure is a cost-function for building dams resp. drying out lakes. An example can be seen in figure 10, which shows the three possible groupings of the image in 8(f).



(a) $\mathcal{X}([-6.02, 6.74))$ (b) $\mathcal{X}([6.74, 10.70))$ (c) $\mathcal{X}([10.7, \omega - d_{\mathcal{X}}(C(\mathcal{X}))$

Figure 10: These three images are possible groupings of the image in figure 8(f) using the tide-partitioning. Again, the intervals are given for distance d, not for significance σ. For an explanation of the interval of figure (c), see section 4.5.

4.5 Infinite components

As a side remark we want to examine what happens with infinite components in the scaled grouping using the tide-partitioning. The question whether an infinite component has a significance measure equal to zero is only of theoretical interest but it can clarify the notion of duality used in this work.

Not every infinite component has an infinite distance transformation, for example an infinite ribbon of the width b has a distance transformation $d_{\mathcal{X}} \leq b$. But the background components are usually infinite and the maximal distance transformation, which is equal to the maximal radius of an inscribed circle.

Let us assume that all the previous definitions were made on the projective plane $\mathbb{P}^2 = (\mathbb{R} \cup \{-\infty, \infty\})^2$ instead of the Euclidean plane by unifying points at infinity with opposite direction. To make it simple, we first look at the one dimensional case: components are intervals $I \subset \mathbb{P}^1$, the maximal distance transformation of an interval is equal to half of its

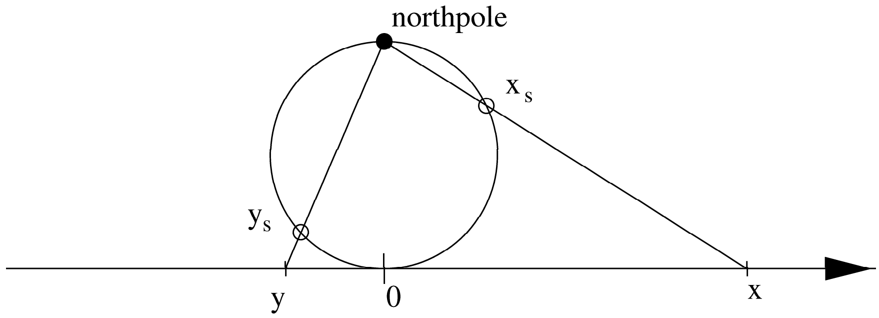


Figure 11: Mapping of $\mathbb{P}^1$ onto a sphere S^1.

length. We can identify $\mathbb{P}^1$ with a unit circle S^1 by mapping every point $x \in \mathbb{P}^1$ onto a point $x_s \in S^1$ as depicted in figure 11. The points $-\infty, \infty \in \mathbb{P}^1$ refer to the north pole of S^1. Furthermore we introduce a metric since the Euclidean distance is not sufficient: let $\varrho_{\mathbb{P}^1}(x, y)$ be the arclength between x_s and y_s on the sphere S^1. We get $\varrho_{\mathbb{P}^1}(-\infty, \infty) = 2\pi$ and a length $\varrho_{\mathbb{P}^1}(I)$ of an interval $I \subseteq \mathbb{P}^1$ is finite.

Returning to the projective plane this means that for the metric $\varrho_{\mathbb{P}^2}$ we do not have infinite components, thus every component has a distance transformation which is finite. Transferring the Euclidean plane to the projective plane, the sphere S^2 has to be large enough so that the set of topological events described so far remains invariant.

For notation purposes we use the following convention: if a component $\mathcal{W}$ is infinite in the above sense, the distance transformation of its complement $\mathcal{W}^c$ is used: $d_{\mathcal{W}}(p(\mathcal{W})) := \omega - d_{\mathcal{W}^c}(p(C(\mathcal{W}^c)))$ where $C(\mathcal{W}^c)$ is the convex hull of $\mathcal{W}^c$ and ω denotes the distance of the origin to infinity, for example $\omega = \pi$ for $\rho_{\mathbb{P}^2}$.

4.6 Duality Theorem

We now observe the duality of fore- and background. Part-lines of the background correspond to connection-lines for foreground components. Similar, the deletion of a background blob is a creation of a foreground blob. We can summarize this statement in the following theorem: partitioning of a figure $\mathcal{X}$ means grouping the dual figure $\mathcal{X}^c$ and vice versa grouping $\mathcal{X}$ is equal to partitioning $\mathcal{X}^c$.

