Uncertainty and Projective Geometry

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Summary. Geometric reasoning in Computer Vision always is performed under uncertainty. The great potential of both, projective geometry and statistics, can be integrated easily for propagating uncertainty through reasoning chains, for making decisions on uncertain spatial relations and for optimally estimating geometric entities or transformations. This is achieved by (1) exploiting the potential of statistical estimation and testing theory and by (2) choosing a representation of projective entities and relations which supports this integration.

The redundancy of the representation of geometric entities with homogeneous vectors and matrices requires a discussion on the equivalence of uncertain projective entities. The multi-linearity of the geometric relations leads to simple expressions also in the presence of uncertainty. The non-linearity of the geometric relations finally requires to analyze the degree of approximation as a function of the noise level and of the embedding of the vectors in projective spaces.

The paper discusses a basic link of statistics and projective geometry, based on a carefully chosen representation, and collects the basic relations in 2D and 3D and for single view geometry.

1 Introduction

Uncertainty is present in Computer Vision in all analysis steps: in image processing, in feature extraction, pose estimation, grouping, but also in recognition and interpretation. Problems are, among others, the adequate representation of uncertainty, propagation of uncertainty, estimation under uncertainty, decision making under uncertainty. Recently, statistical inference has become a major thread of research at all levels of image analysis. This certainly is caused by the rich arsenal of tools, which allows to precisely model uncertainty, to check the validity of the assumptions made, and to reason under uncertainty.

This paper is about uncertainty in geometric reasoning, specifically using projective geometry. Algebraic projective geometry has become the basic tool for representing geometry of multiple views, cf. the two classical text books [10, 18]. The two examples in fig. 1 and 2 show two applications where algebraic projective geometry can be used to advantage.

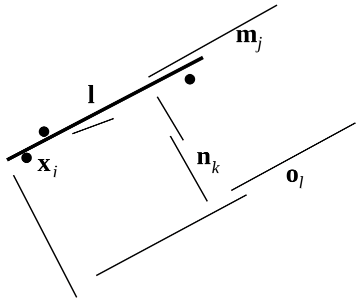


Fig. 1. 2D-Grouping of points and lines, e. g. resulting from a image preprocessing step, consists of two steps: (1) Testing of hypothesized mutual relations, taking uncertainty of image features into account, and (2) joint estimation of geometric features. All points and lines in the figure may be grouped. The result may be an optimal estimate, e. g. of the line $\mathbf{l}$ using incident points $\mathbf{x}_i$, collinear lines $\mathbf{m}_j$, orthogonal $\mathbf{n}_k$ and parallel lines $\mathbf{o}_l$. In algebraic projective geometry all these relations are linear easing statistical testing and estimation.

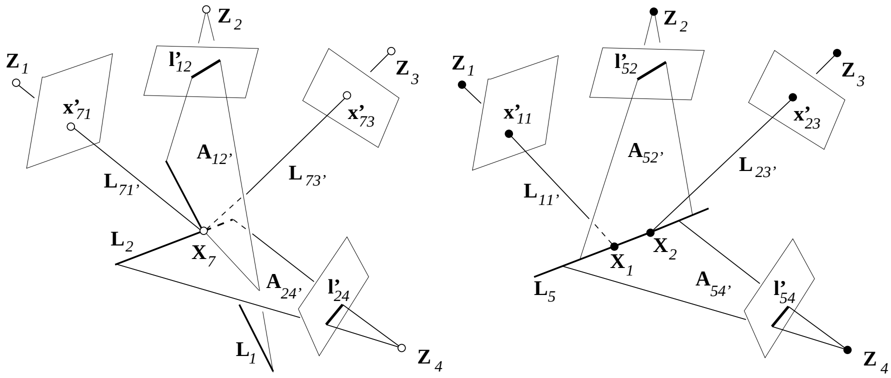


Fig. 2. Triangulation of points and lines: Estimation of 3D-point $\mathbf{X}_7$ (left) and 3D-line $\mathbf{L}_5$ (right) from image points $\mathbf{x}'$ and image lines $\mathbf{l}'$. The spatial relations between image and space points and lines can easily be expressed as a function of the projection matrices and for points and lines resp. Also in this case, using algebraic projective geometry eases statistical testing these relations and the joint optimal estimation of 3D-line.

In both cases the geometric relations can be expressed as multi-linear forms of the entities involved, which would not have been possible when not using projective geometry.

On the other hand rigorous estimation techniques, e. g. used in the bundle adjustment for image orientation minimizing the reprojection error, have been accepted as reference for suboptimal techniques and as a final step in order to obtain statistically optimal results. The need to exploit the full information about the statistics is demonstrated in the example of fig. 3: All geometric entities with a certain probability lie within a certain region, whose shape and size vary individually. Therefore pure geometric measures are not useful for reasoning under uncertainty.

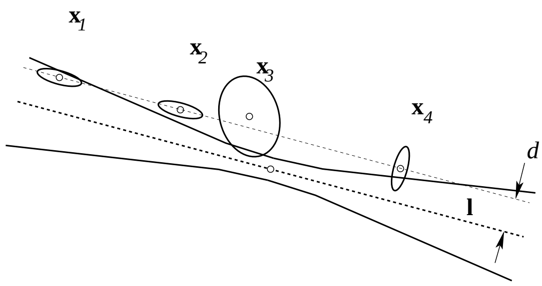


Fig. 3. Testing spatial relations: Necessity of taking the uncertainty into account rigorously, e. g. when testing a point-line incidence: In the presence of uncertainty the geometric distance d of the points $\mathbf{x}_i$ from the line $\mathbf{l}$ is not useful for testing. The uncertainty of a point in a first approximation can be represented by a confidence ellipse, while the uncertainty of a line can be represented by a confidence hyperbola being the collection of the confidence regions across the line of all points. Though the situation is much more complex in 3D, it can easily be handled using the covariance matrices of the entities in concern.

We only want to mention two prominent representative publications where projective geometry and statistics have been integrated to a larger extent:

Kanatani [25] apparently was the first who integrated geometry in 2D and 3D and statistics in a rigorous manner. He aimed at completeness in uncertain geometric reasoning, and discussed motion estimation and optical flow. He proposed rigorous tests and optimal, i. e. maximum likelihood estimates. However, though he used homogeneous vectors for representing geometric entities, he required 2D- and 3D-points to be Euclidian normalized. This was motivated by the otherwise indefinite scaling of the vectors, but does not allow to handle points at infinity. The partitioning of the vectors into a homogeneous and an Euclidean part, which as such is reasonable for interpretation (cf. [3]), lead to cumbersome expressions in the covariance matrices, especially as he aimed at giving explicit expressions including both, the error propagation and the normalization to Euclidean homogenous vectors.

In his thesis, Criminisi [9] integrated uncertainty reasoning into all steps of single and multiple view analysis. For a great number of geometric reasoning tasks, also including the determination of transformations, he gave explicit expressions for covariance matrices. He also analyzed the degree of approximation introduced by linearization.

Unfortunately, in both cases the beauty of projective geometry got lost on the way to the integration of statistical reasoning.

The ease of handling multi-linear, especially bilinear forms was questioned by Haddon and Forsyth [16]: They demonstrated that significant bias and deviations from a Gaussian distribution might occur when partitioning bilinear forms, i. e. when solving for structure and motion from image observations. Their examples, however, seem to be caused by a very low signal to noise ratio, resulting from comparably short base lines in image sequences.

Altogether there appears to be a clear agreement to use both, statistics and projective geometry for spatial reasoning under uncertainty. The main problem left is to find an adequate representation and adequate procedures for geometric reasoning. The approach described in this paper tries to avoid the disadvantages mentioned above

In statistics uncertainty of measurements usually is represented by covariance matrices. Generally speaking, instead of working with the probability densities one uses the first two moments of the distribution. This appears to be widely accepted and is adequate as long as the signal-to-noise ratio is high enough, say much above 10 : 1, which nearly always is the case in the first steps of image analysis. Moreover, propagation of uncertainty can be performed within the calculus of linear algebra to a sufficient accuracy.

Therefore, among the many *representations*, homogeneous vectors and matrices appear to be the right choice for geometric entities and transformations. An embedding into more fundamental concepts, such as the double algebra or Grassman-Cayley algebra [4, 11] or even the Geometric Algebra [21, 22] does not seem to be possible. However, - and this was the motivation for the approach presented here, cf. [15, 23] - the beautiful structures of these algebras, should be kept as far as possible.

Together with the nearly always approximate representation of uncertainty using covariance matrices, there exists a rich and powerful arsenal of tools for statistical reasoning along. This especially holds for estimation techniques (e. g. [31]). The versatility of these tools is the basis for the broad experience in the field of geodesy, where estimation of geometric quantities is a standard task [20, 29, 27]. The key to an easy to use concept of estimation procedures lies in the generic representation of the given functional relations between all observed quantities and all unknowns, not so much in the optimization function nor the optimization procedures e. g. the trust region method to increase convergence [19], which appear to be a - though necessary - second step.

Therefore, among the many *procedures* for estimation, the Maximum Likelihood estimation based on the so-called Gauss-Helmert model appears to be the right choice. It is generic, as it allows to represent all estimation problems with non-linear constraints. Similarly to all other alternative models the estimation processes iteratively improves approximate values for all parameters. The approximate values have to be reasonably good, which in our context can be achieved based on the rich work of the past decade.

The paper is organized as follows: Section two *uncertainty* discusses issues of uncertainty: (1) the representation, especially in the context of homogeneous entities, (2) the propagation, especially the effect of linearization, (3) the estimation under generic constraints and (4) basic elements from testing. Section three *geometric relations* discusses issues of projective geometry: (1) the representation of geometric entities, (2) their construction from given ones, (3) their relations including homogeneous transformations as a basis for uncertainty propagation, statistical testing and estimation. Section four finally discuss when conditioning of the geometric entities and normalization of the covariance matrices is necessary to overcome the proposed approximations.

The goal of the paper is to provide simple-to-use tools for uncertain geometric reasoning. We do not discuss the sources of uncertainty (cf. Kanatani's valuable discussion in this volume) and do not address refinements concerning computational efficiency.

Notation: Vectors are bold face times letters, such as $\boldsymbol{x}$ or $\mathbf{X}$, matrices are bold face sens serif letters, such as $\mathsf{A} = [a_{ij}]$ or H. Homogeneous vectors and matrices are upright letters, such as $\mathbf{x}$ or H, Euclidean vectors are slanted letters, such as $\boldsymbol{x}$ or $\boldsymbol{X}$. Vectors representing geometric 2D-entities are lower case letters, such as $\boldsymbol{x}$ or $\mathbf{l}$, vectors representing geometric 3D-entities are upper case letters, such as $\mathbf{A}$ and $\mathbf{X}$. Planes are denoted with letters $\mathbf{A}$, $\mathbf{B}$, ... from the beginning of the alphabet, lines are denoted with letters $\mathbf{l}$, $\mathbf{L}$, $\mathbf{M}$, ... from the middle of the alphabet, and points are denoted with letters $\mathbf{X}$, $\mathbf{Y}$ ... from the end of the alphabet. The $n \times n$-unit matrix is denoted with I_n. The i-th n-unit vector is denoted with $\boldsymbol{e}_i^{(n)}$. Stochastical variables are underscored, such as $\underline{\boldsymbol{x}}$. The density function of the stochastical variable x, possibly being a vector $\boldsymbol{x}$, is denoted with $p_x(\cdot)$. The expectation, the variance and the covariance operators are $\mathrm{E}(\cdot)$, $\mathrm{V}(\cdot)$ and $\mathrm{Cov}(\cdot,\cdot)$ resp. Covariance matrices are indexed with two indices, e. g. $\mathrm{V}(\underline{\boldsymbol{x}}) = \boldsymbol{\Sigma}_{xx} = [\sigma_{x_i x_j}]$, allowing to densely write the covariance of two different vectors $\mathrm{Cov}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \boldsymbol{\Sigma}_{xy} = [\sigma_{x_i y_j}]$. The determinant of a matrix is $|\mathsf{A}|$.

We will use the vec-operator, column-wise stacking the columns of an $n \times m$-matrix A into a nm-vector $\mathrm{vec}\mathsf{A}$, thus $\mathrm{vec}(\mathsf{A}^\mathsf{T})$ contains the nm elements of A row-wise. We will use the Kronecker product $\mathsf{A} \otimes \mathsf{B} = [A_{ij}\mathsf{B}]$. With the vec-operator we use the two relations $\mathrm{vec}(\mathsf{ABC}) = (\mathsf{C}^\mathsf{T} \otimes \mathsf{A})\mathrm{vec}\mathsf{B}$ and as $\mathrm{vec}(\mathsf{AB}\boldsymbol{c}) = \mathrm{vec}(\boldsymbol{c}^\mathsf{T}\mathsf{B}^\mathsf{T}\mathsf{A}^\mathsf{T})$ and $\mathrm{vec}(\mathsf{AB}\boldsymbol{c}) = (\boldsymbol{c}^\mathsf{T} \otimes \mathsf{A})\mathrm{vec}\mathsf{B} = (\mathsf{A} \otimes \boldsymbol{c}^\mathsf{T})\mathrm{vec}(\mathsf{B}^\mathsf{T})$. Row-wise concatenation of two matrices A and B leads to the matrix $[\mathsf{A}|\mathsf{B}]$.

2 Uncertainty

2.1 Representation and Propagation of Uncertainty

Basics

Probability theory is a classical tool for representing uncertainty. In our context we are concerned with representing the uncertainty of coordinate vectors $\underline{\boldsymbol{x}}$, which is usually done via the probability density function (pdf) $p_x(\boldsymbol{x})$ or the cumulative probability density function (cpdf) $P_x(\boldsymbol{x})$, the function in contrast to the independent variable in brackets. In many cases one reasonably well can use the Gaussian or normal distribution with density

$$g_x(\boldsymbol{x}; \boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx}) = \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{\Sigma}_{xx}|}} \mathrm{e}^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_x)^\mathsf{T} \boldsymbol{\Sigma}_{xx}^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_x)} , \tag{1}$$

which depends on two parameters, the n-vector $\boldsymbol{\mu}_x$ and the symmetric $n \times n$-matrix $\boldsymbol{\Sigma}_{xx}$. Observe, the density function of the normal distribution only is defined for regular $\boldsymbol{\Sigma}_{xx}$. For practical reasons one uses the short notation $\underline{\boldsymbol{x}} \sim N(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ to indicate the stochastical variable $\underline{\boldsymbol{x}}$ to be normally distributed with the parameters $\boldsymbol{\mu}_x$ and $\boldsymbol{\Sigma}_{xx}$.

Often reasoning can be restricted to the so-called moments of the distribution. With the expectation operator $\mathrm{E}(\boldsymbol{f}(\underline{\boldsymbol{x}})) = \int \boldsymbol{f}(\boldsymbol{x}) p_x(\boldsymbol{x}) d\boldsymbol{x}$ we will regularly use the mean, being the first moment of the distribution, the variance, being the central second moment, and the kurtosis, being the central fourth moment μ_{4_x} normalized with $3\sigma_x^4$

$$\mu_x = \mathrm{E}(\underline{x}) = \int x p_x(x)\, dx \ , \quad \sigma_x^2 = \mathrm{V}(\underline{x}) = \mathrm{E}((\underline{x}-\mu_x)^2) \ , \quad \kappa = \frac{\mu_{4_x}}{3\sigma_x^4} = \frac{\mathrm{E}((\underline{x}-\mu_x)^4)}{3\sigma_x^4} \ ,$$

which leads to $\kappa = 1$ for Gaussian variables.

For vector valued stochastical variables we have the covariance matrix defined as $\boldsymbol{\Sigma}_{xx} = [\sigma_{x_i x_j}] = \mathrm{V}(\underline{\boldsymbol{x}}) = \mathrm{E}((\underline{\boldsymbol{x}} - \boldsymbol{\mu}_x)(\underline{\boldsymbol{x}} - \boldsymbol{\mu}_x)^{\mathsf{T}})$. In case two stochastical variables are statistically independent, their joint distribution $p_{xy}(x, y)$ is separable, thus

$$p_{xy}(x,y) = p_x(x)\, p_y(y) \qquad \text{and} \qquad P_{xy}(x,y) = P_x(x)\, P_y(y) \ . \tag{2}$$

For normally distributed variables $\underline{\boldsymbol{x}} \sim N(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ the two parameters $\boldsymbol{\mu}_x$ and $\boldsymbol{\Sigma}_{xx}$ are the mean and the covariance matrix.

