Projective Geometry for Photogrammetric Orientation Procedures

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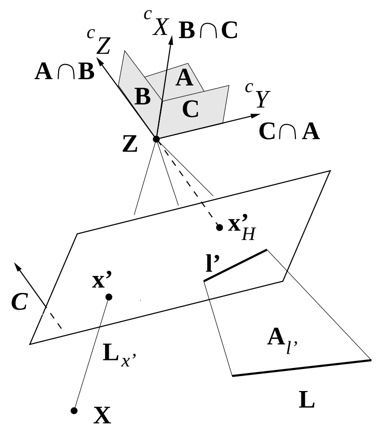
Mappings into a Single Image

- projection of points and lines
- backprojection of points and lines
- constraints between corresponding points and lines
- relation to classical camera models



Projection Matrix for Points

Derivation:

Object point $\mathbf{X}$, image point $\mathbf{X}'$ in a 3D image coordinate system

$$\mathbf{X}' = \mathsf{H}\mathbf{X} \qquad \text{or} \qquad \begin{bmatrix} U' \\ V' \\ W' \\ T' \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^\mathsf{T} \\ \mathbf{A}_2^\mathsf{T} \\ \mathbf{A}_3^\mathsf{T} \\ \mathbf{A}_4^\mathsf{T} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix}$$

As $Z' = 0$, delete 3rd coordinate in $\mathbf{X}'$, with substitution

$$\mathbf{x}' = \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \doteq \begin{bmatrix} U' \\ V' \\ T' \end{bmatrix}$$

Thus

$$\mathbf{x}' = \begin{bmatrix} \mathbf{A}_1^\mathsf{T} \\ \mathbf{A}_2^\mathsf{T} \\ \mathbf{A}_4^\mathsf{T} \end{bmatrix} \mathbf{X}$$

Canonical

$$\underset{3\times 1}{\mathbf{x}'} = \underset{3\times 4}{\mathsf{P}} \; \underset{4\times 1}{\mathbf{X}}$$

with projection matrix

$$\mathsf{P} = \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \\ \mathbf{C}^\mathsf{T} \end{bmatrix}$$

or

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \\ \mathbf{C}^\mathsf{T} \end{bmatrix} \mathbf{X} = \begin{bmatrix} <\mathbf{A},\mathbf{X}> \\ <\mathbf{B},\mathbf{X}> \\ <\mathbf{C},\mathbf{X}> \end{bmatrix}$$

Interpretation of projection matrix P

- homogeneous matrix, 11 parameters
- rows: planes
  - $u' =< \mathbf{A},\mathbf{X} > 0$: plane through y'-axis
  - $v' =< \mathbf{B},\mathbf{X} > 0$: plane through x'-axis
  - $w' =< \mathbf{C},\mathbf{X} > 0$: plane through line at infinity
- Nullspace

  $$\mathbf{0} = \mathsf{P}\mathbf{Z} \qquad \begin{bmatrix} <\mathbf{A},\mathbf{Z}> \\ <\mathbf{B},\mathbf{Z}> \\ <\mathbf{C},\mathbf{Z}> \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

  Point $\mathbf{Z}$

  $$\mathbf{Z} = \mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$$

  is projection centre, planes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ coordinate planes

- Decomposition

  $$\mathsf{P} = [\mathsf{H}_\infty | \mathbf{h}]$$

  From $\mathbf{0} = \mathsf{P}\mathbf{Z} = \mathsf{H}_\infty \boldsymbol{Z}_0 + \mathbf{h} Z_h$ follows

  $$\boldsymbol{Z} = \frac{\boldsymbol{Z}_0}{Z_h} = -\mathsf{H}_\infty^{-1}\mathbf{h}$$

  ideal points $\mathbf{X}_\infty$ are mapped to

  $$\mathbf{x}' = \mathsf{P}\mathbf{X}_\infty = \mathsf{H}_\infty \boldsymbol{X}_{\infty 0} = [\mathsf{H}_\infty | \mathbf{h}] \begin{bmatrix} \boldsymbol{X}_{\infty 0} \\ 0 \end{bmatrix}$$

  have images, independent on $\mathbf{Z}$, infinite homography H_∞
- Columns $\mathbf{x}'_{0i}$

  $$\mathsf{P} = [\mathbf{x}'_{01}, \mathbf{x}'_{02}, \mathbf{x}'_{03}, \mathbf{x}'_{04}]$$

  are images of coordinate points $\boldsymbol{e}_i^{(4)}$
  Horizon $\mathbf{h}'_\infty$ is line through images $\mathbf{x}'_{01}$ and $\mathbf{x}'_{02}$ of coordinate points

$\mathbf{X}_{\infty X}$ and $\mathbf{X}_{\infty Y}$

$$\mathbf{h}'_\infty = \mathbf{x}'_{01} \times \mathbf{x}'_{02}$$

Nadir or zenith point: $\mathbf{x}'_{03}$

- Camera with affine sensor coordinate system

  Viewing direction: normal to $\mathbf{C}$

$$\boldsymbol{C}_h = \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix}$$

- principle point $\mathbf{x}'_H$: Image of point $(\boldsymbol{C}_h^\mathsf{T}, 0)$

$$\mathbf{x}'_H = \mathsf{H}_\infty \boldsymbol{C}_h$$

Construction of projection matrix:

Given

1. projection centre $\mathbf{Z}$
2. rotation matrix R
3. straight line preserving mapping, principle distance c
4. mensuration in affine coordinate system in image plane (shear s, scale difference m, principle point $\boldsymbol{x}'_H$)

$$\mathbf{x}' = \begin{bmatrix} 1 & s & x'_H \\ 0 & 1+m & y'_H \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} I & -\boldsymbol{Z} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_0 \\ X_h \end{bmatrix}$$

$$\mathbf{x}' = \mathsf{K}_c\, {}^c\mathsf{P}_c\, {}^c\mathsf{R}\mathsf{T}\mathbf{X}$$

with calibration matrix

(homogeneous, 5 parameters of interior orientation)

$$\mathsf{K} = \begin{bmatrix} c & cs & x'_H \\ 0 & (1+m)c & y'_H \\ 0 & 0 & 1 \end{bmatrix}$$

we can write

$$\boxed{\mathsf{P} = \mathsf{K}R[I| - \boldsymbol{Z}]}$$

and

$$\boxed{\mathbf{x}' = \mathsf{P}\mathbf{X}}$$

Explicit

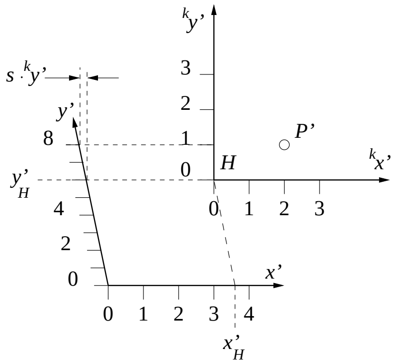
Camera coordinates = reduced image coordinates

$$
\begin{aligned}
{}^c x' &= c \frac{r_{11}(X - X_0) + r_{12}(Y - Y_0) + r_{13}(Z - Z_0)}{r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)} \\
{}^c y' &= c \frac{r_{21}(X - X_0) + r_{22}(Y - Y_0) + r_{23}(Z - Z_0)}{r_{31}(X - X_0) + r_{32}(Y - Y_0) + r_{33}(Z - Z_0)}
\end{aligned}
$$