Theorem 1 (Duality of Grouping and Partitioning)

Given an algorithm to compute complete partitionings of binary figures and consistent significance measures for part-lines and parts. If $\sigma \in \mathbb{R}_0^+$ then

$$\mathcal{X}(+\sigma)^c = \mathcal{X}^c(-\sigma) \qquad ; \qquad \mathcal{X}(-\sigma)^c = \mathcal{X}^c(+\sigma)$$

Proof: We only proof the first equation, the second follows because of the duality.

$$\begin{aligned}
\mathcal{X}(\sigma)^c &= (\mathcal{X} \cup L_c^*(\sigma) \cup K_c^*(\sigma))^c \\
&= \mathcal{X}^c \cap (L_c^*(\sigma))^c \cap (K_c^*(\sigma))^c \\
&= (\mathcal{X}^c \setminus L_c^*(\sigma)) \setminus K_c^*(\sigma) \\
&= \mathcal{X}^c(-\sigma)
\end{aligned}$$

□

5 Hierarchy

In the previous sections we have introduced a method for partitioning and grouping binary figures by proximity, where the output is an ordered list of scaled partitionings resp. groupings. In this section we show that we can enrich this representation using a causality-relation between the components of each scaled image.

5.1 The Partition Graph

First we define a hierarchy for scaled partitionings. For every transition $\mathcal{X}(I_i) \rightsquigarrow \mathcal{X}(I_{i+1})$ we have exactly one topological event, this is ensured by the consistency requirement in the definition of scaled partitionings. Theoretically this event could be one of the four types:

$$(\kappa(\mathcal{X}_i) = \kappa(\mathcal{X}_{i+1}) - 1) \lor (\lambda(\mathcal{X}_i) = \lambda(\mathcal{X}_{i+1}) - 1) \lor (\kappa(\mathcal{X}_i) = \kappa(\mathcal{X}_{i+1}) + 1) \lor (\lambda(\mathcal{X}_i) = \lambda(\mathcal{X}_{i+1}) + 1)$$

The latter case does not occur: it would mean that there is a hole in a high scaled partitioning which does not exist in a lower scaled partitioning, or dually: for the background it would mean that a component is created on a high-scaled grouping. But in our definition of scaled grouping only hole areas or part-lines are added, see definition 1; both could not be a separated component in the grouping image. Returning to the scaled partitioning, the event $\lambda(\mathcal{X}_i) = \lambda(\mathcal{X}_{i+1}) + 1$ is impossible for scaled partitionings.

Looking at the other three events, either a component is split into two components ($\kappa(\mathcal{X}_i) = \kappa(\mathcal{X}_{i+1}) - 1$), a component is transformed to another by removing a part-line ($\lambda(\mathcal{X}_i) = \lambda(\mathcal{X}_{i+1}) - 1$) or a component is completely removed ($\kappa(\mathcal{X}_i) = \kappa(\mathcal{X}_{i+1}) + 1$). Seeing the events as relations between components of different scale, we can use them to define a graph, where the nodes are sets of possible components:

Definition 3 (Partition Graph)

Let $\mathcal{X}$ be a binary figure with a scaled partitioning $\mathcal{X} \setminus L$ with L a set of general part-lines, K the set of parts and $n = |L| + |K|$. We call the (directed) graph $G^- = (V, R)$ with

$$V = \{\mathcal{U} |\ \mathcal{U} \text{ is a component in } \mathcal{X}(I_i), i = 0, \dots, n\}$$

$$R = \{(v_1, v_2)|\ v_1, v_2 \in V \text{ and } \exists\ \alpha \in L : ((v_2 = v_1 \setminus \alpha) \vee (\exists\ v_3 \in V : v_2 = v_1 \setminus \alpha \setminus v_3))\}$$

a *partition graph* of $\mathcal{X}$. We call R the *causality relation* of a scaled partitioning. ■

The structure of the graph is hierarchical: the partition graph is a tree, if $\mathcal{X}$ consists of one connected component, see Heuel 1997. For other $\mathcal{X}$ the graph is a set of trees.

We can identify every event with a node in the graph: if the node has two children, two blobs are split; if a node has one child, a hole is connected to another hole or to the background; if anode has no children, a blob is deleted.

5.2 Grouping Graph

For creating the grouping graph, we again look at the possible topological events that can happen between $\mathcal{X}(I_i)$ and $\mathcal{X}(I_{i+1})$: as above we have three types of events: either two components are connected ($\kappa(\mathcal{X}^i) = \kappa(\mathcal{X}^{i+1}) + 1$), a component is transformed to another component by adding a part-line of the background ($\lambda(\mathcal{X}^i) = \lambda(\mathcal{X}^{i+1}) - 1$) or a hole is added to an existing component ($\lambda(\mathcal{X}^i) = \lambda(\mathcal{X}^{i+1}) + 1$). For the partition graph, the third event was the deletion of a component, this had no effect on the causality relation. Here we have a deletion of a hole which results in a larger component. This has to be taken into account by the causality relation, since a component is transformed into another component.