Propagating uncertainty through chains of non-linear functions in general is intractable. For very specific distributions and simple functions this can be done explicitely, using various techniques, depending on the special situation, cf. the discussion and the many examples given by Papoulos [30]. We, however, may restrict to the propagation of the first two moments.

If two first moments of a stochastical vector are used to describe its $\underline{\boldsymbol{x}} \sim M_x(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ distribution, then the vector valued nonlinear function $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ has a distribution $\underline{\boldsymbol{y}} \sim M_y(\boldsymbol{\mu}_y, \boldsymbol{\Sigma}_{yy})$ with the first two moments (cf. [27])

$$\boxed{\boldsymbol{\mu}_y = \boldsymbol{f}(\boldsymbol{\mu}_x) \qquad \boldsymbol{\Sigma}_{yy} = J_{yx} \boldsymbol{\Sigma}_{xx} J_{yx}^{\mathsf{T}} \quad \text{with} \quad J_{yx} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{x=\mu_x}} \tag{3}$$

This error propagation law holds rigorously for any distributions with finite first and second order moments in case the relation $\boldsymbol{f}(\boldsymbol{x})$ is linear. In case of non-linear function it is an approximation, which we discuss below in our context of geometric reasoning.

The basic idea is to attach a covariance matrix to each uncertain entity during geometric reasoning.

Representing Uncertain Homogeneous Vectors

Attaching a covariance matrix to homogeneous vectors can be done straight forward and has been extensively done by Kanatani and Criminisi. E. g. in case the Euclidean coordinates $\underline{\boldsymbol{x}} = (\underline{x}, \underline{y})^{\mathsf{T}} \sim M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ is given the corresponding covariance matrix of the homogeneous 3-vector $\underline{\mathbf{x}} = (\underline{x}, \underline{y}, 1)^{\mathsf{T}}$ is given by

$$\boldsymbol{\Sigma}_{\mathrm{xx}} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ . \tag{4}$$

This approach is correct and circumvents the problem of discussing projective entities (cf. table 4): The transition from uncertain Euclidean entities, which are primary observations, to uncertain homogeneous entities is simple and can be done statistically rigorous for points. One goal of the paper is to show, that the transition to other uncertain homogeneous entities, e. g. by construction, is simple, and a good approximation. The same holds for the derivation of Euclidean entities from homogeneous ones. Thus instead of working with a non-redundant representation in Euclidean space, e. g. in $\mathbb{R}^2$ for 2D-points, one uses a redundant representation in a higher dimensional Euclidean space, e. g. in $\mathbb{R}^3$ for 2D-points.

The beauty of algebraic projective geometry for geometric reasoning and multiple view analysis shown in the paper of Faugeras/Papadopoulo [11] was the key motivation to use homogeneous coordinates for representing uncertain geometric entities but to stay as close as possible to the concepts of the Grassman-Cayley algebra in order to preserve the transparency of the geometric relations.

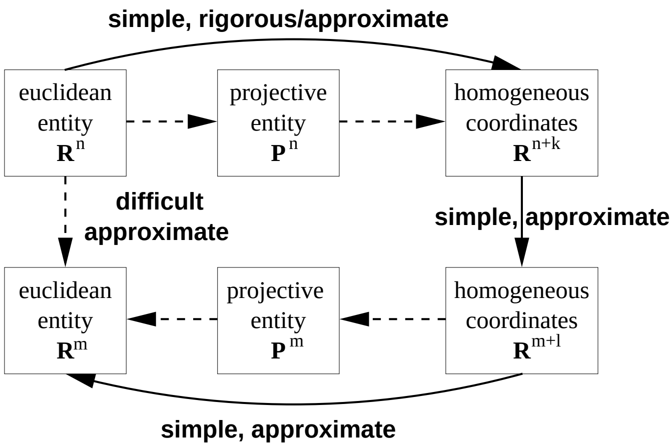


Fig. 4. Reasoning under uncertainty in projective geometry. Instead of performing calculations with Euclidean entities one works with homogeneous entities, only implicitly using them as representations of projective entities.

However, the redundancy k in the representation with homogeneous entities, which is $k = 1$ for all entities, except for 3D-lines, where it is $k = 2$, leads to some difficulties:

1. The redundancy in the representation immediately leads to *singular covariance matrices* as e. g. in (4). Thus there is no proper pdf for homogeneous entities[1]
2. Construction of new entities from given ones, using the classical tools of statistical error propagation in general leads to regular covariance matrices. This is in contrast to the singularity of the covariance matrices derived from Euclidean entities, cf. (4) and prevents a simple comparison of uncertain stochastic vectors having covariance matrices of different rank.

[1] which was a critics of a reviewer of some earlier conference paper using this approach.

Kanatani [25] proposed to use the Euclidean normalized homogeneous coordinates for points. This imposes a constraint, e. g. the component $\underline{w}$ of $\underline{\mathbf{x}} = [\underline{u}, \underline{v}, \underline{w}]^\mathsf{T}$ to be normalized to 1, and results in covariance matrices with the desired rank. Interestingly, he normalizes 2D-lines, 3D-lines and planes spherically.
This normalization, however, will be shown not to be necessary in general. Therefore also homogeneous vectors with a full rank covariance matrix will be allowed.
The *equivalence relation* for homogeneous vectors therefore needs to be redefined for this reason.
Only in case one uses the proposed test statistics for correctly sorting multiple hypotheses in search problems, one needs to condition and normalize the geometric entities. However, always spherical normalization can be applied which enables to include points at infinity.

3. The representation and propagation of uncertainty with the second moments is an approximation.
The degree of approximation needs to be known to safely apply the proposed approach.

On Singular Covariance Matrices

We first discuss the pdf of random vectors containing fixed entities. When representing fixed values, such as the 3rd component in $(\underline{x}, \underline{y}, 1)^\mathsf{T}$, we might track this property through the reasoning chain or just treat the value 1 as stochastical variable with mean 1 and variance 0. The second alternative has implicitly been chosen by Kanatani [25] and Criminisi [9].

One can easily construct a 2-vector with a singular 2×2-covariance matrix. Assume $\underline{x} \sim N(\mu_x, 1)$ and $\underline{y} \sim N(\mu_y, 0)$ are independent stochastical variables, thus

$$\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) .$$

The distribution of $\underline{y}$ can be defined as a limiting process:

$$p_y(y) = \lim_{\sigma_y \to 0} g(y; \mu_y, \sigma_y^2) = \delta(y - \mu_y) .$$

The resulting δ-function is a so-called *generalized function*, only definable via a limiting process. As $\underline{x}$ and $\underline{y}$ are stochastically independent their *joint generalized pdf* is, cf. (2)

$$g_{xy} = g_x(x; \mu_x, 1)\, \delta(y - \mu_y) .$$

Obviously, working with a mixture of Gaussians and δ-functions will be cumbersome in case stochastical variables are not independent.

Again, in most cases reasoning can done using the moments, therefore the complicated distribution is not of primary concern. The propagation of uncertainty using the second moments only relies on the covariance matrices, not on their inverses, and can be derived using the so-called moment generating function (cf. [30]), which is also defined for generalized pdf's. Thus uncertainty propagation can be performed also in mixed cases.

Equivalence Relation for Uncertain Homogeneous Entities

A more critical problem is the equivalence relation of uncertain vectors.

The equivalence of two fixed, i. e. statistically certain, homogeneous n-vectors $\mathbf{x}$ and $\mathbf{y}$ usually is represented as

$$\mathbf{x} \cong \mathbf{y} \Longleftrightarrow \mathbf{x} = \lambda \mathbf{y} \tag{5}$$

for some factor $\lambda \in \mathbb{R} \setminus 0$. In case two stochastic n-vectors $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ are given with their pdf $p_x(\mathbf{x})$ and $p_y(\mathbf{y})$ the equivalence relation would transfer to

$$\underline{\mathbf{x}} \cong \underline{\mathbf{y}} \Longleftrightarrow p_x(\mathbf{x}) = \frac{1}{\lambda^n}\, p_y\left(\frac{\mathbf{y}}{\lambda}\right) \tag{6}$$

for some factor $\lambda \in \mathbb{R} \setminus 0$.

This equivalence relation does not allow to use regular covariances for homogeneous entities as they may occur. As an example, assume

$$\underline{\mathbf{p}} = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \sim N(\boldsymbol{\mu}_p, \boldsymbol{\Sigma}_{pp}) \qquad \underline{\mathbf{q}} = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \sim N(\boldsymbol{\mu}_q, \boldsymbol{\Sigma}_{qq})$$

with

$$\boldsymbol{\Sigma}_{pp} = \boldsymbol{\Sigma}_{qq} = \sigma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Using (3), the covariance matrix of the joining line $\underline{\mathbf{l}} = \underline{\mathbf{p}} \times \underline{\mathbf{q}}$ is (cf. the discussion around table 3.3, p. 30)

$$\boldsymbol{\Sigma}_{ll} = \sigma^2 \begin{bmatrix} 2 & 0 & -(x_1 + x_2) \\ 0 & 2 & -(y_1 + y_2) \\ -(x_1 + x_2) & -(y_1 + y_2) & x_1^2 + x_2^2 + y_1^2 + y_2^2 \end{bmatrix}$$

and has determinant

$$|\boldsymbol{\Sigma}_{ll}| = 2\sigma^6((x_2 - x_1)^2 + (y_2 - y_1)^2) .$$

Thus the covariance matrix of $\underline{\mathbf{l}}$ always has full rank. This is in contrast to the fact that, given the two parameters of a line, the resulting covariance matrix of its homogeneous vector has rank 2.

The reason for this conflict is: The equivalence relation (6) does not allow the factor λ to be uncertain.

E. g., if two 2-vectors $\underline{\mathbf{x}} = \underline{\mathbf{y}} \in \mathbb{P}^1$ follow a Gaussian distribution $p_x(\mathbf{x}) = p_y(\mathbf{y})$ with covariance matrix $\boldsymbol{\Sigma}_{xx} = \boldsymbol{\Sigma}_{yy}$, they certainly are equivalent. If now $\underline{\lambda}$ is 2 and 3 with probability 1/2, then $\underline{\mathbf{z}} = \underline{\lambda}\,\underline{\mathbf{y}}$ follows a mixture of two equally probable Gaussians with $4\boldsymbol{\Sigma}_{yy}$ and $9\boldsymbol{\Sigma}_{yy}$ thus

$$p_z(\mathbf{y}) = \frac{1}{2}\cdot\frac{1}{4}\cdot p_y\left(\frac{\mathbf{y}}{2}\right) + \frac{1}{2}\cdot\frac{1}{9}\cdot p_y\left(\frac{\mathbf{y}}{3}\right) = \frac{1}{8}\, p_y\left(\frac{\mathbf{y}}{2}\right) + \frac{1}{18}\, p_y\left(\frac{\mathbf{y}}{3}\right)$$

shown in fig. 5. This density function is not equivalent to $p_x(\mathbf{x})$ when using the equivalence relation (6). However, any realization comes either from $1/4\, p_y(\mathbf{y}/2)$ or from $1/9\, p_y(\mathbf{y}/3)$, thus is equivalent to $\underline{\mathbf{x}}$. Therefore the equivalence relation (6) is

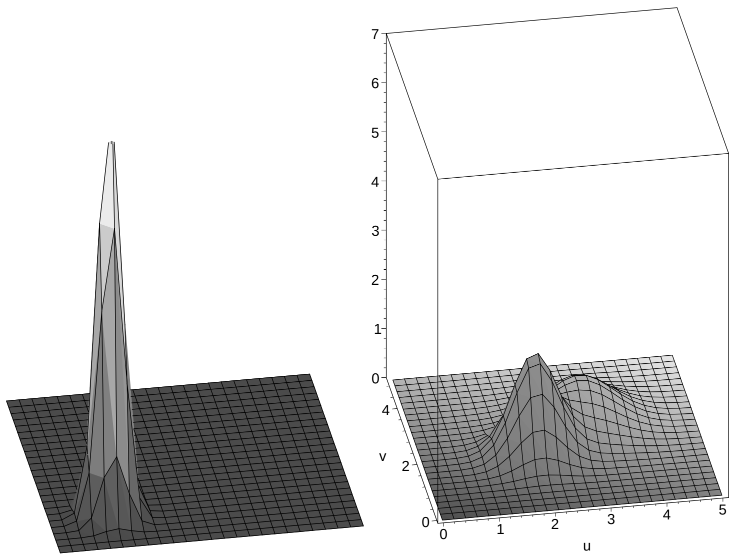


Fig. 5. Two pdf's of a homogeneous 2-vector. Left: original, right: modified. The two distributions represent the same 1D-point $\underline{x} = \underline{u}/\underline{v}$ with $\boldsymbol{\mu}_x = (\mu_u, \mu_v)^{\mathsf{T}} = (1,1)^{\mathsf{T}}$. However, the distributions cannot be easily be related; especially, without further knowledge, the right distribution cannot be derived from the left one.

too restrictive.

We therefore propose to use the following equivalence relation for fixed vectors:

$$\boxed{\mathbf{x} \cong \mathbf{y} \iff \mathbf{x}^s = \mathbf{y}^s} \tag{7}$$

with the spherically normalized vectors

$$\mathbf{x}^s = \frac{\mathbf{x}}{|\mathbf{x}|} \qquad \mathbf{y}^s = \frac{\mathbf{y}}{|\mathbf{y}|} \; .$$

This equivalence relation can be directly transferred to uncertain vectors using the pdf of the normalized vectors

$$\boxed{\underline{\mathbf{x}} \cong \underline{\mathbf{y}} \iff p_{x^s}(\mathbf{x}^s) = p_{y^s}(\mathbf{y}^s)} \tag{8}$$

In case one wants to be sure not to deal with generalized functions, one also can use the equivalence relation based on the cumulative distributions

$$\underline{\mathbf{x}} \cong \underline{\mathbf{y}} \iff P_{x^s}(\mathbf{x}^s) = P_{y^s}(\mathbf{y}^s) \; . \tag{9}$$

In the case of a normally distributed homogeneous vector $\underline{\mathbf{x}} \sim N(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ with a covariance matrix of arbitrary rank, one directly can determine the covariance matrix of the normalized vector from

$$\boldsymbol{\Sigma}_{x^s x^s} = J_s \boldsymbol{\Sigma}_{xx} J_s^{\mathsf{T}}$$

with the Jacobian

$$J_s = \frac{\partial \mathbf{x}^s}{\partial \mathbf{x}} = \frac{1}{|\mathbf{x}|}\left(I - \frac{\mathbf{x}\mathbf{x}^\mathsf{T}}{\mathbf{x}^\mathsf{T}\mathbf{x}}\right) .$$

Obviously, the Jacobian has rank deficiency 1 and null-space $\mathcal{N}(\boldsymbol{\Sigma}_{xx}) = \mathbf{x}$, therefore the covariance matrix at least has rank deficiency 1 and $\mathbf{x}$ is in its null space.

Thus the equivalence relations (8) and (9) explicitely state, that *only the direction* of a homogeneous vector is of concern, and, if the pdf or cpdf of the direction is the same for two homogenous vectors, they are equivalent. This allows to use covariance matrices of any rank for representing uncertain homogeneous vectors.

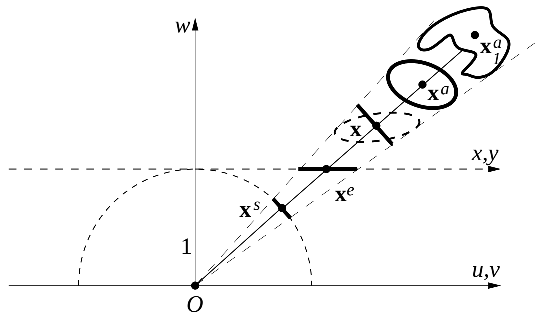


Fig. 6. Equivalent uncertain homogeneous vectors. The Euclidean plane $\mathbb{R}^2$ with Euclidean vectors (x, y) is embedded in $\mathbb{R}^3$ with coordinates $(u, v, w)^\mathsf{T}$. From upper-right to center: (1) homogeneous vector ${}^a\mathbf{x}$ with general and regular covariance matrix, (2) homogeneous vector $\mathbf{x}$ with equivalent covariance matrices, dashed: regular, solid: singular, normalized such that null-space of $\boldsymbol{\Sigma}_{xx}$ is $\mathbf{x}$, (3) Euclidean normalized homogeneous vector ${}^e\mathbf{x}$ with $w = 1$ and original covariance matrix from observations, null-space $\boldsymbol{\Sigma}_{{}^ex\,{}^ex}$ is $(0, 0, 1)^\mathsf{T}$, cf. (4) and [25], (4) spherically normalized homogeneous vector ${}^s\mathbf{x}$, null-space of $\boldsymbol{\Sigma}_{{}^sx\,{}^sx}$ is $\mathbf{x}$. In principle the distribution of a homogeneous vector may have any form (cf. point $\mathbf{x}_1^a$): it still is equivalent to another one, in case after spherical normalization, they have the same distribution, as only the direction is of concern.