Image coordinates

$$
\begin{aligned}
x' &= {}^c x' + s\, {}^c y' + x'_H \\
y' &= (1 + m)\, {}^c y' + y'_H
\end{aligned}
$$

$$x' = {}^c x' + s\, {}^c y' + x'_H \qquad y' = (1 + m)\, {}^c y' + y'_H$$



Interpretation:

- infinite homography H_∞ (for star calibration)

$$\mathsf{H}_\infty = \mathsf{K}R$$

- normalized coordiantes = directions in the camera system

$$^0\mathbf{x}' = \mathsf{H}_\infty^{-1}\mathbf{x}'$$

camera model $R = K = I$
(nadir image, principle distance 1)

$$^0\mathsf{P} = [I | -\boldsymbol{Z}]$$

yields

$$^0\mathbf{x}' = \boldsymbol{X} - \boldsymbol{Z} \qquad \text{or} \qquad {}^0x' = \frac{X - X_0}{Z - Z_0} \qquad {}^0y' = \frac{Y - Y_0}{Z - Z_0}$$

direction in object space = direction in image space expressed in object space

Partitioning: Given P (11), sought K (5), R (3), $\mathbf{Z}$ (3)

1. projection centre, 3 parameters

$$\boldsymbol{Z} = -\mathsf{H}_\infty^{-1}\mathbf{h}$$

2. rotation matrix (3) and calibration matrix from Choleski-decomposition

$$\mathsf{H}_\infty \mathsf{H}_\infty^\mathsf{T} = \mathsf{K}\mathsf{K}^\mathsf{T} \qquad R = \mathsf{K}^{-1}\mathsf{H}$$

3. Normalization of calibration matrix (5)

$$K = \frac{\mathsf{K}}{K_{33}}$$

One-to-one relation

$$\boxed{\mathsf{P} \longleftrightarrow (\mathsf{K}, R, \boldsymbol{Z})}$$

Non-linear Distortions

General camera

$$^g\mathbf{x}' = {}^g\mathsf{K}(\boldsymbol{x}', \boldsymbol{q})\,\mathbf{x}' \qquad \begin{bmatrix} ^g x' \\ ^g y' \end{bmatrix} = \begin{bmatrix} x' + \Delta x'(\boldsymbol{x}', \boldsymbol{q}) \\ y' + \Delta y'(\boldsymbol{x}', \boldsymbol{q}) \end{bmatrix}$$

with

$$^g\mathsf{K} = \begin{bmatrix} 1 & 0 & \Delta x'(\boldsymbol{x}', \boldsymbol{q}) \\ 0 & 1 & \Delta y'(\boldsymbol{x}', \boldsymbol{q}) \\ 0 & 0 & 1 \end{bmatrix}$$

and polynomials $\Delta x'(\boldsymbol{x}'\boldsymbol{q})$ and $\Delta y(\boldsymbol{x}'\boldsymbol{q})$ depending on

- image coordinates $\mathbf{x}'$
- additional parameters $\boldsymbol{q}$

Complete model

$$ {}^g\mathbf{x}' = {}^g\mathsf{P}(\mathbf{x}')\ \mathbf{X} $$

with

$$ {}^g\mathsf{P}(\mathbf{x}') = {}^g\mathsf{K}(\mathbf{x}')\mathsf{P} $$

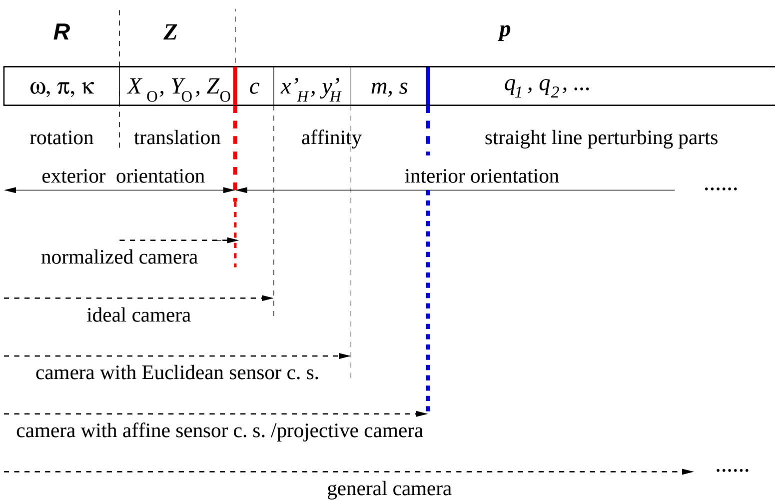
Use:

- Prediction, two-step

$$ 1: \quad \mathbf{x}' = \mathsf{P}\mathbf{X} \qquad 2: \quad {}^g\mathbf{x}' = {}^g\mathsf{K}\mathbf{x}' $$

- local projective model, reference point $\mathbf{x}'_r$, fix P

$$ {}^g\mathbf{x}'(\mathbf{x}'_r) = {}^g\mathsf{P}(\mathbf{x}'_r)\ \mathbf{X} $$

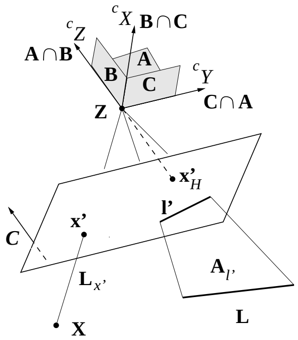


camera model	E. O.	parameters of I. O.	# of E. O. + I. O.
normalized	$\boldsymbol{X}_O, R = I$	-	3
ideal	$\boldsymbol{X}_O, R$	c	7
Euclidean sensor coord. system	$\boldsymbol{X}_O, R$	$c,\ x'_H,\ y'_H$	9
affine sensor coord. system	$\boldsymbol{X}_O, R$	$c,\ x'_H,\ y'_H,\ m,\ s$	11
general	$\boldsymbol{X}_O, R$	$c,\ x'_H,\ y'_H,\ m,\ s,\ \Delta\boldsymbol{x}'(\boldsymbol{q})$	$11{+}N_q$

camera	# EO/im.	# IO/im.	# O/im.	# CP,# CL
calibrated	6	-	6	≥ 3
straight line pres.	6	5	11	≥ 6

Number of parameters and control features required for orientation

Projection Matrix for Lines



projection of points and lines and inversion

Line $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ maps to

$$\mathbf{l}' = \mathbf{x}' \times \mathbf{y}' = \mathsf{P}\mathbf{X} \times \mathsf{P}\mathbf{Y}$$