Definition 4 (Grouping Graph)

Let $\mathcal{X}$ be a binary figure with a scaled partitioning $\mathcal{X}^c \setminus L_c$ with L_c a set of general part-lines, K the set of background parts and $m = |L_c| + |K_c|$. We call the (directed) graph $G^+ = (V, R)$ with

$$
\begin{aligned}
V = \; & \{\mathcal{U} |\, \mathcal{U} \text{ is a component of } \mathcal{X}(I^i), i = 0, \dots, m\} \\
R = \; & \{(v_1, v_2) |\, v_1, v_2 \in V \quad \text{and} \quad \exists\, \beta \in L_c : ((v_1 = v_2 \cup \beta) \vee (\exists\, v_3 \in V : v_1 = v_2 \cup \beta \cup v_3)) \\
& \qquad\qquad\qquad\quad\;\; \text{or} \quad\;\; \exists\, \mathcal{V} \in K_c : (v_1 = v_2 \cup \mathcal{V})\}
\end{aligned}
$$

a *grouping graph* of $\mathcal{X}$. ■

This graph is always a tree since the maximal grouping is one component, namely the image plane $\mathbb{R}^2$. Note that for a grouping graph, the number of nodes is not equal to the number of critical events because the grouping $\mathcal{X}^0 = \mathcal{X}$ consists of the components of $\mathcal{X}$ which are the leaves of the tree. It is to be investigated how this representation relates to a region adjacency graph, Pavlidis 1977.

Now we can combine the grouping and partition graph by identifying the nodes of $\mathcal{X}(I_n) = \mathcal{X} = \mathcal{X}(I^0)$, see definitions in sec. 3 and 4. The leaves of the grouping tree are the roots of the trees in the partition graph. For a binary figure $\mathcal{X}$ we call the fusion of the two graphs a *foreground tree*. Dually, the *background tree* is the fusion of the graphs of $\mathcal{X}^c$.

A binary image can be described by two trees. Using the analogy, the trees represent all topological events when having low- or high-tide: either islands sink (leaves),they are merged (nodes with two children), a lake is created (node with one child) or dries out (node with one child). The distinction between the last two events can be done in the dual tree. Therefore we have a close relationship between the background and the foreground tree. An example for both trees is shown in figure 12.

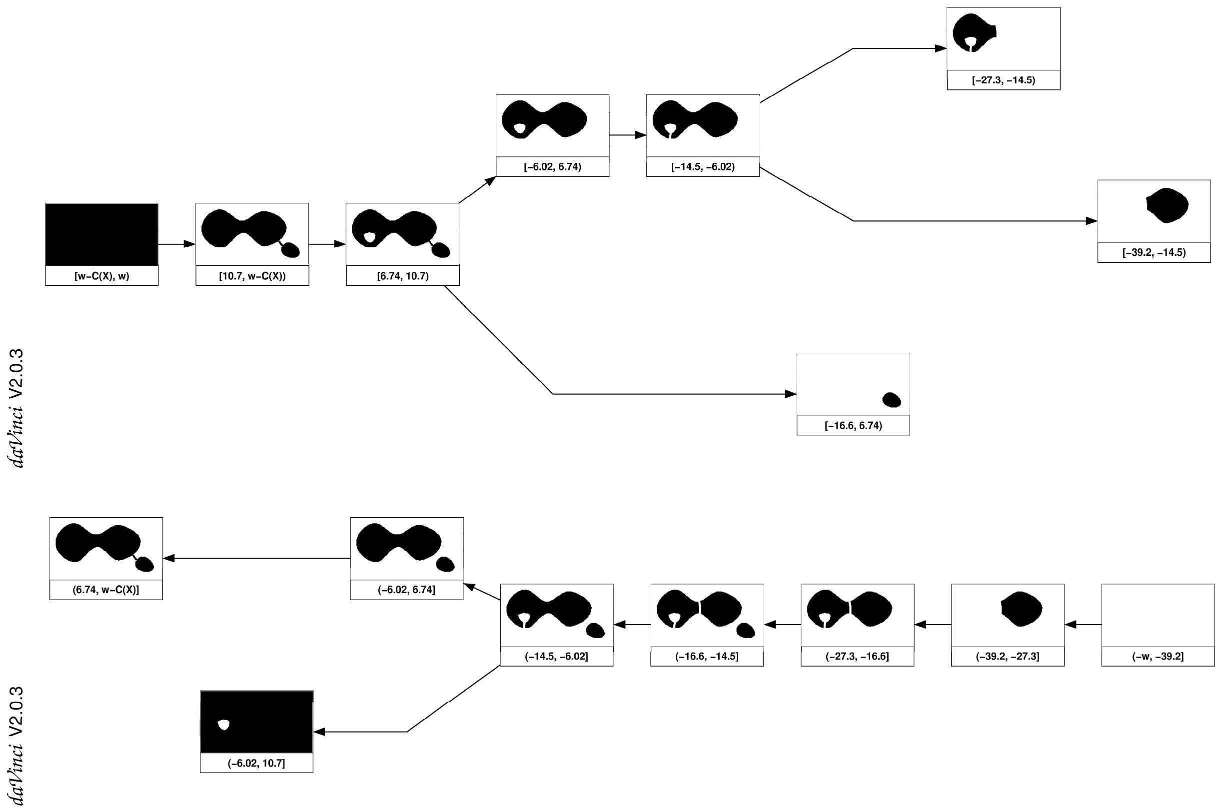


Figure 12: Example for fore- and background tree for the image in figure 8(f)