Comment: There is a close relation of this equivalence relation to the equivalence relation for covariance matrices referring to different gauges (cf. [26]): In both cases only estimable quantities, i. e. quantities which are estimable are of concern. The invariance here refers to arbitrary distributions, not only to Gaussians, though the transformations discussed later all refer to the second moments. The notion of *gauge* is known as *datum* in the geodetic literature (cf. [2] and [12]). Gauge transformations, due to the special application, are called S-transformations ('S' from similarity as in the application in [26]). The problem is already identified 1987 in robotics (cf. [34]).

Bias in Uncertainty Propagation

From table 4 we observe three essential steps where we use an approximation: (1) generating a homogenous vector from Euclidean coordinates, (2) generating a homogenous vector from given ones, and (3) deriving Euclidean parameters for the geometric entity in concern.

We want to show that in most practical cases the induced bias in mean and variance is negligible and only in unlikely cases the distribution of the resulting entities is significantly deviating form a Gaussian. Though this has been discussed earlier in literature (cf. e. g. [30], [8] or [9]) we present it here for completeness, adapted to the problem in concern.

Bias in mean and variance

The above mentioned rule for propagating uncertainty (3) results from a Taylor expansion of the non-linear function $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$. Including higher order terms yields bias terms.

For a scalar function in one variable we obtain the following result: If the pdf of a stochastical variable $\underline{x}$ is symmetrical, the mean and the variance for $y = f(x)$ can be shown to be

$$E(\underline{y}) = \mu_y = f(\mu_x) + \frac{1}{2} f''(\mu_x)\sigma_x^2 + \frac{1}{24} f^{(4)} \mu_{4_x} + O(f^{(n)}, m_n) \quad n > 4 \qquad (10)$$

and for normally distributed variables, with the central fourth moment $\mu_{4_x} = 3\sigma_x^4$

$$V(\underline{y}) = \sigma_y^2 = f'^2(\mu_x)\,\sigma_x^2 + \left(f'(\mu_x) f'''(\mu_x) + \frac{1}{2} f''(\mu_x) \right) \sigma_x^4 + O(f^{(n)}, m_n) \quad n > 4 \; .$$

Obviously the bias, i. e. the second term, depends on the variance and the higher order derivatives: The larger the variance and the higher the curvature or the 3rd derivative, the higher the bias. Higher order terms depend on derivatives and moments of order higher than 4.

If the pdf of a stochastical vector $\underline{\boldsymbol{x}}$ is symmetrical, the mean of the scalar function $y = f(\boldsymbol{x})$ can be shown to be

$$E(\underline{y}) = \mu_y = f(\boldsymbol{\mu}_x) + \frac{1}{2} \text{trace}(H|_{x=\mu_x} \cdot \boldsymbol{\Sigma}_{xx}) + O(f^{(n)}, m_n), \quad n \geq 3 \qquad (11)$$

with the Hessian matrix $H = (\partial f^2 / \partial x_i \partial x_j)$ of the function $f(x)$. This is a generalization of (10).

We now want to discuss two cases: (1) The product $\underline{z} = \underline{xy}$ of two random variables. This is the most simple case of a bilinear form, occurring when constructing new geometric elements with homogeneous coordinates, (2) normalizing a vector to unity ${}^s\mathbf{x} = \mathbf{x}/|\mathbf{x}|$.

Bias and distribution of the bilinear form $z = xy$

The Taylor series at the mean in this case is finite. Therefore we can derive rigorous expressions for the mean for arbitrary distribution M

$$\mu_z = \mathrm{E}(\underline{z}) = \mu_x \mu_y + \sigma_{xy} \qquad (12)$$

As fourth moments are involved in the determination of the variance, we assume M to be normal. Then we obtain the rigorous expression for the variance of $\underline{z}$

$$\sigma_z^2 = \mathrm{V}(\underline{z}) = \mu_y^2 \sigma_x^2 + \mu_x^2 \sigma_y^2 + 2\mu_x \mu_y \sigma_{xy} + \sigma_x^2 \sigma_y^2 + \sigma_{xy}^2 \qquad (13)$$

Obviously the linear approximation of the mean and the variance are

$$\mu_z^{(1)} = \mu_x \mu_y \qquad \sigma_z^{2(1)} = \mu_x^2 \mu_y^2 + \mu_y^2 \sigma_x^2 + 2\mu_x \mu_y \sigma_{xy} \tag{14}$$

The bias in mean is

$$b_{\mu_z} = \mu_z^{(1)} - \mu_z = -\sigma_{xy} \tag{15}$$

It is zero if the two variables are uncorrelated. The bias in variance is

$$b_{\sigma_z^2} = \sigma_z^{2(1)} - \sigma_z^2 = -\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2 = -\sigma_x^2 \sigma_y^2 (1 + \rho_{xy}^2) \tag{16}$$

It is *not* zero for uncorrelated variables. Actually the variance is *underestimated* if one relies on classical error propagation, as $\sigma_z^{2(1)} < \sigma_z^2$ for uncorrelated variables (cf. [16])

In order to get an impression on the size we assume $\sigma_x = \sigma_y = \sigma$ and $\sigma_{xy} = 0$ and obtain the relative bias in variance

$$r_{\sigma_z^2} = \frac{b_{\sigma_z^2}}{\sigma_z^2} = -\frac{\sigma^2}{\mu_x^2 + \mu_y^2 + \sigma^2} \tag{17}$$

Thus only in case $\mu_x^2 + \mu_y^2 < \sigma^2$ the relative bias in variance is larger than 50 % of the variance. This is very unlikely to occur, as the relative precision σ_x / μ_x of homogeneous vectors in computer vision applications usually is better than 1/100.

We finally want to show the type distribution of the product for an extreme case, especially for $\mu_x = \mu_y = 0$. In case of independent zero mean Gaussian variables

$$\underline{x} \sim N(0, \sigma^2) \qquad \underline{y} \sim N(0, \sigma^2)$$

we find the probability density function of $\underline{z}$ from

$$p_z(z) = \int_0^\infty \frac{2}{u} p_x(u) p_y\left(\frac{z}{u}\right) du$$

yielding

$$p_z(z) = \frac{\text{BesselK}\left(0, \frac{|z|}{\sigma^2}\right)}{\pi \sigma^2}$$

with the Bessel function BesselK(x) of the second kind. It definitely is not normally distributed (cf. fig. 7), but has the variance

$$V(\underline{z}) = 2 \int_{z=0}^\infty z^2 p_z(z) dz = \sigma^4$$

in accordance with (13).

Bias of spherical normalization

The findings are confirmed when analyzing the bias of normalization of a vector. Let the vector $\underline{\mathbf{w}} \sim N(\boldsymbol{\mu}_w, \sigma_w^2 I)$ be normally distributed with independent components with the same variance $\sigma_{w_i} = \sigma_w$. The normalized vector is determined from

$$\underline{\boldsymbol{z}} = \frac{\underline{\boldsymbol{w}}}{|\underline{\boldsymbol{w}}|} \qquad \text{or} \qquad \underline{z}_i = \frac{\underline{w}_i}{|\underline{\boldsymbol{w}}|} \, .$$

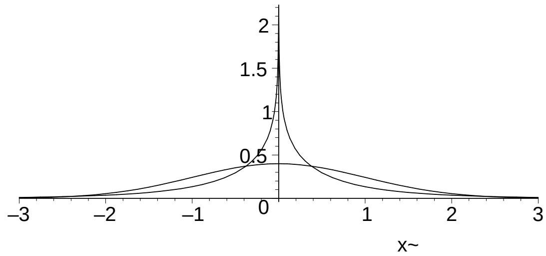


Fig. 7. Bessel function of the second kind and Gaussian with the same variance. The pdf of the product of two zero mean Gaussian random variables is a Bessel function, thus significantly deviates from a Gaussian pdf.

With (11) this leads to the second order approximation of the mean

$$\mathrm{E}(\underline{z}) = \frac{\boldsymbol{\mu}_w}{|\boldsymbol{\mu}_w|} - \frac{1}{2}\frac{\boldsymbol{\mu}_w}{|\boldsymbol{\mu}_w|^3}\sigma_w^2 = \frac{\boldsymbol{\mu}_w}{|\boldsymbol{\mu}_w|}\left(1 - \frac{1}{2}\frac{\sigma_w^2}{|\boldsymbol{\mu}_w|^2}\right) .$$

Thus the relative bias, i. e. the bias related to the standard deviation is

$$\frac{|b_{\mu_z}|}{\sigma_{\mu_z}} = 0.7\frac{\sigma_w}{|\boldsymbol{\mu}_w|}$$

approximately identical to the directional error $\sigma_w/|\boldsymbol{\mu}_w|$, which usually is below 1/100. One can show that also the relative bias in the standard deviation is approximately identical to the directional error

$$\frac{|b_{\sigma_z}|}{\sigma_{\sigma_z}} \sim \frac{\sigma_w}{|\boldsymbol{\mu}_w|} .$$

Remark: These findings are useful for the constructions, for the tests and for the estimation procedures discussed below. The strong bias, found by Haddon and Forsyth [16] refer to partitioning bilinear forms with mean values close to zero. There the relative errors are large, thus the bias cannot be neglected.

2.2 Estimation

Estimation of unknown parameters is a classical task in statistical inference. Maximum-likelihood estimation is one of the standard tools.

In our context, many relations are multi-linear, especially they are linear in the unknown parameters. Therefore direct solutions, minimizing the so-called algebraic error are frequently used. Though they usually lead to sufficiently good approximate values, they are suboptimal. Therefore techniques for improving these approximations have been proposed, such as renormalization [25] and [5], total least squares, the HEIV-method (heteroscedastic errors in variables, meaning variables with errors of different weight [28]) and their improvements [6], [7]. The motivation for these techniques stems from deficiencies in historically older methods: Renormalization compensates for bias, total least squares takes the stochastic properties of the coefficients in a regression model into account, HEIV takes the correct weights in the algebraic minimization scheme. All of them are iterative.

We propose to directly use the given constraints and minimize the weighted sum of residuals of the observations, the weights being inversely proportional to the covariance matrix. The resulting estimates are local ML-estimates in case the given observations are Gaussian. We will show, that the HEIV method, in a modified version, is a special case, and furthermore, the method of minimizing the algebraic error is a further simplification.

The proposed scheme has two definite advantages:

- It can handle *any number of constraints at the same time.* In our context, this allows to simultaneously take into account the fixed length of a homogeneous vector or matrix and additional constraints, such as the singularity of the fundamental matrix or the Plücker constraint for space lines. Also problems, such as space-curve fitting for observed space points $(x, y, z)_l, l = 1, ...L$, can easily be handled, e. g. a space conic, being the intersection of a vertical conical cylinder with the 6 parameters $a_{ij}, 0 \leq i + j \leq 2$ and a plane with the 4 parameters $b_{ijk}, 0 \leq i + j + k \leq 1$ via

$$\sum_{0 \leq i+j \leq 2} a_{ij} x_l^i y_l^j = 0 \qquad \sum_{0 \leq i+j+k \leq 1} b_{ijk} x_l^i y_l^j z_l^k = 0 \qquad \sum_{i+j \leq 2} a_{ij}^2 = 1 \qquad \sum_{i+j+k \leq 1} b_{ijk}^2 = 1$$

  including two constraints, guaranteeing the space conic to be parameterized by 8 parameters. The proposed estimation method covers the technique in [36] as special case.
- It can *handle groups of mutually correlated observations.* In many cases this might not be an issue. However, assume space points are derived from a pair of images by correspondence analysis, including relative orientation and triangulation: then the coordinates of the space points are mutually correlated due to the common relative orientation, e. g. represented by an estimated fundamental matrix $\widehat{\mathsf{F}}$. In case these points are used for further processing, e. g. surface or curve fitting, one may take the mutual correlations of the space coordinates of *all* points into account.

Gauß-Helmert Model

The used *mathematical model* for the estimation may be partitioned into a *functional model* and a *stochastical model.*

The *functional model*, the so-called Gauß-Helmert model, already proposed by Helmert in 1872 [20], with constraints between the unknown parameters, starts from G constraints $\boldsymbol{g} = (g_g)$ among N observations $\boldsymbol{l} = (l_n)$ and U unknown parameters $\boldsymbol{x} = (x_u)$ with additional H constraints $\boldsymbol{h} = (h_h)$ among the unknowns. The constraints should hold for the fitted observations $\widehat{\boldsymbol{l}} = \boldsymbol{l} + \widehat{\boldsymbol{v}}$, including the estimated corrections $\widehat{\boldsymbol{v}}$ and the estimated parameters $\widehat{\boldsymbol{x}}$:

$$\boldsymbol{g}(\boldsymbol{l} + \widehat{\boldsymbol{v}}, \widehat{\boldsymbol{x}}) = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{g}(\widehat{\boldsymbol{l}}, \widehat{\boldsymbol{x}}) = \boldsymbol{0} \tag{18}$$

$$\boldsymbol{h}(\widehat{\boldsymbol{x}}) = \boldsymbol{0} \tag{19}$$

Starting from approximate values $\widehat{\boldsymbol{x}}^{(0)}$ and $\widehat{\boldsymbol{l}}^{(0)}$ by Taylor expansion and neglection of terms of order higher than 2 one obtains the linear Gauß-Helmert model with constraints between the unknown parameters

$$A\widehat{\boldsymbol{\Delta x}} + B^{\mathsf{T}}\widehat{\boldsymbol{v}} = \boldsymbol{w}_g \qquad H^{\mathsf{T}}\widehat{\boldsymbol{\Delta x}} = \boldsymbol{w}_h \tag{20}$$

with the residuals of the constraints and the corrections of the unknown parameters

$$\boldsymbol{w}_g = -\boldsymbol{g}(\widehat{\boldsymbol{l}}^{(0)}, \widehat{\boldsymbol{x}}^{(0)}) - B^{\mathsf{T}}(\boldsymbol{l} - \widehat{\boldsymbol{l}}^{(0)}) \ , \quad \boldsymbol{w}_h = -\boldsymbol{h}(\widehat{\boldsymbol{x}}^{(0)}) \ , \quad \widehat{\boldsymbol{\Delta x}} = \boldsymbol{x} - \widehat{\boldsymbol{x}}^{(0)} \tag{21}$$

and the Jacobians evaluated at the approximate values

$$\underset{G\times U}{A} = \left.\frac{\partial \boldsymbol{g}(\boldsymbol{l},\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\substack{l\,=\,l^{(0)}\\ x\,=\,x^{(0)}}} \qquad \underset{N\times G}{B} = \left.\left(\frac{\partial \boldsymbol{g}(\boldsymbol{l},\boldsymbol{x})}{\partial \boldsymbol{l}}\right)^{\mathsf{T}}\right|_{\substack{l\,=\,l^{(0)}\\ x\,=\,x^{(0)}}} \qquad \underset{U\times H}{H} = \left.\left(\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^{\mathsf{T}}\right|_{x=x^{(0)}} .$$

The matrices B and H are defined as transposed of the Jacobians, to ensure they have more rows than columns.

The *stochastical model* is assumed to be simple: We assume an initial covariance matrix $\boldsymbol{\Sigma}_{ll}^{(0)}$ of the observations $\boldsymbol{l}$ to be known. It may be singular. We assume the true covariance matrix to be

$$\boldsymbol{\Sigma}_{ll} = \sigma_0^2 \boldsymbol{\Sigma}_{ll}^{(0)} \ .$$

We start the estimation with the initial value $\sigma_0^{2(0)} = 1$, thus start with the approximation $\boldsymbol{\Sigma}_{ll} = \boldsymbol{\Sigma}_{ll}^{(0)}$. Generalizing the stochastical model is straight forward, e. g. by assuming variance components (cf. [13, 27]

$$\boldsymbol{\Sigma}_{ll} = \sum_k \sigma_{0k}^2 \boldsymbol{\Sigma}_{ll,k}^{(0)} \ .$$

Then, the variance factors σ_{0k}^2 need to be estimated simultaneously with the unknown parameters $\boldsymbol{x}$.