With projection matrix for lines

$$\underset{3\times 6}{\mathsf{Q}} \doteq \begin{bmatrix} \overline{\mathbf{B}\cap\mathbf{C}}^{\mathsf{T}} \\ \overline{\mathbf{C}\cap\mathbf{A}}^{\mathsf{T}} \\ \overline{\mathbf{A}\cap\mathbf{B}}^{\mathsf{T}} \end{bmatrix} \quad \text{and} \quad \underset{18\times 1}{\mathbf{q}} = \text{vec}(\mathsf{Q}^{\mathsf{T}}) = \begin{bmatrix} \overline{\mathbf{B}\cap\mathbf{C}} \\ \overline{\mathbf{C}\cap\mathbf{A}} \\ \overline{\mathbf{A}\cap\mathbf{B}} \end{bmatrix}$$

we obtain

$$\underset{3\times 1}{\mathbf{l}'} = \underset{3\times 6}{\mathsf{Q}} \underset{6\times 1}{\mathbf{L}} = (\underset{3\times 3}{I} \otimes \underset{1\times 6}{\mathbf{L}^\mathsf{T}}) \underset{18\times 1}{\mathbf{q}}$$

or

$$\mathbf{l}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} < \mathbf{B} \cap \mathbf{C}, \mathbf{L} > \\ < \mathbf{C} \cap \mathbf{A}, \mathbf{L} > \\ < \mathbf{A} \cap \mathbf{B}, \mathbf{L} > \end{bmatrix}$$

Q is *quadratic* in the entries of P !

Proof:

$$\mathbf{l}' = \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{X} \\ \mathbf{B}^\mathsf{T}\mathbf{X} \\ \mathbf{C}^\mathsf{T}\mathbf{X} \end{bmatrix} \times \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{Y} \\ \mathbf{B}^\mathsf{T}\mathbf{Y} \\ \mathbf{C}^\mathsf{T}\mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^\mathsf{T}\mathbf{X}\mathbf{C}^\mathsf{T}\mathbf{Y} - \mathbf{B}^\mathsf{T}\mathbf{Y}\mathbf{C}^\mathsf{T}\mathbf{X} \\ \mathbf{C}^\mathsf{T}\mathbf{X}\mathbf{A}^\mathsf{T}\mathbf{Y} - \mathbf{C}^\mathsf{T}\mathbf{Y}\mathbf{A}^\mathsf{T}\mathbf{X} \\ \mathbf{A}^\mathsf{T}\mathbf{X}\mathbf{B}^\mathsf{T}\mathbf{Y} - \mathbf{A}^\mathsf{T}\mathbf{Y}\mathbf{B}^\mathsf{T}\mathbf{X} \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} \mathbf{X}^\mathsf{T}(\mathbf{B}\mathbf{C}^\mathsf{T} - \mathbf{C}\mathbf{B}^\mathsf{T})\mathbf{Y} \\ \mathbf{X}^\mathsf{T}(\mathbf{C}\mathbf{A}^\mathsf{T} - \mathbf{A}\mathbf{C}^\mathsf{T})\mathbf{Y} \\ \mathbf{X}^\mathsf{T}(\mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T})\mathbf{Y} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{B} \cap \mathbf{C}}^\mathsf{T}(\mathbf{X} \wedge \mathbf{Y}) \\ \overline{\mathbf{C} \cap \mathbf{A}}^\mathsf{T}(\mathbf{X} \wedge \mathbf{Y}) \\ \overline{\mathbf{A} \cap \mathbf{B}}^\mathsf{T}(\mathbf{X} \wedge \mathbf{Y}) \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} \overline{\mathbf{B} \cap \mathbf{C}}^\mathsf{T} \\ \overline{\mathbf{C} \cap \mathbf{A}}^\mathsf{T} \\ \overline{\mathbf{A} \cap \mathbf{B}}^\mathsf{T} \end{bmatrix} \mathbf{X} \wedge \mathbf{Y} \quad (3)$$

Properties of projection matrix Q for lines:

- homogeneous, 17 parameters, but only 11 independent
- rows are (duals of) coordinate axes
- columns, image of coordinate axes

$$\mathsf{Q} = [\mathbf{l}'_{01}, \mathbf{l}'_{02}, \mathbf{l}'_{03}, \mathbf{l}'_{04}, \mathbf{l}'_{05}, \mathbf{l}'_{06})]$$

due to

$$\mathbf{l}'_{0i} = \mathsf{Q}\boldsymbol{e}_i^{[6]}$$

also

$$\mathsf{Q} = [\mathbf{x}'_{04} \times \mathbf{x}'_{01}, \mathbf{x}'_{04} \times \mathbf{x}'_{02}, \mathbf{x}'_{04} \times \mathbf{x}'_{03}, \mathbf{x}'_{02} \times \mathbf{x}'_{03}, \mathbf{x}'_{03} \times \mathbf{x}'_{01}, \mathbf{x}'_{01} \times \mathbf{x}'_{02}]$$

as image of coordinate lines, e. g. $\boldsymbol{e}_1^{[6]} = \boldsymbol{e}_4^{[4]} \wedge \boldsymbol{e}_1^{[4]}$ can be determined from the images $\mathbf{x}'_{0i} = \mathsf{P}\boldsymbol{e}_i^{[4]}$of the coordinate points

- image of horizon: image of $\boldsymbol{e}_6^{[6]}$ (line at infinity, vertical normal)

$$\mathbf{l}'_{06} = \mathsf{Q}\boldsymbol{e}_6^{[6]} = \mathbf{x}'_{01} \times \mathbf{x}'_{02}$$

- partitioning

$$\mathsf{Q} = [\mathsf{M}, \mathsf{N}] = \mathsf{H}_\infty^{-\mathsf{T}}[-\mathsf{S}(\boldsymbol{Z})|\boldsymbol{I}]$$

as

  - image of ideal line $\mathbf{L}_\infty^\mathsf{T} = (\mathbf{0}, \boldsymbol{L}_0^\mathsf{T})$ is $\mathbf{l}'_\infty = \mathsf{N}\boldsymbol{L}_0$ must sit on $\mathbf{x}'_\infty = \mathsf{H}_\infty \mathbf{X}_\infty$ with $\mathbf{X}_\infty \in \mathbf{L}_\infty$
  - line $\mathbf{L} = \mathbf{Z} \wedge \mathbf{X} = \Pi(\mathbf{Z})\mathbf{X}$ maps to $\mathbf{0}$ for any $\mathbf{X}$ as $[-\mathsf{S}(\boldsymbol{Z})|\boldsymbol{I}]\Pi(\mathbf{Z}) = \mathbf{0}$

Compare:

$$\mathbf{x}' = \mathsf{P}\mathbf{X} \qquad \mathbf{l}' = \mathsf{Q}\mathbf{L}$$

and

$$\mathsf{P} = \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \\ \mathbf{C}^\mathsf{T} \end{bmatrix} \qquad \mathsf{Q} = \begin{bmatrix} \overline{\mathbf{B} \cap \mathbf{C}}^\mathsf{T} \\ \overline{\mathbf{C} \cap \mathbf{A}}^\mathsf{T} \\ \overline{\mathbf{A} \cap \mathbf{B}}^\mathsf{T} \end{bmatrix}$$

and

$$\mathbf{x}' = \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} < \mathbf{A}, \mathbf{X} > \\ < \mathbf{B}, \mathbf{X} > \\ < \mathbf{C}, \mathbf{X} > \end{bmatrix} \qquad \mathbf{l}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} < \mathbf{B} \cap \mathbf{C}, \mathbf{L} > \\ < \mathbf{C} \cap \mathbf{A}, \mathbf{L} > \\ < \mathbf{A} \cap \mathbf{B}, \mathbf{L} > \end{bmatrix}$$