Remark. With the above causality, we can relate a hierarchical grouping to a simple morphological scale-space, see for example Jackway and Deriche 1996, for a binary image. The dilation scale space $DR(\mathcal{X}, \rho) := \mathcal{X} \oplus \rho$ with gravity-points as features fulfills the requirements of a scale-space (smoothing, detectable features, causality relation). Because of the duality of erosion and dilation, we have a scale space both for grouping and partitioning.

5.3 Digitization

Up to now we assumed to have the Euclidean plane $\mathbb{R}^2$ (resp. the projective plane $\mathbb{P}^2$). Two main steps are necessary for computing fore- and background trees in digital images. First the Euclidean topology has to be preserved for the raster image, since we extensively used topological terms for partitioning and grouping. Based on ideas of Kovalevsky 1989, Winter 1995 introduced the *hyperraster*, for which a discrete set can be open and closed, there also exist zero- and one dimensional elements (0– and 1–cells), that enable to represent points and lines. In the conventional raster format the crack edge between to pixels or the common point of four pixel are not represented explicitly. A hyperraster consists of a set of 2–cells f (*faces*), 1–cells e (*edges*) and 0-cells n (*nodes*). In figure 13 are two discretized components of a binary figure with their part-line depicted.

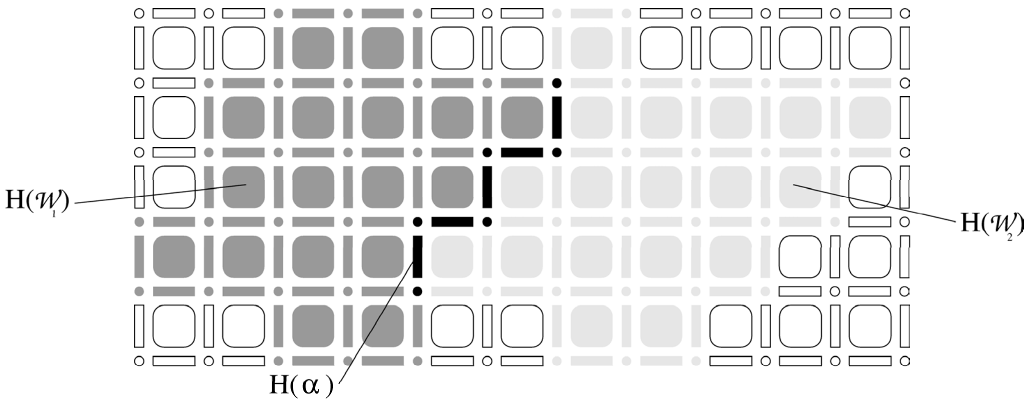


Figure 13: A discretized binary figure with two parts $\boldsymbol{H}(\mathcal{W}_1)$ and $\boldsymbol{H}(\mathcal{W}_2)$ and a part-line $\boldsymbol{H}(\alpha)$.

To compute the critical events and their associated points one could use the distance transformation, for which good approximations exist (Borgefors 1991), and detect topological events as it was defined in the previous sections. This results in a very inefficient algorithm and thus we used the skeleton and exoskeleton of the binary image: defining the three dimensional curve $\Gamma(\mathcal{X}) := \{(x, d_{\mathcal{X}}(x)) | x \in S(\mathcal{X})\}$, we search for relative minima and maxima of $\Gamma(\mathcal{X})$ and $\Gamma(\mathcal{X}^c)$ which are the wanted critical points. In our implementation we used the Voronoi-skeleton of Ogniewicz and Kübler 1995, which can be computed fast and gives real-valued coordinates of the skeleton line.

6 Examples

The first example makes the dual nature of the two trees clear: switching fore- and background means switching fore- and background tree. First, assume a binary figure with a single disc as the foreground. In the projective plane $\mathbb{P}^2$, the background is finite and can also be interpreted as a disc: imagine that the sphere S^2 is turned around and the colors black and white are switched, mapped back to $\mathbb{P}^2$ results in a very large disc.