ML-Estimation

We now give explicit expressions for the estimated parameters, the covariance matrices of the parameters, the corrections and the fitted observations and the estimated variance factor.

We derive the locally best linear estimators, i. e. estimators having the smallest variance in the linearized models. Moreover, in case the observations are samples from a normal distribution with $\underline{\boldsymbol{l}} \sim N(\tilde{\boldsymbol{l}}, \boldsymbol{\Sigma}_{ll})$ the estimates are local Maximum-Likelihood estimators, local, as they depend on the approximate values, and far-off the global optimum might exist.

Minimizing the quadratic form

$$\Omega = (\widehat{\boldsymbol{l}} - \boldsymbol{l})^{\mathsf{T}} \boldsymbol{\Sigma}_{ll}^{+} (\widehat{\boldsymbol{l}} - \boldsymbol{l}) \tag{22}$$

under the constraints $A\widehat{\boldsymbol{\Delta x}} + B^{\mathsf{T}}(\widehat{\boldsymbol{l}} - \boldsymbol{l}) = \boldsymbol{w}_g$ and $H^{\mathsf{T}}\widehat{\boldsymbol{\Delta x}} = \boldsymbol{w}_h$ we have to minimize the form

$$\Phi = (\widehat{\boldsymbol{l}} - \boldsymbol{l})^{\mathsf{T}} \boldsymbol{\Sigma}_{ll}^{+} (\widehat{\boldsymbol{l}} - \boldsymbol{l}) + 2\boldsymbol{\lambda}^{\mathsf{T}}(A\widehat{\boldsymbol{\Delta x}} + B^{\mathsf{T}}(\widehat{\boldsymbol{l}} - \boldsymbol{l}) - \boldsymbol{w}_g) + 2\boldsymbol{\mu}^{\mathsf{T}}(H^{\mathsf{T}}\widehat{\boldsymbol{\Delta x}} - \boldsymbol{w}_h) \ ,$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are Lagrangian multipliers. In case the covariance matrix $\boldsymbol{\Sigma}_{ll}$ of the observations is singular, one needs to take its pseudo inverse. As $\boldsymbol{\Sigma}_{ll}^{+}$ is positive semidefinite and the constraints are linear we obtain a unique minimum.

Setting the partials of Φ zero, we obtain with $\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{l}} - \boldsymbol{l}$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \widehat{\boldsymbol{l}}^{\mathsf{T}}} = \boldsymbol{\Sigma}_{ll}^{+}\widehat{\boldsymbol{v}} + B\boldsymbol{\lambda} = \mathbf{0} \tag{23}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \widehat{\boldsymbol{x}}^{\mathsf{T}}} = A^{\mathsf{T}}\boldsymbol{\lambda} + H\boldsymbol{\mu} = \mathbf{0} \tag{24}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\lambda}^{\mathsf{T}}} = -\boldsymbol{w}_g + A\widehat{\boldsymbol{\Delta x}} + B^{\mathsf{T}}\widehat{\boldsymbol{v}} = \mathbf{0} \tag{25}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \boldsymbol{\mu}^{\mathsf{T}}} = -\boldsymbol{w}_h + H^{\mathsf{T}}\widehat{\boldsymbol{\Delta x}} = \mathbf{0} \tag{26}$$

From (25) follows the relation

$$\widehat{\boldsymbol{v}} = -\boldsymbol{\Sigma}_{ll}B\boldsymbol{\lambda}\ . \tag{27}$$

When substituting (27) into (25), solving for $\boldsymbol{\lambda}$ yields

$$\boldsymbol{\lambda} = (B^{\mathsf{T}}\boldsymbol{\Sigma}_{ll}B)^{-1}(A\widehat{\boldsymbol{\Delta x}} - \boldsymbol{w}_g)\ . \tag{28}$$

Substitution in (26) yields the symmetric normal equation system

$$\begin{bmatrix} A^{\mathsf{T}}(B^{\mathsf{T}}\boldsymbol{\Sigma}_{ll}B)^{-1}A & H \\ H^{\mathsf{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{\Delta x}} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} A^{\mathsf{T}}(B^{\mathsf{T}}\boldsymbol{\Sigma}_{ll}B)^{-1}\boldsymbol{w}_g \\ \boldsymbol{w}_h \end{bmatrix}\ . \tag{29}$$

The Lagrangian multipliers can be obtained from (28) which then yields the estimated residuals in (27).

The estimated variance factor is given by

$$\widehat{\underline{\sigma}}_0^2 = \frac{\widehat{\underline{\boldsymbol{v}}}^{\mathsf{T}}\boldsymbol{\Sigma}_{ll}^{+}\widehat{\underline{\boldsymbol{v}}}}{R} \sim F_{R,\infty} \tag{30}$$

with the redundancy

$$R = G + H - U\ . \tag{31}$$

The redundancy R is the difference of the number $G + H$ of constraints and the number U of of unknown parameters. In case of normally distributed observations $\underline{\boldsymbol{l}}$ and in case the model holds the estimated variance factor is Fisher-distributed with (R, ∞) degrees of freedom. Thus $\widehat{\sigma_0}^2$ may be used to check the validity of the model.

We finally obtain the *estimated* covariance matrix

$$\widehat{\boldsymbol{\Sigma}}_{\widehat{x}\widehat{x}} = \widehat{\sigma}_0^2\boldsymbol{\Sigma}_{\widehat{x}\widehat{x}} \tag{32}$$

of the estimated parameters, where $\boldsymbol{\Sigma}_{\widehat{x}\widehat{x}}$ results from the inverted reduced normal equation matrix

$$\begin{bmatrix} \boldsymbol{\Sigma}_{\widehat{x}\widehat{x}} & S \\ S^{\mathsf{T}} & T \end{bmatrix} = \begin{bmatrix} \overline{N} & H \\ H^{\mathsf{T}} & \mathbf{0} \end{bmatrix}^{-1} \tag{33}$$

using

$$\overline{N} = A^\mathsf{T}(B^\mathsf{T}\boldsymbol{\Sigma}_{ll}B)^{-1}A\,.$$

Eq. (33) can be used even if $\overline{N}$ is singular. The covariance matrix $\boldsymbol{\Sigma}_{\widehat{x}\widehat{x}}$ has null space H.

The estimation needs to be iterated using improved approximate values in the next, say the $\nu+1$. iteration

$$\widehat{\boldsymbol{x}}^{(\nu+1)} = \boldsymbol{x}^{(\nu)} + \widehat{\boldsymbol{\Delta x}}^{(\nu)} \qquad \widehat{\boldsymbol{l}}^{(\nu+1)} = \widehat{\boldsymbol{l}}^{(\nu)} + \widehat{\boldsymbol{v}}^{(\nu)}$$

from (21) and (27). This requires to recompute the Jacobians A, B and H.

We now discuss various specializations of these relations leading to well known statistical tools.

The Special Case of Implicit Error Propagation

Observe, in case the number G of independent constraints $\boldsymbol{g}(\boldsymbol{l},\boldsymbol{x}) = \boldsymbol{0}$ and the number U of unknowns is identical, and there are no other constraints, the redundancy R is zero, the matrices A and B are of size $U \times U$, and the covariance matrix of the unknown parameters is

$$\boldsymbol{\Sigma}_{\widehat{x}\widehat{x}} = A^{-1}B^\mathsf{T}\boldsymbol{\Sigma}_{ll}BA^{-\mathsf{T}}$$

being the implicit error propagation law. Observe the definition of B as the transposed of the Jacobian of $\boldsymbol{g}$ w. r. t. the observations $\boldsymbol{l}$.

Relation to Matei/Meer's HEIV-Method

Up to this point all given derivations are well known. We now want to assume the constraints $\boldsymbol{g}$ to be linear in the unknown parameters and only the constraint h on the length of the unknown parameter vector should hold. Thus we have the model

$$\boldsymbol{g}(\widehat{\boldsymbol{l}},\widehat{\boldsymbol{x}}) = A(\widehat{\boldsymbol{l}})\widehat{\boldsymbol{x}} = 0 \qquad h = \frac{1}{2}(\widehat{\boldsymbol{x}}^\mathsf{T}\widehat{\boldsymbol{x}} - 1)\,.$$

This leads to $H = \widehat{\boldsymbol{x}}$. In the case of convergence we have $\widehat{\boldsymbol{\Delta x}} = \boldsymbol{0}$ and $\boldsymbol{w}_g = A(\boldsymbol{l})\widehat{\boldsymbol{x}}$ and therefore the first equation of (29) leads to the iteration sequence (cf. [14])

$$\mu{\cdot}\widehat{\boldsymbol{x}}^{(\nu)} = A^\mathsf{T}\left(\widehat{\boldsymbol{l}}^{(\nu-1)}\right)\left(B\left(\widehat{\boldsymbol{x}}^{(\nu-1)}\right)\boldsymbol{\Sigma}_{ll}B^\mathsf{T}\left(\widehat{\boldsymbol{x}}^{(\nu-1)}\right)\right)^{-1}A(\boldsymbol{l}){\cdot}\widehat{\boldsymbol{x}}^{(\nu)} \tag{34}$$

This shows the unknown parameter vector to be an eigenvector of an un-symmetric matrix. The Jacobian B is to be evaluated at the fitted values $\widehat{\boldsymbol{x}}^{(\nu-1)}$, causing the iteration process. This method is equivalent to Matei/Meer's HEIV-method [28]. In case of additional constraints, such as the singularity of the fundamental matrix, the authors propose to impose these constraints in a second step.

Imposing Constraints onto a Stochastic Vector

Imposing a set of constraints onto a stochastic vector, already tackled by Helmert in 1872 [20], is a special case of the Gauß-Helmert model mentioned above and is used for normalizing a vector or imposing additional constraints onto the vector, such as the Plücker constraint for space lines or the singularity constraint for the fundamental matrix.

The stochastic vector is treated as observational vector $\underline{\boldsymbol{l}} \sim M(\tilde{\boldsymbol{l}}, \boldsymbol{\Sigma}_{ll})$. Then we only have the G constraints

$$\boldsymbol{g}(\widehat{\boldsymbol{l}}) = \mathbf{0}$$

as no unknowns are involved. The resulting fitted observations $\widehat{\boldsymbol{l}}$ can be derived iteratively, from (27), (28) and (21)

$$\widehat{\boldsymbol{l}}^{(\nu+1)} = \boldsymbol{l} + \widehat{\boldsymbol{v}}^{(\nu)} = \boldsymbol{l} - \boldsymbol{\Sigma}_{ll} B (B^{\mathsf{T}} \boldsymbol{\Sigma}_{ll} B)^{-1} (\boldsymbol{g}(\widehat{\boldsymbol{l}}^{(\nu)}) + B^{\mathsf{T}} (\boldsymbol{l} - \widehat{\boldsymbol{l}}^{(\nu)}))$$

or in case of linear constraints from

$$\widehat{\boldsymbol{l}} = \boldsymbol{l} + \widehat{\boldsymbol{v}} = \boldsymbol{l} - \boldsymbol{\Sigma}_{ll} B (B^{\mathsf{T}} \boldsymbol{\Sigma}_{ll} B)^{-1} \boldsymbol{g}(\boldsymbol{l})$$

The covariance matrix of the fitted observations $\widehat{\underline{\boldsymbol{l}}}$ is

$$\boldsymbol{\Sigma}_{\widehat{l}\widehat{l}} = P_B^{\mathsf{T}} \boldsymbol{\Sigma}_{ll} P_B = \boldsymbol{\Sigma}_{ll} - \boldsymbol{\Sigma}_{ll} B (B^{\mathsf{T}} \boldsymbol{\Sigma}_{ll} B)^{-1} B^{\mathsf{T}} \boldsymbol{\Sigma}_{ll} = \boldsymbol{\Sigma}_{ll} P_B \tag{35}$$

with the $N - G$ rank projection matrix

$$P_B = I - B (B^{\mathsf{T}} \boldsymbol{\Sigma}_{ll} B)^{-1} B^{\mathsf{T}} \boldsymbol{\Sigma}_{ll}$$

fulfilling $P_B B = \mathbf{0}$, leading to a singular covariance matrix $\boldsymbol{\Sigma}_{\widehat{l}\widehat{l}}$ with null-space B. This is an example, demonstrating the generality of the Gauss-Helmert model.

Minimizing the Algebraic Error

Minimizing the algebraic error $\boldsymbol{w}_g = A(\boldsymbol{l})\widehat{\boldsymbol{x}}$ under the constraint $|\widehat{\boldsymbol{x}}| = 1$ leads to the simple eigenvalue problem

$$\mu \cdot \widehat{\boldsymbol{x}} = A^{\mathsf{T}}(\boldsymbol{l}) A(\boldsymbol{l}) \cdot \widehat{\boldsymbol{x}}$$

demonstrating the neglection of the weighting matrix $(B \boldsymbol{\Sigma}_{ll} B^{\mathsf{T}})^{-1}$, when compared to the rigorous solution.

2.3 Testing

We only need very little from testing theory, namely testing a vector to be zero.

Let the observed n-vector be $\boldsymbol{c}$. We want to test the null-hypothesis $H_0 : \boldsymbol{\mu}_c = \mathbf{0}$, that the mean $\boldsymbol{\mu}_c$ of the vector $\underline{\boldsymbol{c}}$ is zero against the alternative hypothesis $H_a : \boldsymbol{\mu}_c \neq \mathbf{0}$ that the mean is not zero. For the test we need the distribution of $\underline{\boldsymbol{c}}|H_0$ of the vector, provided the hypothesis H_0 holds. In case we can assume $\underline{\boldsymbol{c}}|H_0 \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{cc})$ with a full rank covariance matrix, we obtain the test optimal statistic (cf. [27])

$$T = \boldsymbol{c}^{\mathsf{T}} \boldsymbol{\Sigma}_{cc}^{-1} \boldsymbol{c} \sim \chi_d^2$$

If $\mathrm{rank}(\boldsymbol{\Sigma}_{cc}) = r \leq d$ then we obtain the test statistic

$$T = \boldsymbol{c}^{\mathsf{T}} \boldsymbol{\Sigma}_{cc}^{+} \boldsymbol{c} \sim \chi_r^2$$

The test compares the test statistic T with a critical value. Specifying a (smaller) significance number α or a (large) significance level $S = 1 - \alpha$ one uses the $1 - \alpha$-percentile of the distribution of the test statistic as critical value. If

$$T > \chi^2_{r,1-\alpha}$$

then we may reject H_0. Thus there is reason to assume that the difference of $\boldsymbol{c}$ from $\mathbf{0}$ cannot be explained by random errors, leading to deviations of $\boldsymbol{c}$ from $\mathbf{0}$. Otherwise, H_0 cannot be rejected, i. e. there is no reason to assume H_0 to be incorrect. This does not say, H_0 is accepted, as other hypotheses H_{0i} might be valid, which are not tested for.

3 Geometric Relations

3.1 Representations

We represent all geometric entities and transformations with homogeneous vectors or matrices.

Points $\mathbf{x}$ and lines $\mathbf{l}$ in 2D are 3-vectors

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ \hline w \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ x_h \end{bmatrix} \quad \text{and} \quad \mathbf{l} = \begin{bmatrix} a \\ b \\ \hline c \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ \hline l_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{l}_h \\ l_0 \end{bmatrix}$$

resp.

Sometimes it is useful to distinguish the Euclidean part, indexed 0, and the homogeneous part, indexed h of a vector or a matrix (cf. [3]): In case the homogeneous part is normalized to 1, the Euclidean part can be interpreted metrically. In case the homogeneous part of a geometric entity is zero, the entity is at infinity.

Analogous, in 3D points $\mathbf{X}$ and planes $\mathbf{A}$ are represented with 4-vectors

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ \hline T \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \hline X_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_0 \\ X_h \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} A \\ B \\ C \\ \hline D \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \hline A_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_h \\ A_0 \end{bmatrix} .$$

3D-lines $\mathbf{L}$ are represented with so-called Plücker coordinates

$$\mathbf{L} = [L_i] = \begin{bmatrix} \boldsymbol{L}_h \\ \boldsymbol{L}_0 \end{bmatrix} ,$$

where the first 3-vector $\boldsymbol{L}_h$ is the direction of the line and the second 3-vector $\boldsymbol{L}_0$ is the normal on the plane through the line and the origin, such that the three vectors $\boldsymbol{L}_h$, $\boldsymbol{L}_0$ and the normal from the origin onto the 3D-line span a right handed orthogonal coordinate system. Any 6-vector fulfilling the so-called Plücker constraint

$$\boldsymbol{L}_h^{\mathsf{T}} \boldsymbol{L}_0 = 0$$

represents a 3D-line.