Inverse Projection

- relation of image space to object space
- monoplotting
- reconstruction

projection plane

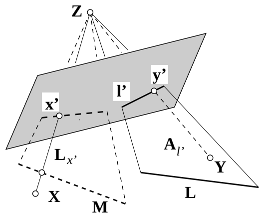
$$\mathbf{A}_{l'} = \mathsf{P}^{\mathsf{T}}\mathbf{l}' = a'\mathbf{A} + b'\mathbf{B} + c'\mathbf{C}$$

as for any $\mathbf{Y} \in \mathbf{A}_{l'}$ we have $\mathbf{A}_{l'}^{\mathsf{T}}\mathbf{Y} = \mathbf{l}'^{\mathsf{T}}\mathsf{P}\mathbf{Y} = \mathbf{l}'^{\mathsf{T}}\mathbf{y}' = 0$

projection line

$$\mathbf{L}_{x'} = \overline{\mathsf{Q}}^{\mathsf{T}}\mathbf{x}' = u'(\mathbf{B} \cap \mathbf{C}) + v'(\mathbf{C} \cap \mathbf{A}) + w'(\mathbf{A} \cap \mathbf{B})$$

as for any $\mathbf{M}$ coplanar to $\mathbf{L}_{x'}$ we have $\mathbf{L}_{x'}^{\mathsf{T}}\overline{\mathbf{M}} = \mathbf{x}'^{\mathsf{T}}\ \overline{\mathsf{Q}}\ \overline{\mathbf{M}} = \mathbf{x}'^{\mathsf{T}}\mathbf{m}' = 0$



Remark: Relation of Q to P

$$\mathbf{L}_{x'} = \mathbf{Z} \wedge \mathbf{X} = \Pi(\mathbf{Z})\mathbf{X} = \overline{\mathsf{Q}}^{\mathsf{T}}\mathbf{x}' = \overline{\mathsf{Q}}^{\mathsf{T}}\mathsf{P}\mathbf{X}$$

thus

$$\overline{\mathsf{Q}}^{\mathsf{T}}\mathsf{P} = \begin{bmatrix} -I \\ -\mathsf{S}(\boldsymbol{Z}) \end{bmatrix} \mathsf{H}_\infty^{-1}\ \ \mathsf{H}_\infty[I| - \boldsymbol{Z}] = -\begin{bmatrix} I & -\boldsymbol{Z} \\ \mathsf{S}(\boldsymbol{Z}) & \mathbf{0} \end{bmatrix} \cong \Pi(\mathbf{Z})$$

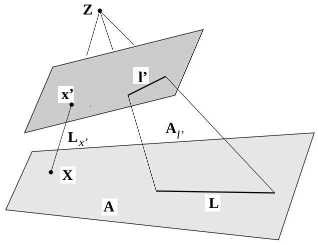
independent on H_∞

backprojection of image point to plane

$$\mathbf{X} = \mathbf{A} \cap \mathbf{L}_{x'} = \Pi^{\mathsf{T}}(\mathbf{A})\overline{\mathsf{Q}}^{\mathsf{T}}\mathbf{x}' = \mathsf{P}_A^{+}\mathbf{x}'$$

backprojection of image line to plane

$$\mathbf{L} = \mathbf{A} \cap \mathbf{A}_{l'} = \overline{\Pi}(\mathbf{A})\mathsf{P}^{\mathsf{T}}\mathbf{l}' = \mathsf{Q}_A^{+}\mathbf{l}'$$



Back projection of points and lines onto plane

backprojection to plane A: $Z = 0$

projection matrix for points and of 2D-point $\mathbf{y}'$ in reference plane

$$\mathsf{P} = [\mathbf{x}'_{01}, \mathbf{x}'_{02}, \mathbf{x}'_{03}, \mathbf{x}'_{04}] \qquad \mathbf{X} = \begin{bmatrix} U \\ V \\ 0 \\ T \end{bmatrix} \quad \rightarrow \quad \mathbf{y}' = \begin{bmatrix} U \\ V \\ T \end{bmatrix}$$

yields

$$\mathbf{x}' = \mathsf{H}_r \mathbf{y}' \qquad \text{with} \qquad \mathsf{H}_r = [\mathbf{x}'_{01}, \mathbf{x}'_{02}, \mathbf{x}'_{04}]$$

Inversion

$$\mathbf{y}' = \mathsf{H}_r^{-1} \mathbf{x}'$$

projection matrix of lines and 2D-line $\mathbf{m}'$ in reference plane

$$\mathsf{Q} = [\mathbf{l}'_{01}, \mathbf{l}'_{02}, \mathbf{l}'_{03}, \mathbf{l}'_{04}, \mathbf{l}'_{05}, \mathbf{l}'_{06}] \qquad \mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ 0 \\ 0 \\ 0 \\ L_6 \end{bmatrix} \quad \rightarrow \quad \mathbf{m}' = \begin{bmatrix} L_1 \\ L_2 \\ L_6 \end{bmatrix}$$

yields

$$\mathbf{l}' = \mathsf{H}_s \mathbf{m}' \qquad \text{with} \qquad \mathsf{H}_s = [\mathbf{l}'_{01}, \mathbf{l}'_{02}, \mathbf{l}'_{06}] = [\mathbf{x}'_{04} \times \mathbf{x}'_{01}, \mathbf{x}'_{04} \times \mathbf{x}'_{02}, \mathbf{x}'_{01} \times \mathbf{x}'_{02}]$$