Now we analyze the dual figure $\mathcal{X}_{\text{dual}}$ depicted in figure 14, where both $\mathcal{X}_{\text{dual}}$ and $(\mathcal{X}_{\text{dual}})^c$ have two components. A combination of grouping and partitioning results in six events: two blob deletions, two hole creations and two splits. The resulting trees have the same structure, only the intervals differ, because the two large components of fore- and background differ in size.

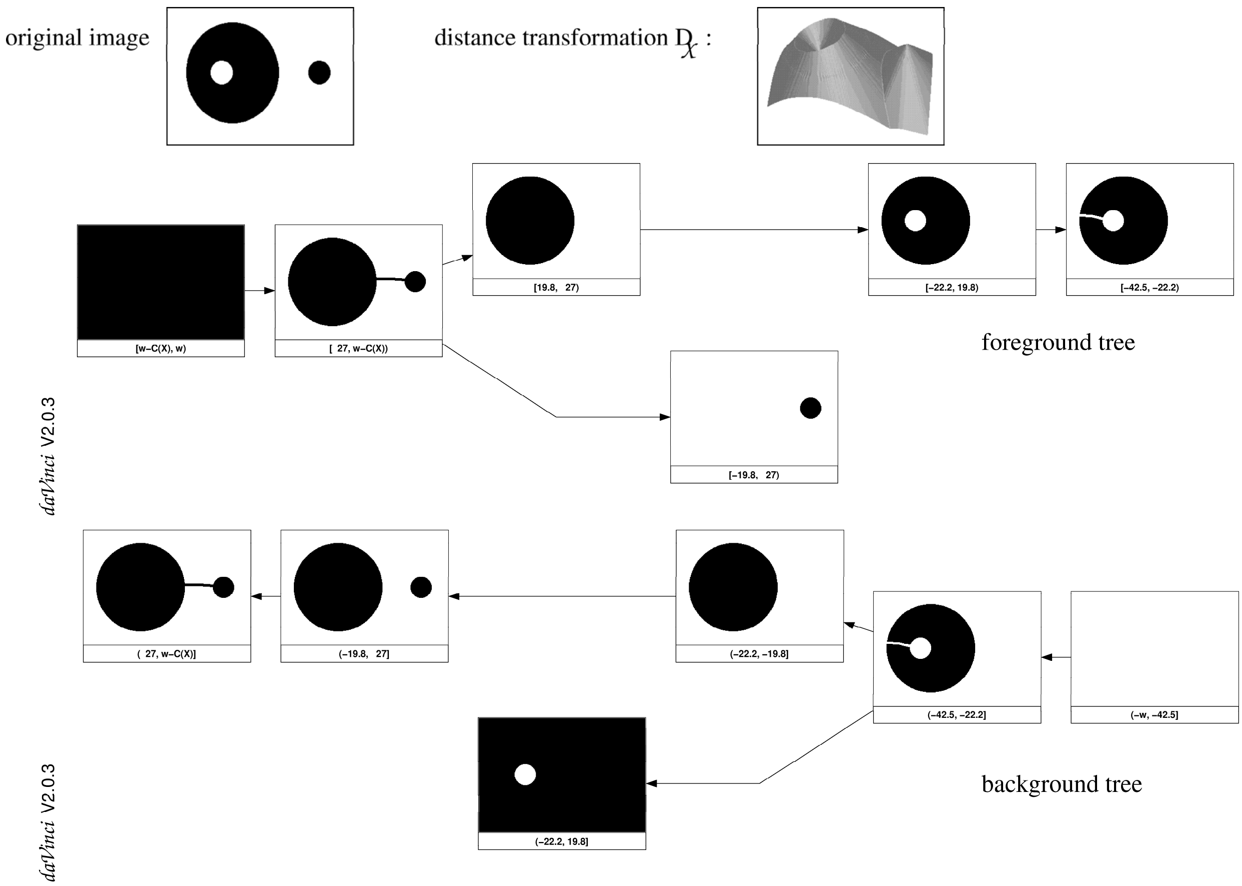
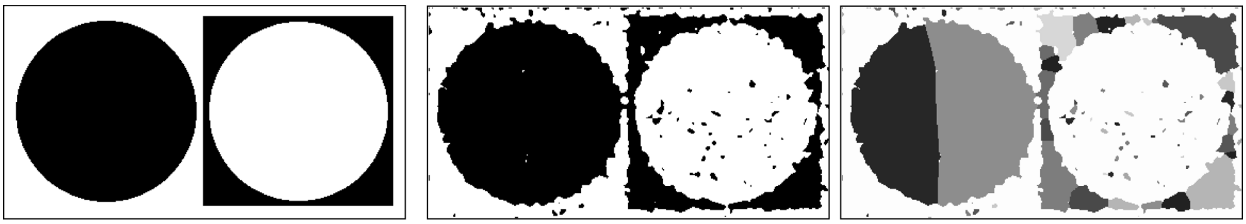


Figure 14: $\mathcal{X}_{\text{dual}}$, distance transformation $D_{\mathcal{X}_{\text{dual}}}$ and foreground- and background tree. Observe the topology of the corresponding images in the foreground and the background tree to be identical.