Each geometric element $\mathbf{g}$ has a dual, denoted by $\overline{\mathbf{g}}$. Vise versa, $\mathbf{g}$ is the dual of $\overline{\mathbf{g}}$, thus $\mathbf{g} = \overline{\overline{\mathbf{g}}}$.

In 2D the dual $\overline{\mathbf{x}}$ of a point $\mathbf{x}$ is the line $[u, v, w]^\mathsf{T}$ with the same coordinates, and vice versa, the dual $\overline{\mathbf{l}}$ of a line $\mathbf{l}$ is the point $[a, b, c]^\mathsf{T}$ with the same coordinates:

$$\overline{\mathbf{x}} = I_3\,\mathbf{x} \qquad \overline{\mathbf{l}} = I_3\,\mathbf{l}\,.$$

In 3D the dual $\overline{\mathbf{X}}$ of a point $\mathbf{X}$ is the plane $[U, V, W, T]^\mathsf{T}$ with the same coordinates, and vice versa, the dual $\overline{\mathbf{A}}$ of a plane $\mathbf{A}$ is the point $[A, B, C, D]^\mathsf{T}$ with the same coordinates:

$$\overline{\mathbf{X}} = I_4\,\mathbf{X} \qquad \overline{\mathbf{A}} = I_4\,\mathbf{A}\,.$$

The dual $\overline{\mathbf{L}}$ of a 3D-line $\mathbf{L} = (\boldsymbol{L}_h^\mathsf{T}, \boldsymbol{L}_0)^\mathsf{T}$ is the 3D-line $(\boldsymbol{L}_0^\mathsf{T}, \boldsymbol{L}_h^\mathsf{T})^\mathsf{T}$ with the homogeneous and the Euclidean part exchanged:

$$\overline{\mathbf{L}} = D_6\,\mathbf{L} = \begin{bmatrix} \boldsymbol{L}_0 \\ \boldsymbol{L}_h \end{bmatrix}$$

with the dualizing matrix

$$D_6 = \begin{bmatrix} \mathbf{0} & I_3 \\ I_3 & \mathbf{0} \end{bmatrix}\,.$$

Remark: This representation of 3D-entities is consistent with the geometric algebra G_4 (cf. [22]) when using the bases $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$ for 3D-points, $(\boldsymbol{e}_{41}, \boldsymbol{e}_{42}, \boldsymbol{e}_{43}, \boldsymbol{e}_{23}, \boldsymbol{e}_{31}, \boldsymbol{e}_{12})$ for 3D-lines and $(\boldsymbol{e}_{234}, \boldsymbol{e}_{314}, \boldsymbol{e}_{124}, \boldsymbol{e}_{321})$ for planes, which can be easily verified with the GA-package of Ashdown [1].

Representation and Visualization of the Uncertainty of Geometric Entities

All homogenous vectors and matrices involved will get a covariance matrix attached to it. Thus we obtain the pairs

$$(\underline{\mathbf{x}}, \boldsymbol{\Sigma}_{xx})\,, \quad (\underline{\mathbf{l}}, \boldsymbol{\Sigma}_{ll})\,, \quad (\underline{\mathbf{X}}, \boldsymbol{\Sigma}_{XX})\,, \quad (\underline{\mathbf{L}}, \boldsymbol{\Sigma}_{LL})\,, \quad (\underline{\mathbf{A}}, \boldsymbol{\Sigma}_{XX}) \tag{36}$$

for points and lines in 2D and for points, lines and planes in 3D. We will later also transfer this representation to transformation matrices.

The uncertainty of the geometric entities can be visualized by the confidence regions. In fig. 3 we already draw confidence regions for 2D-points, being ellipses, and 2D-lines, being hyperbola, namely the set of all one-dimensional confidence regions of points sitting on the line. They directly transfer to 3D-points, being ellipsoids, and planes, being hypeboloids of two sheets, cf. fig. 8. The situation is more complicated for 3D-lines. The set of confidence ellipses of the 3D-points sitting on the 3D-line, measured across the line, yields a shape as in fig. 9 left. It has different minima in different planes through the line, thus in general is no hyperboloid of one sheet and is closely related to the ray configuration of an astigmatism.

3.2 Constructions

Constructions in 2D

Geometric entities easily can be constructed from given ones.

(1) A 2D line $\mathbf{l}$ joining two points $\mathbf{x}$ and $\mathbf{y}$ is given by

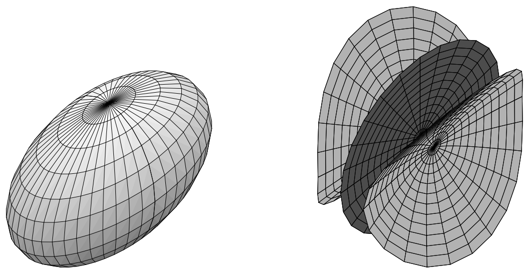


Fig. 8. Confidence regions for a 3D-point and a plane. The hyperboloid of two sheets is the set of the 1D-confidence regions of all points in the plane measured across the plane.

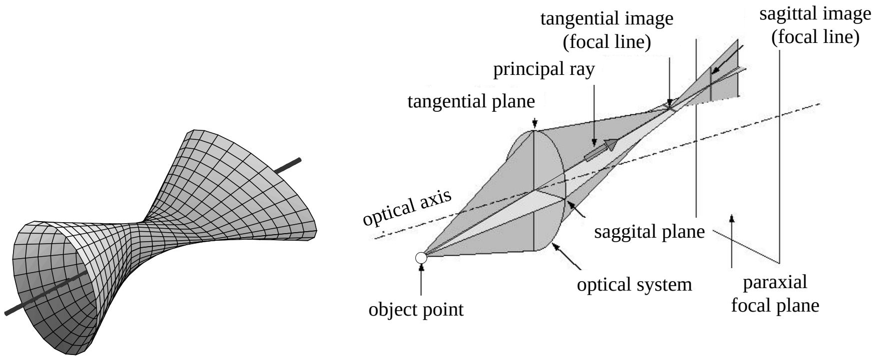


Fig. 9. Confidence region for a 3D-line (left): It is the set of the 2D-confidence regions of all points on the 3D-line measured across the line. Compare the structure of the iso-surface with the structure of the bundle of rays for astigmatism (right, after http://www.mellesgriot.com/): In case the 3D-line only is uncertain in direction, in general there are two points, where the elliptic confidence region degenerates to a straight line segment, corresponding to the two focal lines of an astigmatism. However, the straight line segments need not be perpendicular, whereas the two focal lines are.

$$\boxed{\mathbf{l} = \mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x} : \quad \mathbf{l} = \mathbf{x} \times \mathbf{y} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}} \tag{37}$$

Herein, we used the Jacobians

$$\mathsf{S}(\mathbf{x}) = \frac{\partial(\mathbf{x} \times \mathbf{y})}{\partial \mathbf{y}} \qquad \text{and} \qquad \mathsf{S}(\mathbf{y}) = \frac{\partial(\mathbf{y} \times \mathbf{x})}{\partial \mathbf{x}}$$

and the skew symmetric matrices of a 3-vector, denoted by

$$\mathsf{S}(\mathbf{x}) = \mathsf{S}_x = [\mathbf{x}]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} . \tag{38}$$

This representation of constructions as matrix-vector products is given explicitely for the most important constructions in 2D and 3D and very useful for statistical error propagation.

(2) The 2D-point $\mathbf{x}$ being the intersection of two lines $\mathbf{l}$ and $\mathbf{m}$ is given by

$$\boxed{\mathbf{x} = \mathbf{l} \cap \mathbf{m} = -\mathbf{m} \cap \mathbf{l} : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}} \tag{39}$$

Here the Jacobians are

$$\mathsf{S}(\mathbf{l}) = \frac{\partial(\mathbf{l} \times \mathbf{m})}{\partial \mathbf{m}} \quad \text{and} \quad \mathsf{S}(\mathbf{m}) = \frac{\partial(\mathbf{m} \times \mathbf{l})}{\partial \mathbf{l}}$$

The constructions are collected in table 1

As the expressions for constructions are bilinear we easily can find the covariance matrix of the generated elements. Therefore, we always give the two expressions

$$\mathbf{c} = \mathsf{U}(\mathbf{a})\mathbf{b} = \mathsf{V}(\mathbf{b})\mathbf{a} \tag{40}$$

for the bilinear form, where the matrices

$$\mathsf{U}(\mathbf{a}) = \frac{\partial \mathbf{c}}{\partial \mathbf{b}} \quad \text{and} \quad \mathsf{V}(\mathbf{b}) = \frac{\partial \mathbf{c}}{\partial \mathbf{a}}$$

are the Jacobians of $\mathbf{c}$ with respect to $\mathbf{b}$ and $\mathbf{a}$ resp. They are used to determine a first order approximation of the covariance matrix $\boldsymbol{\Sigma}_{cc}$ of $\underline{\mathbf{c}}$, in case $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are given together with their covariance matrices $\boldsymbol{\Sigma}_{aa}$ and $\boldsymbol{\Sigma}_{bb}$:

$$\boxed{\boldsymbol{\Sigma}_{cc} = [\mathsf{V}(\boldsymbol{b})\ \mathsf{U}(\boldsymbol{a})] \begin{bmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{bmatrix} \begin{bmatrix} \mathsf{V}^{\mathsf{T}}(\boldsymbol{b}) \\ \mathsf{U}^{\mathsf{T}}(\boldsymbol{a}) \end{bmatrix}} \tag{41}$$

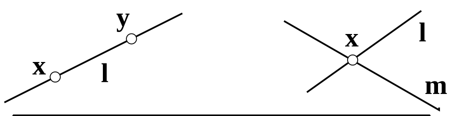
or

$$\boldsymbol{\Sigma}_{cc} = \mathsf{U}(\mathbf{a})\boldsymbol{\Sigma}_{bb}\mathsf{U}(\mathbf{a})^{\mathsf{T}} + \mathsf{V}(\mathbf{b})\boldsymbol{\Sigma}_{aa}\mathsf{V}(\mathbf{b})^{\mathsf{T}} + \mathsf{U}(\mathbf{a})\boldsymbol{\Sigma}_{ba}\mathsf{V}(\mathbf{b})^{\mathsf{T}} + \mathsf{V}(\mathbf{b})\boldsymbol{\Sigma}_{ab}\mathsf{U}(\mathbf{a})^{\mathsf{T}}$$

allowing the given entities to be statistically dependent.

For example, from column 3 in table 1 one easily can read out the Jacobians being the matrices in the bilinear forms, e. g. for the intersection point $\mathbf{x}$ we have $\partial\mathbf{x}/\partial\mathbf{m} = \mathsf{S}(\mathbf{l})$ and $\partial\mathbf{x}/\partial\mathbf{l} = -\mathsf{S}(\mathbf{m})$, cf. after (39).

Table 1. Construction of new 2D geometric entities. The structure of the matrix S is given in eq. (38). All forms are linear in the coordinates of the given entities allowing simple error propagation.



new entity	algebraic construction	eq.
$\mathbf{l} = \mathbf{x} \wedge \mathbf{y}$	$\mathbf{l} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$	(37)
$\mathbf{x} = \mathbf{l} \cap \mathbf{m}$	$\mathbf{x} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$	(39)

Constructions in 3D

Here we mention six different cases, which we give without proof.

(1) A 3D-line $\mathbf{L}$ joining two 3D-points $\mathbf{X}$ and $\mathbf{Y}$ is given by

$$\boxed{\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = -\mathbf{Y} \wedge \mathbf{X} : \mathbf{L} = \begin{bmatrix} X_h \boldsymbol{Y}_0 - Y_h \boldsymbol{X}_0 \\ \boldsymbol{X}_0 \times \boldsymbol{Y}_0 \end{bmatrix} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}} \quad (42)$$

Here the Jacobians are

$$\Pi(\mathbf{X}) = \frac{\partial(\mathbf{X} \wedge \mathbf{Y})}{\partial \mathbf{Y}} \qquad \text{and} \qquad \Pi(\mathbf{Y}) = \frac{\partial(\mathbf{Y} \wedge \mathbf{X})}{\partial \mathbf{X}}$$

or explicitely e. g.

$$\underset{6\times 4}{\Pi(\mathbf{X})} = \begin{bmatrix} X_h I_3 & -\boldsymbol{X}_0 \\ S(\boldsymbol{X}_0) & \mathbf{0} \end{bmatrix} = \left[\begin{array}{ccc|c} X_4 & 0 & 0 & -X_1 \\ 0 & X_4 & 0 & -X_2 \\ 0 & 0 & X_4 & -X_3 \\ \hline 0 & -X_3 & X_2 & 0 \\ X_3 & 0 & -X_1 & 0 \\ -X_2 & X_1 & 0 & 0 \end{array}\right] . \quad (43)$$

(2) A 3-line $\mathbf{L}$ as the intersection of two planes $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\boxed{\mathbf{L} = \mathbf{A} \cap \mathbf{B} = -\mathbf{B} \cap \mathbf{A} : \mathbf{L} = \begin{bmatrix} \boldsymbol{A}_h \times \boldsymbol{B}_h \\ A_0 \boldsymbol{B}_h - B_0 \boldsymbol{A}_h \end{bmatrix} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A}} \quad (44)$$

Here the Jacobians are

$$\overline{\Pi}(\mathbf{A}) = \frac{\partial(\mathbf{A} \cap \mathbf{B})}{\partial \mathbf{B}} \qquad \text{and} \qquad \overline{\Pi}(\mathbf{B}) = \frac{\partial(\mathbf{B} \cap \mathbf{A})}{\partial \mathbf{A}}$$

or explicitely

$$\overline{\Pi}(\mathbf{A}) = D_6 \Pi(\mathbf{A}) = \begin{bmatrix} S(\boldsymbol{A}_h) & \mathbf{0} \\ A_0 I_3 & -\boldsymbol{A}_h \end{bmatrix} . \quad (45)$$

Observe, we might have obtained $\mathbf{L} = \overline{\Pi}(\mathbf{A})\mathbf{B}$ from $\mathbf{L} = \Pi(\mathbf{X})\mathbf{Y}$ using dualing, using $\mathbf{A} = \overline{\mathbf{X}}$ and $\mathbf{B} = \overline{\mathbf{Y}}$, namely $\overline{\mathbf{L}} = \Pi(\overline{\mathbf{X}})\overline{\mathbf{Y}} = D_6\mathbf{L} = \Pi(\mathbf{A})\mathbf{B}$, and noting $D_6 = D_6^{-1}$.

Remark: The letter $'P'$ in the name $'Pi'$ of the Greek capital letter Π indicates this matrix referring to *p*oints and *p*lanes.

(3) 3D-point $\mathbf{X}$ as intersection of the 3D-line $\mathbf{L}$ and the plane $\mathbf{A}$ from

$$\boxed{\mathbf{X} = \mathbf{L} \cap \mathbf{A} = \mathbf{A} \cap \mathbf{L} : \mathbf{X} = \begin{bmatrix} \boldsymbol{L}_0 \times \boldsymbol{A}_h + A_0 \boldsymbol{L}_h \\ -\boldsymbol{L}_h^{\mathsf{T}} \boldsymbol{A}_h \end{bmatrix} = \Gamma^{\mathsf{T}}(\mathbf{L})\mathbf{X} = \Pi^{\mathsf{T}}(\mathbf{X})\mathbf{L}} \quad (46)$$

The Jacobians are

$$\Gamma^{\mathsf{T}}(\mathbf{L}) = \frac{\partial(\mathbf{A} \cap \mathbf{L})}{\partial \mathbf{L}} \qquad \text{and} \qquad \Pi^{\mathsf{T}}(\mathbf{A}) = \frac{\partial(\mathbf{L} \wedge \mathbf{A})}{\partial \mathbf{L}} = \begin{bmatrix} A_0 I_3 & -S(\boldsymbol{A}_h) \\ -\boldsymbol{A}_h^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} \end{bmatrix} .$$

Explicitely we have the Plücker matrix of the line $\mathbf{L}$

$$\mathsf{\Gamma}(\mathbf{L}) = \begin{bmatrix} -S(\boldsymbol{L}_0) & -\boldsymbol{L}_h \\ \boldsymbol{L}_h^{\mathsf{T}} & 0 \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & L_6 & -L_5 & -L_1 \\ -L_6 & 0 & L_4 & -L_2 \\ L_5 & -L_4 & 0 & -L_3 \\ \hline L_1 & L_2 & L_3 & 0 \end{array}\right] . \tag{47}$$

One can show that the Plücker-matrix of the line $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ is the skew-symmetric form having rank 2

$$\mathsf{\Gamma}(\mathbf{L}) = \mathsf{\Gamma}(\mathbf{X} \wedge \mathbf{Y}) = \mathbf{X}\mathbf{Y}^{\mathsf{T}} - \mathbf{Y}\mathbf{X}^{\mathsf{T}} . \tag{48}$$

Remark: The Greek letter $\mathsf{\Gamma}$ is the mirror of $\mathbf{L}$ and indicates it refers to 3D-lines.