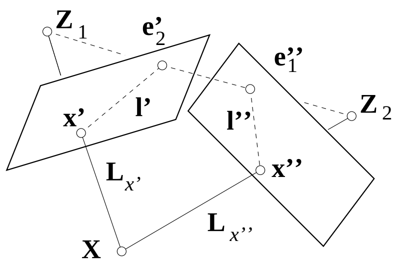
Inversion

$$\mathbf{m}' = \mathsf{H}_s^{-1} \mathbf{l}'$$

link	expression
$\mathbf{X} \underset{\mathsf{P}}{\rightarrow} \mathbf{x}'$	$\mathbf{x}' = \mathsf{P}\,\mathbf{X} = (I_3 \otimes \mathbf{X}^\mathsf{T})\ \mathrm{vec}(\mathsf{P}^\mathsf{T})$
$\mathbf{L} \underset{\mathsf{Q}}{\rightarrow} \mathbf{l}'$	$\mathbf{l}' = \mathsf{Q}\,\mathbf{L} = (I_3 \otimes \mathbf{L}^\mathsf{T})\ \mathrm{vec}(\mathsf{Q}^\mathsf{T})$
$\mathbf{x}' \underset{\overline{\mathsf{Q}}^\mathsf{T}}{\rightarrow} \mathbf{L}'$	$\mathbf{L}' = \overline{\mathsf{Q}}^\mathsf{T}\,\mathbf{x}' = (\mathbf{x}'^\mathsf{T} \otimes I_6)\ \mathrm{vec}(\overline{\mathsf{Q}}^\mathsf{T})$
$\mathbf{l}' \underset{\mathsf{P}^\mathsf{T}}{\rightarrow} \mathbf{A}'$	$\mathbf{A}' = \mathsf{P}^\mathsf{T}\,\mathbf{l}' = (\mathbf{l}'^\mathsf{T} \otimes I_4)\ \mathrm{vec}(\mathsf{P}^\mathsf{T})$
$\mathbf{x}' \underset{\mathsf{P}_A^+}{\rightarrow} \mathbf{X}$	$\mathbf{X} = \mathsf{P}_A^+\,\mathbf{x}' = (I_4 \otimes \mathbf{x}'^\mathsf{T})\ \mathrm{vec}(\mathsf{P}_A^{+\mathsf{T}})$
$\mathbf{l}' \underset{\mathsf{Q}_A^+}{\rightarrow} \mathbf{L}$	$\mathbf{L} = \mathsf{Q}_A^+\,\mathbf{l}' = (I_6 \otimes \mathbf{l}'^\mathsf{T})\ \mathrm{vec}(\mathsf{Q}_A^{+\mathsf{T}})$

Mappings into Two Images

- coplanarity constraint
- epipolar line
- point transfer via plane
- line transfer via plane



Model

$$\mathbf{x}' = \mathsf{P}_1 \mathbf{X} \qquad \mathbf{x}'' = \mathsf{P}_2 \mathbf{X}$$

with

$$\mathsf{P}_1 = \mathsf{K}_1 R_1 [I | -\boldsymbol{Z}_1] = \begin{bmatrix} \mathbf{A}_1^\mathsf{T} \\ \mathbf{B}_1^\mathsf{T} \\ \mathbf{C}_1^\mathsf{T} \end{bmatrix} \qquad \mathsf{P}_2 = \mathsf{K}_2 R_2 [I | -\boldsymbol{Z}_2] = \begin{bmatrix} \mathbf{A}_2^\mathsf{T} \\ \mathbf{B}_2^\mathsf{T} \\ \mathbf{C}_2^\mathsf{T} \end{bmatrix}$$

Coplanarity Constraint (I)

Projection rays for corresponding points

$$\mathbf{L}_{x'} = \overline{\mathsf{Q}}_1^{\mathsf{T}} \mathbf{x}' = u'(\mathbf{B}_1 \cap \mathbf{C}_1) + v'(\mathbf{C}_1 \cap \mathbf{A}_1) + w'(\mathbf{A}_1 \cap \mathbf{B}_1)$$

$$\mathbf{L}_{x''} = \overline{\mathsf{Q}}_2^{\mathsf{T}} \mathbf{x}'' = u''(\mathbf{B}_2 \cap \mathbf{C}_2) + v''(\mathbf{C}_2 \cap \mathbf{A}_2) + w''(\mathbf{A}_2 \cap \mathbf{B}_2)$$

Coplanarity constraint

$$< \mathbf{L}_{x'}, \mathbf{L}_{x''} > = \mathbf{L}_{x'}^{\mathsf{T}} \, \overline{\mathbf{L}}_{x''} = 0$$

or

$$\mathbf{x}'^{\mathsf{T}} \overline{\mathsf{Q}}_1 \mathsf{Q}_2^{\mathsf{T}} \mathbf{x}'' = 0$$

fundamental matrix

$$\underset{3\times3}{\mathsf{F}} \doteq \underset{3\times3}{\mathsf{F}_{12}} = \underset{3\times6}{\overline{\mathsf{Q}_1}} \; \underset{6\times3}{\mathsf{Q}_2^{\mathsf{T}}}$$

coplanarity constraint

$$\boxed{\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0}$$

Epipolar lines: mapping of projection ray in other image

$$\mathbf{l}'' = \mathsf{Q}_2 \overline{\mathsf{Q}_1}^{\mathsf{T}} \mathbf{x}'$$

and

$$\mathbf{l}' = \mathsf{Q}_1 \overline{\mathsf{Q}_2}^{\mathsf{T}} \mathbf{x}''$$

or

$$\mathbf{l}'' = \mathsf{F}^{\mathsf{T}} \mathbf{x}'$$

and

$$\mathbf{l}' = \mathsf{F} \mathbf{x}''$$

Epipoles: points where epipolar lines are indefinite

$$\mathsf{F}^{\mathsf{T}} \mathbf{e}_2' = \mathbf{0} \qquad \mathsf{F} \mathbf{e}_1'' = \mathbf{0}$$

Explicit expressions

from

$$\begin{aligned}\overline{\mathbf{L}}_{x'} \cdot \mathbf{L}_{x''} &= \overline{u'(\mathbf{B}_1 \cap \mathbf{C}_1) + v'(\mathbf{C}_1 \cap \mathbf{A}_1) + w'(\mathbf{A}_1 \cap \mathbf{B}_1)} \cdot \\ &\quad \cdot [u''(\mathbf{B}_2 \cap \mathbf{C}_2) + v''(\mathbf{C}_2 \cap \mathbf{A}_2) + w''(\mathbf{A}_2 \cap \mathbf{B}_2)] \\ &= 0\end{aligned}$$

and

$$[u', v', w'] \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix} = 0$$

F-matrix

$$\mathsf{F} = \begin{bmatrix} |\mathbf{B}_1, \mathbf{C}_1; \mathbf{B}_2, \mathbf{C}_2| & |\mathbf{B}_1, \mathbf{C}_1; \mathbf{C}_2, \mathbf{A}_2| & |\mathbf{B}_1, \mathbf{C}_1; \mathbf{A}_2, \mathbf{B}_2| \\ |\mathbf{C}_1, \mathbf{A}_1; \mathbf{B}_2, \mathbf{C}_2| & |\mathbf{C}_1, \mathbf{A}_1; \mathbf{C}_2, \mathbf{A}_2| & |\mathbf{C}_1, \mathbf{A}_1; \mathbf{A}_2, \mathbf{B}_2| \\ |\mathbf{A}_1, \mathbf{B}_1; \mathbf{B}_2, \mathbf{C}_2| & |\mathbf{A}_1, \mathbf{B}_1; \mathbf{C}_2, \mathbf{A}_2| & |\mathbf{A}_1, \mathbf{B}_1; \mathbf{A}_2, \mathbf{B}_2| \end{bmatrix}$$

and Epipoles: images of other projection centre

$$\mathbf{e}'_2 = \mathsf{P}_1 \mathbf{Z}_2 = \begin{bmatrix} \mathbf{A}_1^\mathsf{T} \\ \mathbf{B}_1^\mathsf{T} \\ \mathbf{C}_1^\mathsf{T} \end{bmatrix} [\mathbf{A}_2 \cap \mathbf{B}_2 \cap \mathbf{C}_2] = \begin{bmatrix} |\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2| \\ |\mathbf{B}_1, \mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2| \\ |\mathbf{C}_1, \mathbf{A}_2, \mathbf{B}_2, \mathbf{C}_2| \end{bmatrix}$$

and

$$\mathbf{e}''_1 = \mathsf{P}_2 \mathbf{Z}_1 = \begin{bmatrix} \mathbf{A}_2^\mathsf{T} \\ \mathbf{B}_2^\mathsf{T} \\ \mathbf{C}_2^\mathsf{T} \end{bmatrix} [\mathbf{A}_1 \cap \mathbf{B}_1 \cap \mathbf{C}_1] = \begin{bmatrix} |\mathbf{A}_2, \mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1| \\ |\mathbf{B}_2, \mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1| \\ |\mathbf{C}_2, \mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1| \end{bmatrix}$$

direct from result of bundle adjustment!