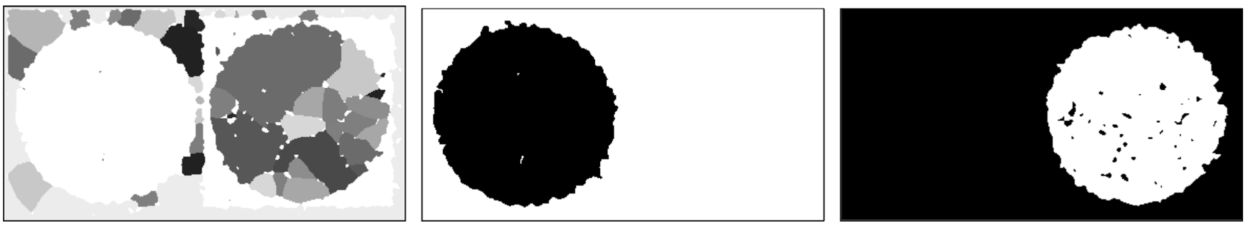
The second example in figure 15 shows an artificial image (a), where two discs are visible, one in the foreground and the other one in the background. Note that usually the surrounding box of the right disc is not immediately perceived. To this example noise was added (b) in the same way as in figure 5. Figures 15(c) and (d) show the parts of the tide-partitioning for fore- and background labeled by grey value. Analyzing the trees of the noisy image one can find nodes which approximately show the black and white disc, to find the discs one only has to look at the partitioning trees, since both discs have to be separated from other blobs resp. the background: figure 15(e) is a component referring to a foreground node which is valid within a distance scale of [-19.5,-2.06). To find the white disc one has to look at the background tree: the node depicted in (f) is valid within a scale of [1.12,1.58). The length of the latter interval is much smaller, because of the large amount of spurious elements within the circle, but up to a scale of $d = 8.2$ the disc keeps its rough shape.



(a) Two discs.

(b) Noisy version of (a)

(c) Tide-partitioning of the foreground



(d) Tide-partitioning of the background

(e) Image of a node in the foreground tree

(f) Image of a node in the background tree

Figure 15: Example of the simultaneous representation of foreground and background objects, see text for details.

Figure 16 depicts an aerial grey scale image (a) and the homogeneous regions (b) according to the grey level. The tide-partitioning results are shown in (c) and (d). There are 19 blobs in figure 16(b), the total number of nodes of the foreground tree is 136, from which the partitioning graph has 61 and the grouping graph has 94 nodes. A clipping of the partitioning tree, in which the right part of the roof is correctly split from the ground blob is shown in figure (e). Because the connection between these two blobs is relatively large, the split applies on a low significance resp. large distance. The connection of the two blobs of the upper roof part is very significant and already appears for a distance $d = 1.5$, see (f). However a grouping of roof blobs using the tide-partitioning is not always successful: here the average distance between two blobs depends on the filter size used in the feature extraction, but it does not reflect whether a separating edge between two blobs has a strong image support. This information has to be included to the significance measure in order to obtain a more intuitive grouping of the parts.

7 Summary and Conclusions

Perceptual organization relies on basic image features, their relations and the evaluation of their attributes with respect to a specified aggregation to higher level primitives. We have proposed a rich representation of image features for supporting perceptual organization. It starts from a partitioning of the image into foreground and background, and a set of image primitives and their relations, describing the foreground. The representation consists of a pair of trees dually treating foreground and background allowing to represent objects with holes and composed of non connected parts. The tree establishes a hierarchy with a scale parameter depending on the significance of parts or holes and links or partitionings.

The representation has been realized for binary images on a topologically consistent cell structure. The results show promising features of the representation, which result from its close link to morphological scale space, however, keeping the detailedness of the original resolution of the image.

The representation can be generalized in several ways:

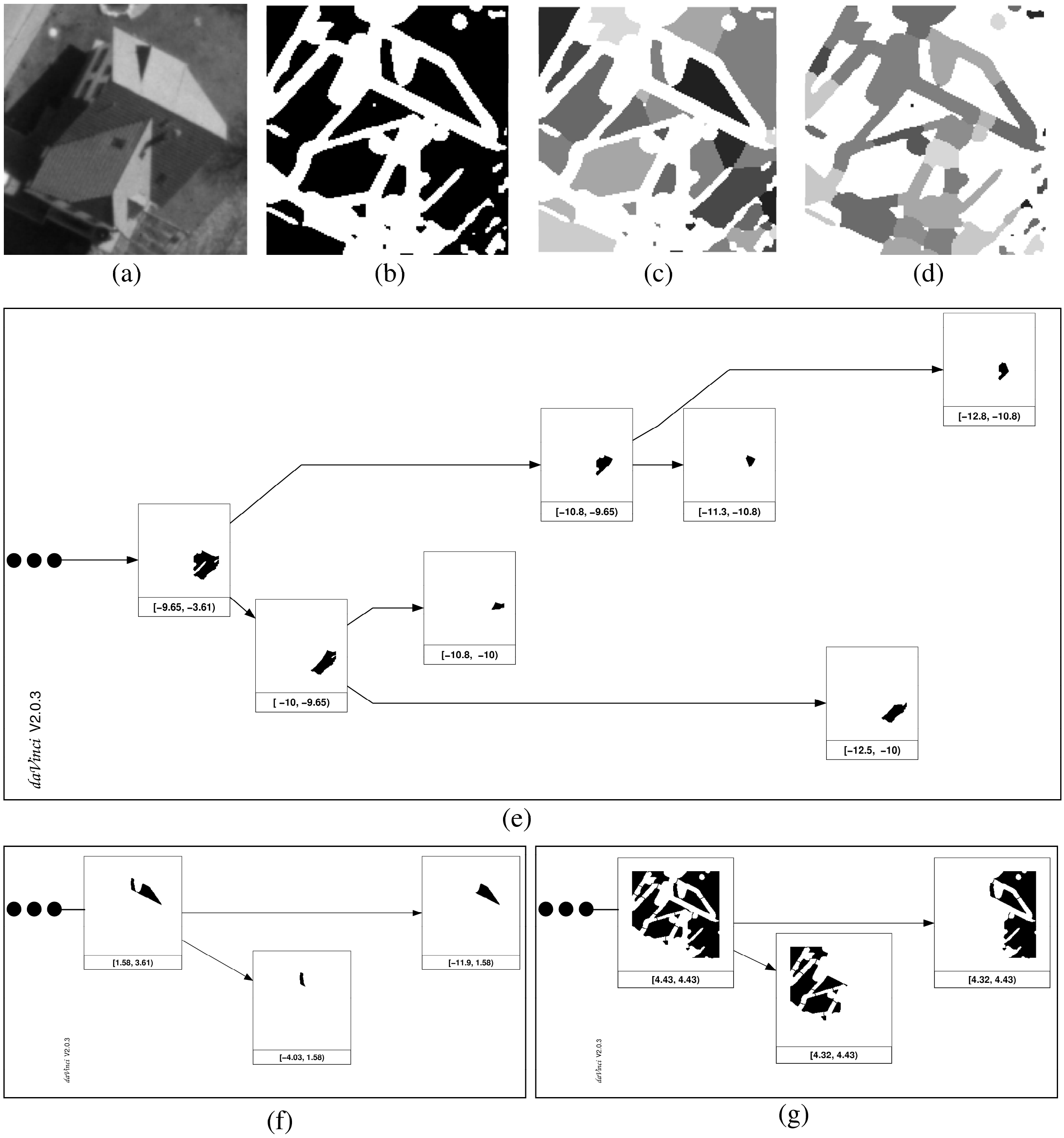


Figure 16: Figure (a) shows a aerial grey scale image, figure (b) is the result of the extraction of homogeneous image regions, see FÖRSTNER 1994. The tide-partitioning for (b) is depicted in (c) and (d). Figures (e), (f) and (g) show parts of the foreground tree: (e) partitions the roof with the lawn area, (f) and (g) shows the connection of two roof parts.

- The significance measure for the partitioning and grouping can be based on all attributes of the relation, not only the minimum distance. This would allow to include e. g. the strength of the geometric link, derived from the length of the common boundary of the Voronoi diagram, or the geometric or radiometric attributes of the features.
- The significance measure may be related to some probability measure, indicating the likelihood of the basic processes of grouping, partitioning, generation and deletion and at the same time take the attributes of the features into account.
- The primitives of the grouping process may be easily generalized to points or lines, not only blobs. This would enable to unify several grouping techniques in a common system. In the case of linear features on could use directional information for measuring the significance of features or their relation, as has been proposed in earlier work on grouping of straight line segments. The hierarchical structure would allow to perform the grouping more specifically.

Quite some image analysis tasks may take advantage of the proposed representation: Matching algorithms and object recognition could attack the problem of occlusion more efficiently by exploiting the hierarchical structure and the scaling parameter. An interesting and open question is how the representation generalizes from binary images with two groups of features to images with three or four groups of features which will interact in a more complex way.

References

AHUJA, N.; TUCERYAN, M. (1989): Extraction of Early Perceptual Structure in Dot Patterns: Integrating Region, Boundary, and Component Gestalt. *Computer Vision, Graphics and Image Processing*, 48:304–356, 1989.