(4) Dually, we obtain the plane $\mathbf{A}$ joining a 3D-line $\mathbf{L}$ and a 3D-point $\mathbf{X}$ is given by

$$\boxed{\mathbf{A} = \mathbf{L} \wedge \mathbf{X} = \mathbf{X} \wedge \mathbf{L} : \mathbf{A} = \begin{bmatrix} \boldsymbol{L}_h \times \boldsymbol{X}_0 + X_h \boldsymbol{L}_0 \\ -\boldsymbol{L}_0^{\mathsf{T}} \boldsymbol{X}_0 \end{bmatrix} = \overline{\mathsf{\Gamma}}^{\mathsf{T}}(\mathbf{L})\mathbf{X} = \overline{\mathsf{\Pi}}^{\mathsf{T}}(\mathbf{X})\mathbf{L}} \tag{49}$$

with the Jacobians

$$\overline{\mathsf{\Gamma}}^{\mathsf{T}}(\mathbf{L}) = \frac{\partial(\mathbf{A} \cap \mathbf{L})}{\partial \mathbf{A}} \qquad \text{and} \qquad \overline{\mathsf{\Pi}}^{\mathsf{T}}(\mathbf{A}) = \frac{\partial(\mathbf{L} \cap \mathbf{A})}{\partial \mathbf{L}} = \begin{bmatrix} -S(\boldsymbol{X}_0) & X_h I \\ \mathbf{0}^{\mathsf{T}} & -\boldsymbol{X}_0^{\mathsf{T}} \end{bmatrix}$$

and the dual Plücker-matrix of the 3D-line $\mathbf{L}$

$$\overline{\mathsf{\Gamma}}(\mathbf{L}) = \mathsf{\Gamma}(\overline{\mathbf{L}}) = \begin{bmatrix} -S(\boldsymbol{L}_h) & -\boldsymbol{L}_0 \\ \boldsymbol{L}_0^{\mathsf{T}} & 0 \end{bmatrix} . \tag{50}$$

One can show, the dual Plücker-matrix of the line $\mathbf{L} = \mathbf{A} \cap \mathbf{B}$ is the skew-symmetric form of rank 2

$$\overline{\mathsf{\Gamma}}(\mathbf{L}) = \overline{\mathsf{\Gamma}}(\mathbf{A} \cap \mathbf{B}) = \mathbf{A}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{A}^{\mathsf{T}} . \tag{51}$$

(5) and (6) Finally we obtain the plane $\mathbf{A}$ joining three points $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ from

$$\boxed{\mathbf{A} = \mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z} = (\mathbf{X} \wedge \mathbf{Y}) \wedge \mathbf{Z}} \tag{52}$$

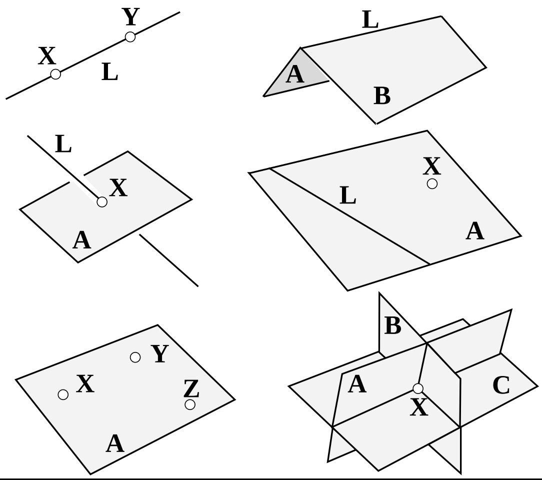
and the point $\mathbf{X}$ as the intersection of three planes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ from

$$\boxed{\mathbf{X} = \mathbf{A} \cap \mathbf{B} \cap \mathbf{C} = (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}} \tag{53}$$

with similar expressions by cyclic exchange of the geometric entities.

The following tables collects these 6 cases. Observe the similarity in the representation with the operators of the Grassman-Cayley algebra and the ease of reading out the Jacobians for statistical error propagation.

Table 2. Construction of new 3D geometric entities. The matrices Π, $\overline{\Pi}$, Γ and $\overline{\Gamma}$ are given in eqs. (43), (45), (47) and (50) resp. All forms are linear in the coordinates of the given entities making error propagation easy.



new entity	algebraic construction	eq.
$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = -\mathbf{Y} \wedge \mathbf{X}$	$\mathbf{L} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$	(42)
$\mathbf{L} = \mathbf{A} \cap \mathbf{B} = -\mathbf{B} \cap \mathbf{A}$	$\mathbf{L} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A}$	(44)
$\mathbf{X} = \mathbf{L} \cap \mathbf{A} = \mathbf{A} \cap \mathbf{L}$	$\mathbf{X} = \Gamma^\mathsf{T}(\mathbf{L})\mathbf{A} = \Pi^\mathsf{T}(\mathbf{A})\mathbf{L}$	(46)
$\mathbf{A} = \mathbf{L} \wedge \mathbf{X} = \mathbf{X} \wedge \mathbf{L}$	$\mathbf{A} = \overline{\Gamma}^\mathsf{T}(\mathbf{L})\mathbf{X} = \overline{\Pi}^\mathsf{T}(\mathbf{X})\mathbf{L}$	(49)
$\mathbf{A} = \mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z}$	$\overline{\Gamma}^\mathsf{T}(\Pi(\mathbf{X})\mathbf{Y})\mathbf{Z} = \overline{\Gamma}^\mathsf{T}(\Pi(\mathbf{Y})\mathbf{Z})\mathbf{X} = \overline{\Gamma}^\mathsf{T}(\Pi(\mathbf{Z})\mathbf{X})\mathbf{Y}$	(52)
$\mathbf{X} = \mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$	$\overline{\Gamma}^\mathsf{T}(\Pi(\mathbf{A})\mathbf{B})\mathbf{C} = \overline{\Gamma}^\mathsf{T}(\Pi(\mathbf{B})\mathbf{C})\mathbf{A} = \overline{\Gamma}^\mathsf{T}(\Pi(\mathbf{C})\mathbf{A})\mathbf{B}$	(53)

Constructions Containing Mappings

Geometric entities also can be generated by transformations. Table 3 collects the most important ones useful in geometric reasoning in Computer Vision.

- 2D-homography H for 2D-points (1) and 2D-lines (2)
- 3D-homography H for 3D-points (3), planes (4) and 3D-lines (5)
- projection P of 3D-points (6) and Q of 3D-lines (7) into an image yielding 2D-points and 2D-lines
- back-projection of image points (8) and lines (9) yielding projection rays (3D-lines) and projection planes
- 2D-2D-correlation F (10) for mapping a point of one image to a line in the other, knowing the epipolar geometry, or the relative orientation via the fundamental matrix F, used in the epipolar constraint $\mathbf{x}'^\mathsf{T}\mathsf{F}\mathbf{x}'' = 0$.

All mappings are bilinear in the given geometric entity and the transformation matrix. Therefore again we give two relations, making the individual Jacobians explicit, which are necessary for statistical error propagation.

The table needs some explanations:

Table 3. Mappings. 2D- and 3D-homographies H for points, homography H_L for space lines, projection matrices P and Q for points and lines. Projection ray $\mathbf{L}_{x'}$ and projection plane $\mathbf{A}_{l'}$. Fundamental matrix $\mathsf{F} \doteq \mathsf{F}_{12}$ for coplanarity constraint $\mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' = 0$

#	mapping	relation 1	relation 2
1	2D homography	$\mathbf{x}' = \mathsf{H}\mathbf{x}$	$\mathbf{x}' = (I_3 \otimes \mathbf{x}^{\mathsf{T}})\text{vec}(\mathsf{H}^{\mathsf{T}})$
2		$\mathbf{l}' = \mathsf{H}^{-\mathsf{T}}\mathbf{l}$	$\mathbf{l}' = (I_3 \otimes \mathbf{l}^{\mathsf{T}})\text{vec}(\mathsf{H}^{-1})$
3	3D homography	$\mathbf{X}' = \mathsf{H}\mathbf{X}$	$\mathbf{X}' = (I_4 \otimes \mathbf{X}^{\mathsf{T}})\text{vec}(\mathsf{H}^{\mathsf{T}})$
4		$\mathbf{A}' = \mathsf{H}^{-\mathsf{T}}\mathbf{A}$	$\mathbf{A}' = (I_3 \otimes \mathbf{A}^{\mathsf{T}})\text{vec}(\mathsf{H}^{-1})$
5		$\mathbf{L}' = \mathsf{H}_L\mathbf{L}$	$\mathbf{L}' = (I_6 \otimes \mathbf{L}^{\mathsf{T}})\text{vec}(\mathsf{H}_L^{\mathsf{T}})$
6	2D-3D projection	$\mathbf{x}' = \mathsf{P}\mathbf{X}$	$\mathbf{x}' = (I_3 \otimes \mathbf{X}^{\mathsf{T}})\text{vec}(\mathsf{P}^{\mathsf{T}})$
7		$\mathbf{l}' = \mathsf{Q}\mathbf{L}$	$\mathbf{l}' = (I_3 \otimes \mathbf{L}^{\mathsf{T}})\text{vec}(\mathsf{Q}^{\mathsf{T}})$
8	3D-2D back-projection	$\mathbf{A}_{l'} = \mathsf{P}^{\mathsf{T}}\mathbf{l}'$	$\mathbf{A}_{l'} = (\mathbf{l}'^{\mathsf{T}} \otimes I_4)\text{vec}(\mathsf{P}^{\mathsf{T}})$
9		$\mathbf{L}_{x'} = \overline{\mathsf{Q}}^{\mathsf{T}}\mathbf{x}'$	$\mathbf{L}_{x'} = (\mathbf{x}'^{\mathsf{T}} \otimes D_6)\text{vec}(\mathsf{Q}^{\mathsf{T}})$
10	2D-2D correlation	$\mathbf{l}'' = \mathsf{F}^{\mathsf{T}}\mathbf{x}'$	$\mathbf{l}'' = (\mathbf{x}'^{\mathsf{T}} \otimes I_3)\text{vec}(\mathsf{F}^{\mathsf{T}})$

1. We use the vec-operator to represent uncertain homogeneous matrices as uncertain homogeneous vectors, e. g.

$$\mathbf{h} = \text{vec}(\mathsf{H}^{\mathsf{T}})$$

Thus we might want to work with the pairs

$$(\underline{\mathbf{h}}, \boldsymbol{\Sigma}_{hh}) \ , \quad (\underline{\mathbf{h}}_L, \boldsymbol{\Sigma}_{h_L h_L}) \ , \quad (\underline{\mathbf{p}}, \boldsymbol{\Sigma}_{pp}) \ , \quad (\underline{\mathbf{q}}, \boldsymbol{\Sigma}_{qq}) \ , \quad (\underline{\mathbf{f}}, \boldsymbol{\Sigma}_{ff}) \ .$$

of uncertain transformations collected in table 3.

2. We use the Kronecker product, the vec-operator and the rule $\text{vec}(A\boldsymbol{b}) = (\boldsymbol{b}^{\mathsf{T}} \otimes I)\text{vec}A = \text{vec}(\boldsymbol{b}^{\mathsf{T}}A^{\mathsf{T}}) = (I \otimes \boldsymbol{b}^{\mathsf{T}})\text{vec}(A^{\mathsf{T}})$ we can express the result $\mathbf{x}'$ of the 2D-2D homography as a function of the vector $\mathbf{h} = \text{vec}(\mathsf{H}^{\mathsf{T}})$.We obtain

$$\mathbf{x}' = \mathsf{H}\mathbf{x} = (I_3 \otimes \mathbf{x}^{\mathsf{T}})\mathbf{h}$$

Is is useful for deriving the covariance matrix of the transformed point $\mathbf{x}'$ in case the covariance matrices $\boldsymbol{\Sigma}_{xx}$ and $\boldsymbol{\Sigma}_{hh}$ of the point $\mathbf{x}$ and of the elements $\mathbf{h}$ of H are known

$$\boldsymbol{\Sigma}_{x'x'} = \mathsf{H}\boldsymbol{\Sigma}_{xx}\mathsf{H}^{\mathsf{T}} + (I_3 \otimes \mathbf{x}^{\mathsf{T}})\boldsymbol{\Sigma}_{hh}(I_3 \otimes \mathbf{x})$$

in this special case assuming statistical independence of $\underline{\mathbf{x}}$ and $\underline{\mathsf{H}}$.

3. If we want to derive the covariance matrix of transformed 2D-lines, we need covariance matrix of the transposed inverse $\mathsf{M} = \mathsf{H}^{-\mathsf{T}}$. This can easily be derived: From $\mathsf{H}\mathsf{H}^{-1} = I$ we have $d\mathsf{H}\ \mathsf{H}^{-1} + \mathsf{H}\ d\mathsf{H}^{-1} = \mathbf{0}$ thus $\mathsf{M}^{\mathsf{T}}\ d\mathsf{H}\ \mathsf{M}^{\mathsf{T}} + d\mathsf{M}^{\mathsf{T}} = \mathbf{0}$ therefore with $\mathbf{m} = \text{vec}(\mathsf{M}^{\mathsf{T}}) = \text{vec}(\mathsf{H}^{-1})$ we obtain $(\mathsf{M} \otimes \mathsf{M}^{\mathsf{T}})d\mathbf{h} + d\mathbf{m} = \mathbf{0}$. The covariance matrix of $\mathbf{m}$ therefore is

$$\boldsymbol{\Sigma}_{mm} = (\mathsf{M} \otimes \mathsf{M}^{\mathsf{T}})\boldsymbol{\Sigma}_{hh}(\mathsf{M} \otimes \mathsf{M}^{\mathsf{T}})$$

Finally we obtain the covariance matrix of the transformed lines

$$\boldsymbol{\Sigma}_{l'l'} = \mathsf{H}^{-\mathsf{T}} \boldsymbol{\Sigma}_{ll} \mathsf{H}^{-1} + (I_3 \otimes \mathbf{l}^{\mathsf{T}}) \boldsymbol{\Sigma}_{mm} (I \otimes \mathbf{l})$$

or only in terms of the given values

$$\boldsymbol{\Sigma}_{l'l'} = \mathsf{H}^{-\mathsf{T}} \boldsymbol{\Sigma}_{ll} \mathsf{H}^{-1} + (\mathsf{H}^{-\mathsf{T}} \otimes \mathbf{l}'^{\mathsf{T}}) \boldsymbol{\Sigma}_{hh} (\mathsf{H}^{-1} \otimes \mathbf{l}')$$