Coplanarity Constraint (II)

Coplanarity of basis and image vectors

Basis

$$\mathbf{b}' = \boldsymbol{B} = \boldsymbol{Z}_2 - \boldsymbol{Z}_1$$

Image vectors in object coordinate system

$$^0\mathbf{x}' = (\mathsf{K}_1 R_1)^{-1}\mathbf{x}' = R_1^{\mathsf{T}}\mathsf{K}_1^{-1}\mathbf{x}' \qquad ^0\mathbf{x}'' = (\mathsf{K}_2 R_2)^{-1}\mathbf{x}'' = R_2^{\mathsf{T}}\mathsf{K}_2^{-1}\mathbf{x}''$$

Coplanarity constraint

$$[\,^0\mathbf{x}', \mathbf{b}',\, ^0\mathbf{x}''] = \,^0\mathbf{x}' \cdot (\mathbf{b}' \times \,^0\mathbf{x}'') = 0$$

or

$$^0\mathbf{x}'^{\mathsf{T}}\, \mathsf{S}_{b'}\, ^0\mathbf{x}'' = \mathbf{x}'^{\mathsf{T}}\mathsf{K}_1^{-\mathsf{T}} R_1 \mathsf{S}_{b'} R_2^{\mathsf{T}}\mathsf{K}_2^{-1}\mathbf{x}'' = 0$$

fundamental matrix

$$\mathsf{F} = \mathsf{K}_1^{-\mathsf{T}} R_1 \mathsf{S}_{b'} R_2^{\mathsf{T}}\mathsf{K}_2^{-1}$$

thus

$$\mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' = 0$$

Calibrated cameras: K_1 and K_2 known

Image directions in camera coordinate system

$$\mathbf{m}' = \mathsf{K}_1^{-1}\mathbf{x}' \qquad \mathbf{m}'' = \mathsf{K}_2^{-1}\mathbf{x}''$$

or for Euclidean cameras

$$\mathbf{m}' = \begin{bmatrix} x' - x'_H \\ y' - y'_h \\ c' \end{bmatrix} \qquad \mathbf{m}'' = \begin{bmatrix} x'' - x''_H \\ y'' - y''_h \\ c'' \end{bmatrix}$$

coplanarity constraint

$$\mathbf{m}'^{\mathsf{T}}\mathsf{E}\mathbf{m}'' = 0$$

with **essential matrix**

$$\mathsf{E} = R_1 \mathsf{S}_{b'} R_2^{\mathsf{T}}$$

Special cases

- dependent images: $R_1 = I$

$$\mathsf{E} = \mathsf{S}_{b'} R_2$$

  - general parametrization

$$\mathbf{b}' = \begin{bmatrix} B_X \\ B_Y \\ B_Z \end{bmatrix} \qquad \text{with} \qquad |\mathbf{b}'| = 1$$

  parameters

$$(\omega'', \phi'', \kappa'', B_X, B_Y, B_Z) \qquad \text{with} \qquad B_X^2 + B_Y^2 + B_Z^2 = 1$$

  - classical, special photogrammetric parametrization

$$\mathbf{b}' = \begin{bmatrix} B_X \\ B_Y \\ B_Z \end{bmatrix} \qquad \text{with} \qquad B_X = \text{const.}$$

  parameters

$$(\omega'', \phi'', \kappa'', B_Y, B_Z)$$

- independent images

$$\mathsf{E} = R_1 \mathsf{S}_{b'} R_2^{\mathsf{T}}$$

with

$$\mathbf{b}' = \begin{bmatrix} B_X \\ 0 \\ 0 \end{bmatrix} \qquad \omega' = -\omega'' = -\frac{\Delta\omega}{2}$$

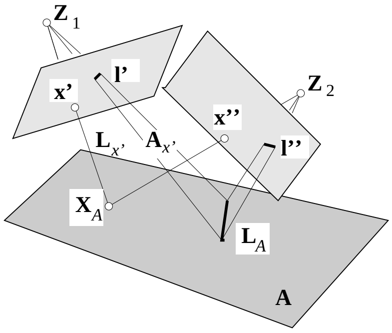
parameters

$$(\Delta\omega, \phi', \phi'', \kappa', \kappa'')$$

cameras	# O/im.	# O/pair	# RO	# AO	# CP	# CL
calibrated	6	12	5	7	≥ 3	≥ 2
un-calibrated*	11	22	7	15	≥ 5	≥ 4

Tabelle 1: Number of parameters of the orientation (O=IO+EO) of an image pair with *straight line preserving cameras. The relative orientation (RO), the absolute orientation (AO) and the minimum number of control points (CP) and control lines (CL) for an image pair.

Point Transfer via Plane



$$\mathbf{x}'' = \mathsf{H}_A \mathbf{x}' \qquad \mathbf{l}'' = \mathsf{H}_A^{-\mathsf{T}} \mathbf{l}'$$

direct

$$\mathbf{x}''_A = \mathsf{P}_2 \mathbf{X}_A = \mathsf{P}_2(\mathbf{A} \cap \mathbf{L}_{x'}) = \mathsf{P}_2 \Pi^{\mathsf{T}}(\mathbf{A}) \overline{\mathsf{Q}_1} \mathbf{x}' = \mathsf{P}_2 \mathsf{P}_{1}{}^{+}_{A} \mathbf{x}' = \mathsf{H}_A \mathbf{x}'$$

with

$$\mathsf{H}_A = \mathsf{P}_2 \mathsf{P}_{1}{}^{+}_{A} = \mathsf{P}_2 \Pi^{\mathsf{T}}(\mathbf{A}) \overline{\mathsf{Q}_1}$$

line transfer via plane **A** (dual homography)

$$\mathbf{l}''_A = \mathsf{Q}_2 \mathbf{L}_A = \mathsf{Q}_2(\mathbf{A} \cap \mathbf{A}_{l'}) = \mathsf{Q}_2 \overline{\Pi}(\mathbf{A}) \mathsf{P}_1^{\mathsf{T}} \mathbf{l}' = \mathsf{Q}_2 \mathsf{Q}_{1}{}^{+}_{A} = \mathsf{H}_A^{-\mathsf{T}} \mathbf{l}'$$

with

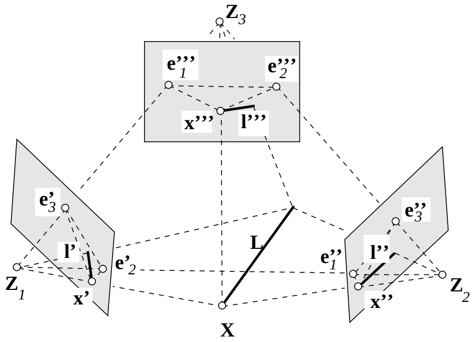
$$\mathsf{H}_A^{-\mathsf{T}} = \mathsf{Q}_2 \mathsf{Q}_{1}{}^{+}_{A} = \mathsf{Q}_2 \overline{\Pi}(\mathbf{A}) \mathsf{P}_1^{\mathsf{T}}$$