BORGEFORS, G. (September 1991): Another Comment on A Note on 'Distance Transformations in Digital Images'. *CVGIP*, 54(2):301–306, September 1991.

CASTAÑO, R.L.; HUTCHINSON, S. (November 1996): A Probabilistic Approach to Perceptual Grouping. *CVIU*, 64(3):399–419, November 1996.

Förstner, W. (1994): A Framework for Low Level Feature Extraction. In: Eklundh, J.-O. (Ed.), *Computer Vision - ECCV '94, Vol. II*, pages 383–394. Lecture Notes in Computer Science, 801, Springer-Verlag, 1994.

Fuchs, C.; Förstner, W. (1995): Polymorphic Grouping for Image Segmentation. In: *5th ICCV '95, Boston*, pages 175–182. IEEE Computer Society Press, 1995.

Fuchs, C. (1998): *Extraktion polymorpher Bildstrukturen und ihre topologische und geometrische Gruppierung*. PhD thesis, Universität Bonn, 1998.

Guy, G.; Medioni, G.G. (1996): Inferring Global Perceptual Contours from Local Features. *IJCV*, 20(1/2):113–133, 1996.

Heitger, F.; von der Heydt, R. (1993): A Computational Model of Neural Contour Processing: Figure-Ground Segregation and Illusory Contours. In: *IEEE Proceedings*, pages 32–40, 1993.

Heuel, S. (September 1997): Duale, Skalierbare und Hierarchische Partitionierung und Gruppierung von Binärfiguren. Master's thesis, University of Bonn, September 1997.

Jackway, P.T.; Deriche, M. (January 1996): Scale-Space Properties of the Multiscale Morphological Dilation Erosion. *PAMI*, 18(1):38–51, January 1996.

Kanizsa, G. (1979): *Organization in Vision*. New York, Praeger, 1979.

Kimia, B.B.; Tannenbaum, A.R.; Zucker, S.W. (July 1995): Shapes, Shocks, and Deformations I: The Components of 2-Dimensional Shape and the Reaction-Diffusion Space. *IJCV*, 15(3):189–22, July 1995.

Koenderink, J.J.; van Doorn, A.J. (1997): Image Structure. In: Paulus, E.; Wahl, F. M. (Eds.), *Mustererkennung 1997*, pages 2–35. DAGM, Springer, 1997.

Kovalevsky, V. A. (1989): Finite topology as applied to image analysis. *CVGIP 46*, pages 141–161, 1989.

Lin, C.; Huertas, A.; Nevatia, R. (1994): Detection of Buildings Using Perceptual Groupings and Shadows. In: *CVPR94*, pages 62–69, 1994.

Lindeberg, T. (1993): Detecting Salient Blob-like Image Structures and their Scales with a Scale-Space Primal Sketch: A Method for Focus-of-Attention. *International Journal of computer Vision*, 11(3):283–318, 1993.

Lowe, D.G. (1985): *Perceptual Organization and Visual Recognition*. Kluwe, 1985.

McCafferty, J.D. (1990): *Human And Machine Vision.* Ellis Horwood, 1990.

Ogniewicz, R. L.; Kübler, O. (March 1995): Hierarchic Voronoi Skeletons. *Pattern Recognition*, 28(3):343–359, March 1995.

Pavlidis (1977): *Structural Pattern Recognition.* Springer Series in Electrophysics 1. Springer, 1977.

Sarkar, S.; Boyer, K.L. (March 1993): Integration, Inference, and Management of Spatial Information Using Bayesian Networks: Perceptual Organization. *PAMI*, 15(3):256–274, March 1993.

Sarkar, S.; Boyer, K.L. (March/April 1993): Perceptual Organization in Computer Vision: A Review and a Proposal for a Classificatory Structure. *Transaction on Systems, Man, and Cybernetics*, 23:382–399, March/April 1993.

Serra, Jean (Ed.) (1982): *Image Analysis and Mathematical Morphology*, Band 1. Academic Press, 1982.

Siddiqi, K.; Kimia, B.B. (March 1995): Parts of Visual Form: Computational Aspects. *PAMI*, 17(3):239–251, March 1995.

Siddiqi, K.; Shokoufandeh, A.; Dickinson, S.; Zucker, S. (January 1998): Shock Graphs and Shape Matching. In: *International Conference on Computer Vision*, January 1998. Bombay, India.

Tari, Z.S.G.; Shah, J.; Pien, H. (May 1997): Extraction Of Shape Skeletons From Grayscale Images. *CVIU*, 66(2):133–146, May 1997.

Winter, Stephan (1995): Topological Relations between Discrete Regions. In: Egenhofer, Max J.; Herring, John R. (Eds.), *Advances in Spatial Databases*, pages 310–327. Springer, LNCS 951, 1995.