4. The 3D-line transformation is not made explicit in the table.
Starting from the transformation $\mathbf{X}' = \mathsf{H}\mathbf{X}$ of 3D-points $\mathbf{X}$, one obtains an expression for the transformation matrix H_L for 3D-lines in terms of their Plücker coordinates

$$\boxed{\mathbf{L}' = \mathsf{H}_L\, \mathbf{L} = (I_6 \otimes \mathbf{L}^{\mathsf{T}}) \mathrm{vec}(\mathsf{H}_L^{\mathsf{T}})} \tag{54}$$

with the transformation matrix (cf. appendix)

$$\mathsf{H}_L = \frac{1}{2} J_{\Gamma L}^{\mathsf{T}} (\mathsf{H} \otimes \mathsf{H}) J_{\Gamma L}$$

using the 16×6-Jacobian

$$\underset{16\times 6}{J_{\Gamma L}} = \frac{\partial \mathrm{vec}(\Gamma(\mathbf{L}))}{\partial \mathbf{L}}$$

which via $\mathbf{L} = \frac{1}{2} J_{\Gamma L}^{\mathsf{T}}\ \mathrm{vec}(\Gamma(\mathbf{L}))$ maps the columns of the Plücker-matrix to the Plücker coordinates.
With the Jacobian of the transformed line with respect to the elements $\mathbf{h}$ of the transformation matrix (cf. appendix)

$$J_{L'h} = \frac{\partial \mathbf{L}'}{\partial \mathbf{h}} = \frac{1}{2} J_{\Gamma L}^{\mathsf{T}} (I_4 \otimes (\Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}} - \mathsf{H}\Gamma^{\mathsf{T}}(\mathbf{L})))$$

we obtain the covariance matrix of the transformed line from statistical error propagation:

$$\boldsymbol{\Sigma}_{L'L'} = J_{L'h} \boldsymbol{\Sigma}_{hh} J_{L'h}^{\mathsf{T}} + \mathsf{H}_L \boldsymbol{\Sigma}_{LL} \mathsf{H}_L^{\mathsf{T}}\ .$$

5. Finally we discuss how to derive the covariance matrix of the projection matrix for space lines Q from the covariance matrix of the projection matrix for space points P. The projection matrix P for points and its elements row-wise are

$$\underset{3\times 4}{\mathsf{P}} = \begin{bmatrix} \mathbf{A}^{\mathsf{T}} \\ \mathbf{B}^{\mathsf{T}} \\ \mathbf{C}^{\mathsf{T}} \end{bmatrix} \qquad \underset{12\times 1}{\mathbf{p}} = \mathrm{vec}(\mathsf{P}^{\mathsf{T}}) = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix}$$

with the coordinate planes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of the camera coordinate system, intersecting in the projection center $\mathbf{Z}$. The corresponding projection matrix Q for 3D-lines and its elements row-wise are given by

$$\underset{3\times 6}{\mathsf{Q}} = \begin{bmatrix} (\mathbf{B} \wedge \mathbf{C})^{\mathsf{T}} \\ (\mathbf{C} \wedge \mathbf{A})^{\mathsf{T}} \\ (\mathbf{A} \wedge \mathbf{B})^{\mathsf{T}} \end{bmatrix} \qquad \underset{18\times 1}{\mathbf{q}} = \mathrm{vec}(\mathsf{Q}^{\mathsf{T}}) = \begin{bmatrix} \mathbf{B} \wedge \mathbf{C} \\ \mathbf{C} \wedge \mathbf{A} \\ \mathbf{A} \wedge \mathbf{B} \end{bmatrix}$$

with the dual lines $\mathbf{B} \cap \mathbf{C}$, $\mathbf{C} \cap \mathbf{A}$, and $\mathbf{A} \cap \mathbf{B}$ of the coordinate axes fo the camera system.
Therefore

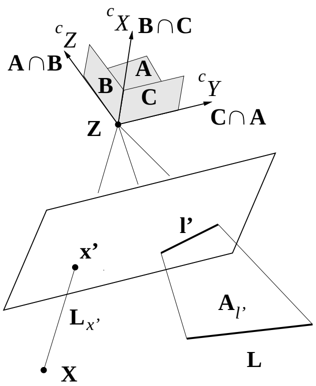


Fig. 10. Single image geometry. Rows of P are coordinate planes, rows of Q are duals of coordinate axes of the camera coordinate system (^cX, ^cY, ^cZ). The projection ray $\mathbf{L}_{x'} = D_6\mathsf{Q}^\mathsf{T}\mathbf{x}'$ and the projection plane $\mathbf{A}_{l'} = \mathsf{P}^\mathsf{T}\mathbf{l}'$ can be determined directly from the image entities using the projection matrices (cf. [11], [18]).

$$\underset{18\times12}{J_{qp}} = \begin{bmatrix} \mathbf{0} & -\Pi(\mathbf{C}) & \Pi(\mathbf{B}) \\ \Pi(\mathbf{C}) & \mathbf{0} & -\Pi(\mathbf{A}) \\ -\Pi(\mathbf{B}) & \Pi(\mathbf{A}) & \mathbf{0} \end{bmatrix}$$

and thus the covariance matrix of $\mathbf{q}$ of the elements of the projection matrix Q for lines reads as

$$\boldsymbol{\Sigma}_{qq} = J_{qp}\boldsymbol{\Sigma}_{pp}J_{qp}^\mathsf{T} \,. \tag{55}$$

In case the projection center of the image is denoted with $\mathbf{Z}$ we have

$$\mathbf{L}_{x'} = D_6\mathsf{Q}^\mathsf{T}\,\mathbf{x}' = \overline{\mathsf{Q}}^\mathsf{T}\,\mathbf{x}' = \overline{\mathsf{Q}}^\mathsf{T}\mathsf{P}\,\mathbf{X} = \Pi(\mathbf{Z})\,\mathbf{X} = \mathbf{Z}\wedge\mathbf{X}$$

as the projection line $\mathbf{L}_{x'}$ is the join of projection center with the 3D-point, thus independent on the other parts of the orientation. Thus

$$\overline{\mathsf{Q}}^\mathsf{T}\mathsf{P} = \Pi(\mathbf{Z}) \,.$$

3.3 Constraints

Constraints Between Geometric Entities in 2D

The incidence of a 2D-point and a 2D-line can be checked using

$$c = \mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{l}^\mathsf{T}\mathbf{x} \overset{!}{=} 0$$

which should vanish. The identity of two points or two lines can be checked using the 3-vectors

$$\mathbf{c} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x} \overset{!}{=} \mathbf{0} \qquad \text{or} \qquad \mathbf{c} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l} \overset{!}{=} \mathbf{0}$$

which should vanish in the case of identity. The reasoning behind this type of constraint is: in case two points are identical the generating line $\mathbf{x} \times \mathbf{y}$ is not defined, similarly, in case two lines are identical, the intersection point $\mathbf{l} \times \mathbf{m}$ is not defined.

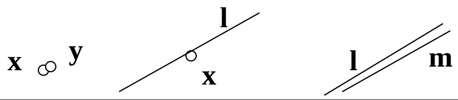
Observe, these are three constraints, but only two of them are independent, as the skew symmetric matrices have rank 2. One may select those two constraints (m, n) where the entry S_{mn} in the skew symmetric matrices is largest absolute value. Then the independence of the selected constraints is guaranteed. This leads to a set of reduced constraints, e. g.:

$$\mathbf{c}^{[\mathrm{r}]} = \underbrace{\begin{bmatrix} \boldsymbol{e}_m^{(3)\mathsf{T}} \\ \boldsymbol{e}_n^{(3)\mathsf{T}} \end{bmatrix} \mathsf{S}(\mathbf{x})}_{\mathsf{S}^{[\mathrm{r}]}(\mathbf{x})} \; \mathbf{y} = -\begin{bmatrix} \boldsymbol{e}_m^{(3)\mathsf{T}} \\ \boldsymbol{e}_n^{(3)\mathsf{T}} \end{bmatrix} \mathsf{S}(\mathbf{y})\mathbf{x} \overset{m=1,n=2}{=} \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (56)$$

which are linearly independent, except for points $\mathbf{x}$ at infinity, thus in case $x_3 \neq 0$.

Table 3.3 collects the three cases mentioned.

Table 4. Relationships between points and lines useful for 2D grouping, together with the degree of freedom and the essential part of the test statistic. The bullet in the last column indicates, that a selection may be performed.



No.	2D-entities	relation	dof	test	select.
1	points $\mathbf{x}$, $\mathbf{y}$	$\mathbf{x} \equiv \mathbf{y}$	2	$\mathbf{c} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$	•
2	point $\mathbf{x}$, line $\mathbf{l}$	$\mathbf{x} \in \mathbf{l}$	1	$c = \mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{l}^\mathsf{T}\mathbf{x}$	
3	lines $\mathbf{l}$, $\mathbf{m}$	$\mathbf{l} \equiv \mathbf{m}$	2	$\mathbf{c} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$	•

Constraints Between Geometric Entities in 3D

In a similar manner one may construct constraints for geometric relations between 3D entities collected in table 3.3, except for the following cases:

7. The identity of two lines $\mathbf{L}$ and $\mathbf{M}$ can be checked using the interpretation of the rows or columns of the Plücker-matrix and their dual: the rows and columns of $\Gamma(\mathbf{L})$ are the intersection points of $\mathbf{L}$ with the coordinate planes $\boldsymbol{e}_i^{(4)}$ and the rows and columns of $\overline{\Gamma}(\mathbf{L})$ are the planes parallel to the coordinate axes. As the intersection points of $\mathbf{L}$ with the coordinate planes lie on the planes through $\mathbf{M}$ parallel to the coordinate axes, in case $\mathbf{L} \equiv \mathbf{M}$ the product $\mathbf{C} = \overline{\Gamma}(\mathbf{L})\Gamma(\mathbf{M})$ must vanish.
8. The incidence of two lines $\mathbf{L}$ and $\mathbf{M}$ can be checked by assuming $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ and $\mathbf{M} = \mathbf{Z} \wedge \mathbf{T}$. Then $|\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}| = -\overline{\mathbf{L}}^\mathsf{T}\mathbf{M} = 0$ only if the four points are coplanar.

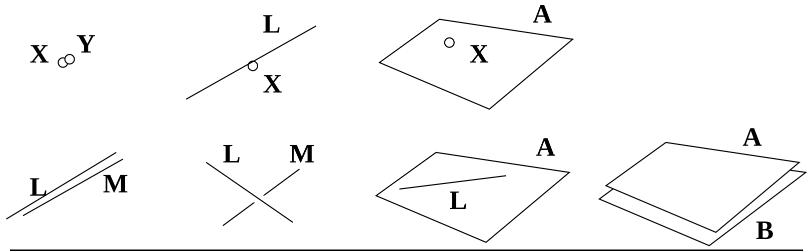
Also here selection of constraints can be performed (numbers refer to rows in table 3.3):

4., 10. From the 6 constraints only 3 are independent. One can select those 3 constraints where the largest element occurs in the matrices $\Pi(\mathbf{X})$ or $\overline{\Pi}(\mathbf{A})$.

5., 9. From the 4 constraints only 2 are independent. One can select those 2 constraints where the largest element occurs in the matrices $\overline{\Gamma}(\mathbf{L})$ or $\Gamma(\mathbf{L})$. The selection transfers to the corresponding matrices $\overline{\Pi}^{\mathsf{T}}(\cdot)$ and $\Pi^{\mathsf{T}}(\cdot)$

7. From the 16 constraints only 4 are independent. They can be selected by taking the largest element of $\overline{\Gamma}(\mathbf{L})$ with index (m,n) and the largest element of $\Gamma(\mathbf{M})$ with index (k,l) and check the entries $\{(m,k),(m,l),(n,k),(n,l)\}$ of C.

Table 5. Relationships between points, lines and planes useful for 3D grouping, together with the degree of freedom and the essential part of the test statistic. The bullet in the last column indicates the possibility for the selection of independent constraints.



No.	3D-entities	relation	dof	test	select.
4	points $\mathbf{X}$, $\mathbf{Y}$	$\mathbf{X} \equiv \mathbf{Y}$	3	$\mathbf{C} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$	•
5	point $\mathbf{X}$, line $\mathbf{L}$	$\mathbf{X} \in \mathbf{L}$	2	$\mathbf{C} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L} = \overline{\Gamma}^{\mathsf{T}}(\mathbf{L})\mathbf{X}$	•
6	point $\mathbf{X}$, plane $\mathbf{A}$	$\mathbf{X} \in \mathbf{A}$	1	$c = \mathbf{X}^{\mathsf{T}}\mathbf{A} = \mathbf{A}^{\mathsf{T}}\mathbf{X}$	
7	lines $\mathbf{L}$, $\mathbf{M}$	$\mathbf{L} \equiv \mathbf{M}$	4	$\mathsf{C} = \overline{\Gamma}(\mathbf{L})\Gamma(\mathbf{M})$	•
8		$\mathbf{L} \cap \mathbf{M} \neq \emptyset$	1	$c = \overline{\mathbf{L}}^{\mathsf{T}}\mathbf{M} = \overline{\mathbf{M}}^{\mathsf{T}}\mathbf{L}$	
9	line $\mathbf{L}$, plane $\mathbf{A}$	$\mathbf{L} \in \mathbf{A}$	2	$\mathbf{C} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L} = \Gamma^{\mathsf{T}}(\mathbf{L})\mathbf{A}$	•
10	planes $\mathbf{A}$, $\mathbf{B}$	$\mathbf{A} \equiv \mathbf{B}$	3	$\mathbf{C} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A}$	•

Constraints Containing Mappings

We now easily can derive arbitrary constraints containing mappings. As an example, the constraint

$$\mathbf{c} = \mathbf{x}' \times \mathsf{H}\mathbf{x} = \mathsf{S}(\mathbf{x}')\mathsf{H}\mathbf{x} = -\mathsf{S}(\mathsf{H}\mathbf{x})\mathbf{x}' = (\mathsf{S}(\mathbf{x}') \otimes \mathbf{x}^{\mathsf{T}})\mathbf{h} \stackrel{!}{=} \mathbf{0}$$

would require the mapped point $\mathsf{H}\mathbf{x}$ to be identical to $\mathbf{x}'$. The fifth expression uses $\mathbf{h} = \mathrm{vec}(\mathsf{H}^{\mathsf{T}})$.

Observe, the expression for $\mathbf{c}$ is trilinear, here in $\mathbf{x}$, in H and in $\mathbf{x}'$. Therefore we give three different expressions making the Jacobians of $\mathbf{c}$ with respect to the three generating entities explicit:

$$\frac{\partial \mathbf{c}}{\partial \mathbf{x}} = \mathsf{S}(\mathbf{x}')\mathsf{H}\ , \qquad \frac{\partial \mathbf{c}}{\partial \mathbf{x}'} = -\mathsf{S}(\mathsf{H}\mathbf{x})\ , \qquad \frac{\partial \mathbf{c}}{\partial \mathbf{h}} = \mathsf{S}(\mathbf{x}') \otimes \mathbf{x}^{\mathsf{T}}\ .$$

Other constraints easily can be found by combining the relations given in the tables. Observe, the constraints involving the mappings with H_L and Q for 3D-lines are not linear in the corresponding mappings H and P for 3D-points. This leads to more involving expressions.

The tests involving mappings from 3D to 2D can be read in two ways: testing incidence in the image or testing incidence in space. As the constraint is the same, both tests lead to the same result. As an example, take the incidence test of a space point $\mathbf{X}$ and image line $\mathbf{l}'$. It reads as $c = \mathbf{l}^{\mathsf{T}}\mathsf{P}\mathbf{X}$. If we take the predicted point to be $\mathbf{x}'_p = \mathsf{P}\mathbf{X}$ and the projection plane to be $\mathbf{A}_{l'} = \mathsf{P}^{\mathsf{T}}\mathbf{l}'$ then the constraint can be written as

$$c = \mathbf{l}'^{\mathsf{T}}\mathbf{x}'_p \stackrel{!}{=} 0 \qquad \text{or} \qquad c = \mathbf{A}_{l'}^{\mathsf{T}}\mathbf{X} \stackrel{!}{=} 0\ .$$

which explicitely shows the equivalence of testing the relation in image and 3D-space. Actually, the relation for the projection plane has been derived from this identity.

Comments: The essence of the constructions and constraints have been published at many places, e. g. [33]. They have also been given in [25]. They can also been found in [18]. The tables in the appendix of [35] give all constructions explicitely, even for elements in $\mathbb{P}^5$. The representation chosen here, initially given in [15], allows to easy remember the relations and explicitly have the corresponding Jacobians available. For space limitations we did not give the relations for geometric entities to be orthogonal or parallel, werefer to [15].

3.4 Testing Uncertain Geometric Relations

We directly use the mentioned constraints for statistical testing. The idea is to test the c-quantities in tables 3.3 and 3.3 having different dimensions. In case the relation holds they should be zero, thus formally, the null-hypothesis $H_0 : c = 0$ is tested versus the alternative $H_a : c \neq 0$.