link	expression
$\mathbf{x}' \underset{\mathsf{F}}{\rightarrow} \mathbf{l}''$	$\mathbf{l}'' = \mathsf{F}^\mathsf{T}\mathbf{x}' = (\mathbf{x}'^\mathsf{T} \otimes I_3)\mathrm{vec}(\mathsf{F}^\mathsf{T})$
$\mathbf{x}' \underset{\mathsf{H}_A}{\rightarrow} \mathbf{x}''$	$\mathbf{x}'' = \mathsf{H}_A\mathbf{x}' = (I_3 \otimes \mathbf{x}'^\mathsf{T})\mathrm{vec}(\mathsf{H}_A^\mathsf{T})$
$\mathbf{l}' \underset{\mathsf{H}_A^{-T}}{\rightarrow} \mathbf{l}''$	$\mathbf{l}'' = \mathsf{H}_A^{-T}\mathbf{l}' = (\mathbf{l}'^\mathsf{T} \otimes I_3)\mathrm{vec}(\mathsf{H}_A^{-T})$

Mappings into Three Images

- point transfer, point constraints
- line transfer, line constraints
- estimation of relative orientation of triplet

Geometry of the Triplet

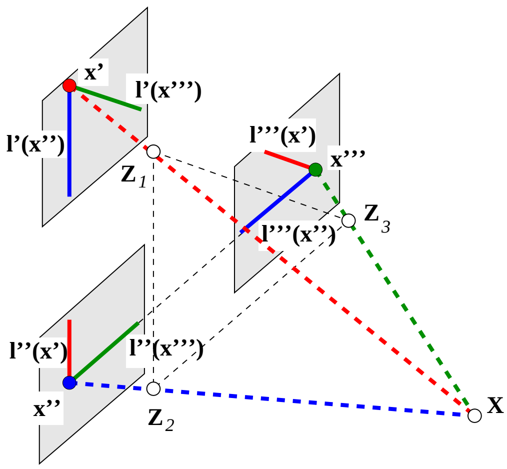


number of parameters

cameras	# O/im.	# O/triplet	# RO	# AO	# CP	# CL
calibrated	6	18	11	7	≥ 3	≥ 2
un-calibrated*	11	33	18	15	≥ 5	≥ 4

Tabelle 2: Number of parameters of the orientation of an image triplet (O=EO+IO) with *straight line preserving cameras. The relative orientation (RO), the absolute orientation (AO) and the minimum number of control points (CP) and control lines (CL) for an image pair.

Transfer of points



three fundamental matrices, three coplanarity constraints:

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}} \mathsf{F}_{12} \mathbf{x}'' &= 0 \\ \mathbf{x}''^{\mathsf{T}} \mathsf{F}_{23} \mathbf{x}''' &= 0 \\ \mathbf{x}'''^{\mathsf{T}} \mathsf{F}_{31} \mathbf{x}' &= 0 \end{aligned}$$

Prediction of point

$$\mathbf{x}' = \mathbf{l}'(\mathbf{x}'') \cap \mathbf{l}'(\mathbf{x}''') = \mathsf{F}_{12}\mathbf{x}'' \times \mathsf{F}_{13}\mathbf{x}'''$$

or two constraints

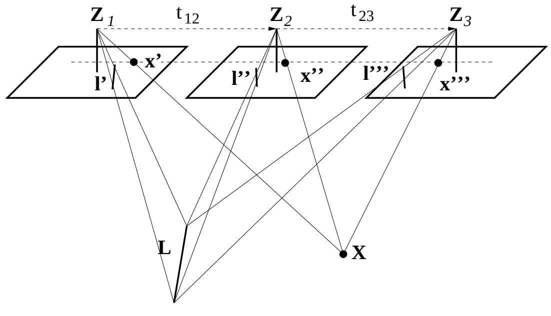
$$\mathbf{x}' \times (\mathsf{F}_{12}\mathbf{x}'' \times \mathsf{F}_{13}\mathbf{x}''') = \mathbf{0}$$

+ third constraint:

$$\mathbf{x}''^{\mathsf{T}} \mathsf{F}_{23} \mathbf{x}''' = 0$$

Problem with

- points in the trifocal plane $(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$
- collinear projection centres !



Line Prediction

$$\mathbf{l}' = \mathcal{T}(\mathbf{l}'', \mathbf{l}''') \doteq \mathsf{Q}_1 \mathbf{L} = \mathsf{Q}_1(\mathbf{A}'' \cap \mathbf{A}''') = \mathsf{Q}_1 \Pi(\mathsf{P}_2^\mathsf{T} \mathbf{l}'') \mathsf{P}_3^\mathsf{T} \mathbf{l}'''$$

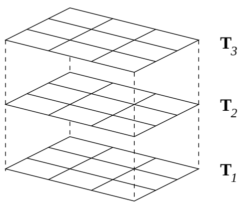
linear in $\mathbf{l}''$, $\mathbf{l}'''$, P_2 and P_3

can be written as three bilinear forms

$$\mathbf{l}' = \begin{bmatrix} \mathbf{l}''^\mathsf{T} \mathsf{T}_1 \mathbf{l}''' \\ \mathbf{l}''^\mathsf{T} \mathsf{T}_2 \mathbf{l}''' \\ \mathbf{l}''^\mathsf{T} \mathsf{T}_3 \mathbf{l}''' \end{bmatrix}$$

with *trifocal tensor* with $3 \times 3 \times 3$ elements, three 3×3 matrices T_i (referring to 1st image)

$$\mathcal{T} = (\mathsf{T}_i) = (T_{i,jk})$$



$$\mathbf{l}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$$

and e. g.

$$a' = (\mathbf{B}_1 \cap \mathbf{C}_1)(a''\mathbf{A}_2 + b''\mathbf{B}_2 + c''\mathbf{C}_2)^\mathsf{T}(a'''\mathbf{A}_3 + b'''\mathbf{B}_3 + c'''\mathbf{C}_3)$$

or

$$a' = (a'', b'', c'') \begin{bmatrix} |\mathbf{B}_1, \mathbf{C}_1, \mathbf{A}_2, \mathbf{A}_3| & |\mathbf{B}_1, \mathbf{C}_1, \mathbf{A}_2, \mathbf{B}_3| & |\mathbf{B}_1, \mathbf{C}_1, \mathbf{A}_2, \mathbf{C}_3| \\ |\mathbf{B}_1, \mathbf{C}_1, \mathbf{B}_2, \mathbf{A}_3| & |\mathbf{B}_1, \mathbf{C}_1, \mathbf{B}_2, \mathbf{B}_3| & |\mathbf{B}_1, \mathbf{C}_1, \mathbf{B}_2, \mathbf{C}_3| \\ |\mathbf{B}_1, \mathbf{C}_1, \mathbf{C}_2, \mathbf{A}_3| & |\mathbf{B}_1, \mathbf{C}_1, \mathbf{C}_2, \mathbf{B}_3| & |\mathbf{B}_1, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3| \end{bmatrix} \begin{bmatrix} a''' \\ b''' \\ c''' \end{bmatrix}$$

or

$$a' = \mathbf{l}''^\mathsf{T} \mathsf{T}_1 \mathbf{l}'''$$

T_2 and T_3 with first two elements of det. exchanged by $\mathbf{C}_1, \mathbf{A}_1$ and $\mathbf{A}_1, \mathbf{B}_1$