The generic procedure is the following:

1. Determine the difference $\boldsymbol{c}$, using one of the two equations in column 5 in table 3.3 or 3.3
2. If necessary, select independent constraints leading to the reduced vector of differences $\boldsymbol{c}'$. The number of independent constraints is the degree of freedom, cf. column 4. The selection is different for the individual tests and indicated in column 6 where necessary.
3. Determine the covariance matrix of the difference $\boldsymbol{c}$ or the reduced difference $\boldsymbol{c}'$ using error propagation (41) and the two Jacobians from table 3.3 or 3.3 in column 5, possibly taking the selection into account. The Jacobians can be taken from these equations all having the structure of eq. (40).
4. Determine the test statistic T

$$T = \frac{c'^2}{\sigma_c'^2} \sim \chi_1^2 \qquad \text{or} \qquad T = \boldsymbol{c}'^{\mathsf{T}} \boldsymbol{\Sigma}_{c'c'}^{-1} \boldsymbol{c}' \sim \chi_r^2 \tag{57}$$

being χ_1^2- or χ_r^2-distributed, the degrees of freedom r given in column 4 of tables 3.3 and 3.3.

5. Choose a significance number α and compare T with the critical value $\chi^2_{r,1-\alpha}$. If $T > \chi^2_{r,1-\alpha}$ then the hypothesis that the spatial relation holds can be rejected.

The proposed error propagation in step 3 using (41) is simple; but it only holds *approximately*, in case the tested relation is not fulfilled.

The reason is, that the Jacobians of the bilinear relations are not consistent, as they depend on the observed entities, not on the true or unbiased estimated entities as required in (3), p. 6.

A rigorous procedure would be to first impose the constraints of the relation onto the two observed entities using the estimation procedure of sect. 2.2, p.16 and test the estimated variance factor $\widehat{\sigma}_0^2$. This approach has been taken by Kanatani [25].

In case the test is not rejected, the covariance matrix of the differences, however, is a very good approximation. It depends on the rigor needed, when applying statistical tests on geometric relations. As the assumed variances of the initial geometric entities will not be more precise than 10 or 20 %, at the best, the test statistic also will have this uncertainty. In most applications this paper is motivated by, the tests will be used for deleting erroneous correspondences or for controlling search in grouping processes. Especially in the latter application the monotonicity of the test statistic with respect to the rigorous one is essential. We will discuss this problem below.

An Example for Testing

We want to demonstrate the test procedure for the point-line incidence. Let the 3D-line $\mathbf{L}$ and the 3D-point $\mathbf{X}$ be given by:

$$\mathbf{L} = [3, 0, 0, 0, 3, -3]^\mathsf{T} \qquad \mathbf{X} = [1, 1, 1, -1]^\mathsf{T}$$

The line is parallel to the X-axis and passes through the point $[0, 1, 1]^\mathsf{T}$. The 3D-point has the Euclidean coordinates $[-1, -1, -1]^\mathsf{T}$. The distance of $\mathbf{X}$ and $\mathbf{L}$ therefore is $d_{XL} = 2\sqrt{2} \approx 2.8$. For demonstration purposes we assume the covariance matrices to be a multiple of the unit matrix:

$$\boldsymbol{\Sigma}_{LL} = 4 I_6 \qquad \boldsymbol{\Sigma}_{XX} = I_4$$

This the standard deviations of the homogeneous point coordinates are 1, thus the standard deviations of the Euclidean coordinates are larger than 1, which could be verified by applying error propagation to $X = U/W$ etc. The line has a similar precision in the vicinity of the origin of the coordinate system, however, a very large uncertainty in direction. This could be verified by intersecting the line with the planes $X = 0$ and $X = -1$, being parallel to the YZ-plane, and determining the standard deviations of the Y- and Z-coordinate of the intersection points. Thus we can expect the distance of 2.8 not to be significant.

We now follow the above mentioned steps:

1. The difference $\boldsymbol{c}$ is

$$c = \overline{\Gamma}^{\mathsf{T}}(\mathbf{L})\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 3 & 0 & -3 \\ 0 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L} = \begin{bmatrix} 0 & 1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 6 \\ 0 \end{bmatrix}$$

2. We select rows 3 and 4, as $L_6 = -3$ does not vanish (We could have taken rows 2 and 3 or rows 2 and 4 also.). Thus we obtain the reduced vector

$$c' = R^{\mathsf{T}} c = \begin{bmatrix} e_3^{(4)\mathsf{T}} \\ e_4^{(4)\mathsf{T}} \end{bmatrix} c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -6 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

with the reduction matrix $R^{\mathsf{T}} = [e_3^{(4)}, e_4^{(4)}]$.

3. The covariance matrix of c' is obtained from

$$\Sigma_{c'c'} = R^{\mathsf{T}} \left(\overline{\Gamma}^{\mathsf{T}}(\mathbf{L}) \Sigma_{XX} \overline{\Gamma}(\mathbf{L}) +^{\mathsf{T}} \overline{\Pi}^{\mathsf{T}}(\mathbf{X}) \Sigma_{LL} \overline{\Pi}(\mathbf{X}) \right) R = \begin{bmatrix} 30 & -5 \\ -5 & 30 \end{bmatrix}$$

4. The test statistic is

$$T = c'^{\mathsf{T}} \Sigma_{c'c'}^{-1} c' = \frac{216}{175} \approx 1.23$$

It is χ_2^2-distributed.

5. We choose a significance number $\alpha = 0.05$. The critical value is $\chi_{2,1-\alpha} = 5.99$. As T is smaller than the critical value, we have no reason to reject the hypothesis, that the point $\mathbf{X}$ sits on the line $\mathbf{L}$.

An Example for Setting up the Estimation

We want to demonstrate the use of the mentioned relations for the estimation task in fig. 2 right, p.2. We assume the orientation of the images to be given. Here we only need the 4 projection matrices Q_k for space lines.

We have $N = 12$ observations, namely the 4 homogeneous 3-vectors of the image points $\mathbf{x}'$ and lines $\mathbf{l}'$ in the four images and and $U = 6$ unknown for the Plücker coordinates $\mathbf{L}_5$ of the 3D-line. Thus, referring to the Gauss-Helmert model, we have the vector $\boldsymbol{l}$ of the observations and the vector $\boldsymbol{x}$ of the unknown parameters:

$$\underset{12\times 1}{\boldsymbol{l}} = \begin{bmatrix} \mathbf{x}'_{11} \\ \mathbf{l}'_{52} \\ \mathbf{x}'_{23} \\ \mathbf{l}'_{54} \end{bmatrix} \qquad \underset{6\times 1}{\boldsymbol{x}} = \mathbf{L}_5 \; .$$

There are 2 constraints for the measured image points and $2 \times 2 = 4$ independent constraints for the observed image lines, using the reduced skew symmetric matrices

$\mathsf{S}^{[r]}(\cdot)$ (cf. (56)), altogether yielding 6 constraints $\boldsymbol{g}$ between the observations and the unknown parameters. Moreover we have 2 constraints $\boldsymbol{h}$ on the unknown parameters, namely the length constraint and the Plücker constraint on $\mathbf{L}_5$. Thus we obtain the Gauss-Helmert model for this estimation task, where all relations should hold for the fitted, i. e. estimated values:

$$\underset{6\times 1}{\boldsymbol{g}(\boldsymbol{l},\boldsymbol{x})} = \begin{bmatrix} \mathbf{x}'^{\mathsf{T}}_{11}\mathsf{Q}_1\mathbf{L}_5 \\ \mathsf{S}^{[\mathbf{r}]}(\mathbf{l}'_{52})\mathsf{Q}_2\mathbf{L}_5 \\ \mathbf{x}^{\mathsf{T}}_{23}\mathsf{Q}_3\mathbf{L}_5 \\ \mathsf{S}^{[\mathbf{r}]}(\mathbf{l}'_{54})\mathsf{Q}_4\mathbf{L}_5 \end{bmatrix} = \mathbf{0} \qquad \underset{2\times 1}{\boldsymbol{h}(\boldsymbol{x})} = \begin{bmatrix} \frac{1}{2}\left(\mathbf{L}_5^{\mathsf{T}}\mathbf{L}_5 - 1\right) \\ \frac{1}{2}\overline{\mathbf{L}}_5^{\mathsf{T}}\mathbf{L}_5 \end{bmatrix} = \mathbf{0}\,.$$

The three Jacobians A, B and H, for solving for the ML-estimates using the Gauß-Helmert model, can easily be derived using the tables given above. The initial solution $\widehat{\boldsymbol{x}}^{(0)}$ for $\boldsymbol{x} = \mathbf{L}_5$ can be obtained from the right eigenvector of A. Then the Jacobians can be evaluated at approximate values. In the first iteration one uses $\widehat{\boldsymbol{l}}^{(0)} = \boldsymbol{l}$ and the initial solution $\widehat{\boldsymbol{x}}^{(0)}$. In the following iterations the Jacobians have to be evaluated at the *fitted* observations and unknown parameters in order to avoid bias and the necessity to renormalize (cf. [24]). The redundancy of the system is $R = G + H - U = 6 + 2 - 6 = 2$. Obviously the setup can directly be transferred to the other two problems shown in figs. 1 and 2.

4 Conditioning and Normalization

There are three reasons why a blind use of the approach described so far may lead to problems: (1) entries in the homogeneous vectors or matrices with highly different orders of magnitude lead to numerical instability, (2) when testing extreme deviations from the null-hypothesis leads to wrong test statistics, and (3) large deviations from the null-hypothesis may lead to test statistics which do not increase monotonically with the geometric distance of the involved entities.

These problems can be cured by conditioning and normalization:

1. *Conditioning*, as the name indicates, aims at improving the condition numbers of the matrices in concern. It is achieved by centering and scaling the data such that the Euclidean coordinates of points are in the range $[-k, +k]$. Hartley [17][2] proposed $k = 1$. The monotonicity of the test statistic with the geometric distance is guaranteed when choosing $k < 1$, e. g. $k = 1/3$, cf. [23].
2. *Normalization* effects at least the covariance matrix of uncertain homogeneous entities. It imposes the constraints on the length of the complete vector or on the Euclidean part of the vector. Thus for some uncertain homogeneous vector $\underline{\mathbf{x}} = (\underline{\boldsymbol{x}}_h, \underline{\boldsymbol{x}}_0)^{\mathsf{T}}$ with Euclidean part $\boldsymbol{x}_h$ and homogenous part $\boldsymbol{x}_0$ we have the constraint (cf. [23])

$$|\underline{\mathbf{x}}| = |\mathbf{x}| \qquad \text{or} \qquad |\underline{\boldsymbol{x}}_h| = |\boldsymbol{x}_h|\,.$$

   In addition we might have the Plücker constraint $\overline{\mathbf{L}}^{\mathsf{T}}\mathbf{L} = 0$ for 3D-lines or the singularity constraint $|\mathsf{F}| = 0$ for the fundamental matrix. When imposing these constraints the resulting covariance matrix can be determined from (35), p. 19. Kanatani [25] applied the Euclidean normalization together with a scaling to of the Euclidean part of points to 1.

[2] calling this procedure normalization

5 Conclusions

The paper discussed (1) an approach for uncertain geometric reasoning, which tries to keep as close to algebraic projective geometry as possible and (2) a generic estimation scheme for uncertain geometric entities and transformation which can handle any number of constraints. Though the various ingredients are well known, building a software system is simplified by using the presented representations, both, in geometry and statistics.

The basic idea behind the approach is to exploit the multi-linearity of all geometric constructions and constraints. As these multi-linearities are to be found also in all variations of geometric algebra, e. g. the conformal geometric algebra [32], it appears to be feasible to reformulate the geometric expressions in terms of coordinate vectors with a covariance matrix attached to it, and thus to extend the approach to a much wider field than projective geometry.

6 Appendix

Uncertainty of Tranformed 3D-Lines

Starting from the transformation $\mathbf{X}' = \mathsf{H}\mathbf{X}$ of 3D-points $\mathbf{X}$, we observe

$$\Gamma(\mathbf{L}') = \mathsf{H}\Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}} \tag{58}$$

as $\mathsf{H}(\mathbf{X}\mathbf{Y}^{\mathsf{T}} - \mathbf{Y}\mathbf{X}^{\mathsf{T}})\mathsf{H}^{\mathsf{T}} = \mathbf{X}'\mathbf{Y}'^{\mathsf{T}} - \mathbf{Y}'\mathbf{X}'^{\mathsf{T}} = \Gamma(\mathbf{L}')$. This shows, the transformation of lines is quadratic in the entries of the transformation matrix H for points.

We now first derive an expression for the transformation matrix H_L for 3D-lines in terms of their Plücker coordinates $\mathbf{L}' = \mathsf{H}_L\mathbf{L}$, and then derive the the Jacobians which are necessary to derive the covariance matrix of the transformed line $\mathbf{L}'$ The transformation (58) may be written in terms of the elements of $\Gamma(\mathbf{L})$ $\text{vec}(\Gamma(\mathbf{L}')) = (\mathsf{H}\otimes\mathsf{H})\text{vec}(\Gamma(\mathbf{L}))$ containing the Plücker coordinates of the lines. We now map the 16 values of $\text{vec}(\Gamma(\mathbf{L}))$ to the 6-Plücker-vector $\mathbf{L}$ using the 16×6-Jacobian $\underset{16\times 6}{J_{\Gamma L}} = \partial\text{vec}(\Gamma(\mathbf{L}))/\partial\mathbf{L}$. Then we have

$$\mathbf{L} = \frac{1}{2}J_{\Gamma L}^{\mathsf{T}}\text{vec}(\Gamma(\mathbf{L})) \qquad \text{and} \qquad \text{vec}(\Gamma(\mathbf{L})) = J_{\Gamma L}\mathbf{L} \tag{59}$$

and obtain the transformation

$$\mathbf{L}' = \mathsf{H}_L\,\mathbf{L} = (I_6 \otimes \mathbf{L}^{\mathsf{T}})\text{vec}(\mathsf{H}_L^{\mathsf{T}}) \tag{60}$$

with the 6×6-transformation matrix for 3D-lines $\mathsf{H}_L = \frac{1}{2}J_{\Gamma L}^{\mathsf{T}}(\mathsf{H}\otimes\mathsf{H})J_{\Gamma L}$. We now want to determine the Jacobians $J_{L'h} = \partial\mathbf{L}'/\partial\mathbf{h}$. We start from the differential of $\Gamma(\mathbf{L}')$ in (58) $d\Gamma(\mathbf{L}') = d\mathsf{H}\Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}} + \mathsf{H}\Gamma(d\mathbf{L})\mathsf{H}^{\mathsf{T}} + \mathsf{H}\Gamma(\mathbf{L})d\mathsf{H}^{\mathsf{T}}$. With (59) we obtain

$$\begin{aligned} d\mathbf{L}' &= \frac{1}{2}J_{\Gamma L}^{\mathsf{T}}\ \text{vec}(d\Gamma(\mathbf{L}')) \\ &= \frac{1}{2}J_{\Gamma L}^{\mathsf{T}}(I_4 \otimes \Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}})d\mathbf{h} + \frac{1}{2}J_{\Gamma L}^{\mathsf{T}}(\mathsf{H}\otimes\mathsf{H})J_{\Gamma L}\ d\mathbf{L} + \frac{1}{2}J_{\Gamma L}^{\mathsf{T}}(I_4 \otimes \mathsf{H}\Gamma(\mathbf{L}))d\mathbf{h} \\ &= \underbrace{\frac{1}{2}J_{\Gamma L}^{\mathsf{T}}(I_4 \otimes (\Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}} - \mathsf{H}\Gamma^{\mathsf{T}}(\mathbf{L})))}_{J_{L'h}}\ d\mathbf{h} + \underbrace{\frac{1}{2}J_{\Gamma L}^{\mathsf{T}}(\mathsf{H}\otimes\mathsf{H})J_{\Gamma L}}_{J_{L'L}=\mathsf{H}_L}\ d\mathbf{L} \end{aligned}$$

or short

$$d\mathbf{L}' = J_{L'h} d\mathbf{h} + \mathsf{H}_L d\mathbf{L} \qquad \text{with} \qquad J_{L'h} = \frac{1}{2} J_{\Gamma L}^{\mathsf{T}} (I_4 \otimes (\Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}} - \mathsf{H}\Gamma^{\mathsf{T}}(\mathbf{L})))$$

which we can use for statistical error propagation:

$$\boldsymbol{\Sigma}_{L'L'} = J_{L'h} \boldsymbol{\Sigma}_{hh} J_{L'h}^{\mathsf{T}} + \mathsf{H}_L \boldsymbol{\Sigma}_{LL} \mathsf{H}_L^{\mathsf{T}}$$

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