Change to tensor notation

Rules:

- Einstein's summation rule
  $\mathbf{x}^\mathsf{T}\mathbf{l} = d \qquad x_1 l_1 + x_2 l_2 + x_3 l_3 = d = \sum_{i=1}^3 x_i l_i$ replaced by: summation over identical indices
  $$d = x_i l_i$$
  Examples: Index $i = 1, 2, 3$, index $m = 1, 2, 3, 4$
  $$\mathbf{x}' = \mathsf{P}\mathbf{X} \qquad \text{or} \qquad x'_i = P_{im} X_m$$
  or
  $$\mathbf{x}'^\mathsf{T}\mathsf{F}\mathbf{x}'' = 0 \qquad \text{or} \qquad x'_i F_{ij} x''_j = 0 \qquad \text{or} \qquad F_{ij} x'_i x''_j = 0$$
- Transposition of a matrix: exchange of indices
  $$\mathbf{A}_{l'} = \mathsf{P}^\mathsf{T}\mathbf{l}' \qquad \text{or} \qquad A_m = P_{mi} l'_i$$

- Convention: Indices
  $$i, j, k \in \{1, 2, 3\}$$

Relation between three corresponding lines and Tensor

$$l'_i \qquad l''_j \qquad l'''_k \qquad T_{ijk}$$

now

$$l'_i = T_{ijk} l''_j l'''_k$$

as

$$a' = l_1 = \sum_{jk} T_{1jk} l''_j l'''_k = \mathbf{l}''^\mathsf{T}\mathsf{T}_1\mathbf{l}'''$$
$$b' = l_2 = \sum_{jk} T_{2jk} l''_j l'''_k = \mathbf{l}''^\mathsf{T}\mathsf{T}_2\mathbf{l}'''$$
$$c' = l_3 = \sum_{jk} T_{3jk} l''_j l'''_k = \mathbf{l}''^\mathsf{T}\mathsf{T}_3\mathbf{l}'''$$

corresponding points in three images

We fist have for point $\mathbf{x}'$ on $\mathbf{l}'$

$$\mathbf{x}'^\mathsf{T}\mathbf{l}' = 0 \qquad x'_i T_{ijk} l''_j l'''_k = 0 \qquad \underbrace{T_{ijk} x'_i l''_j}_{x'''_k} l'''_k = 0$$

For point $\mathbf{x}'''$ on $\mathbf{l}'''$ we have

$$\mathbf{x}'''^\mathsf{T}\mathbf{l}''' = 0 \qquad \text{or} \qquad x'''_k l'''_k = 0$$

By comparison we obtain prediction of $\mathbf{x}'''$ from $\mathbf{x}'$ and *any* line $\mathbf{l}''$ through $\mathbf{x}''$

$$x'''_k = T_{ijk} x'_i l''_j$$

if projection centres colliner
and $\mathbf{l}''$ is perpendicular to epipolar line:
coplanarity constraint of $\mathbf{x}'$ and $\mathbf{x}'''$

linear in $T_{ijk} \rightarrow$

Tensor for prediction and estimation

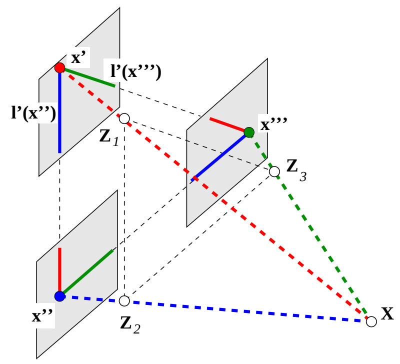
Points outside the trifocal plane

- predicition of point

$$\mathbf{x}' = \mathbf{l}'(\mathbf{x}'') \cap \mathbf{l}'(\mathbf{x}''') = \mathsf{F}_{12}\mathbf{x}'' \times \mathsf{F}_{13}\mathbf{x}'''$$

- constraints for points

$$\mathbf{x}'^{\mathsf{T}}\mathsf{F}_{12}\mathbf{x}'' = 0 \qquad \mathbf{x}''^{\mathsf{T}}\mathsf{F}_{23}\mathbf{x}''' = 0 \qquad \mathbf{x}'''^{\mathsf{T}}\mathsf{F}_{31}\mathbf{x}' = 0$$



Points close to the trifocal plane

- predicition of point

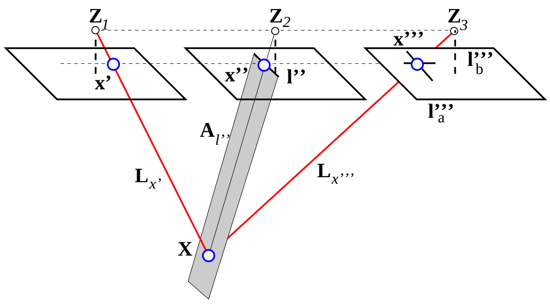
$$\mathbf{x}''' = \mathsf{P}_3(\mathbf{L}_{x'} \cap \mathbf{A}_{l''}) = \mathsf{P}_3 \Gamma(\overline{\mathsf{Q}}_1^{\mathsf{T}} \mathbf{x}') \mathsf{P}_2^{\mathsf{T}} \mathbf{l}'' \quad \text{or} \quad x_k''' = T_{ijk} x_i' l_j''$$

- constraints for points

$$\mathbf{x}''' \times \mathsf{P}_3(\mathbf{L}_{x'} \cap \mathbf{A}_{l''}) = \mathsf{S}(\mathbf{x}''') \mathsf{P}_3 \Gamma(\overline{\mathsf{Q}}_1^{\mathsf{T}} \mathbf{x}') \mathsf{P}_2^{\mathsf{T}} \mathbf{l}'' \mathbf{0} \qquad \text{or} \qquad T_{ijk} x_i' l_j'' l_k''' = 0$$

with two lines $\mathbf{l}_k'''$ passing through $\mathbf{x}'''$ and

$$\mathbf{x}'^{\mathsf{T}} \mathsf{F}_{12} \mathbf{x}'' = 0$$



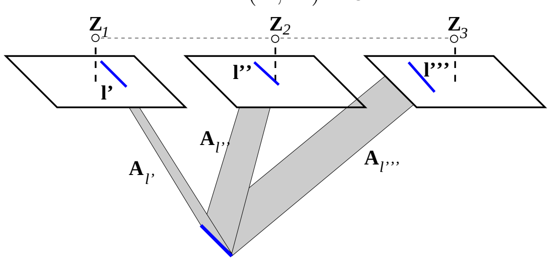
Lines off the trifocal plane

- predicition of line

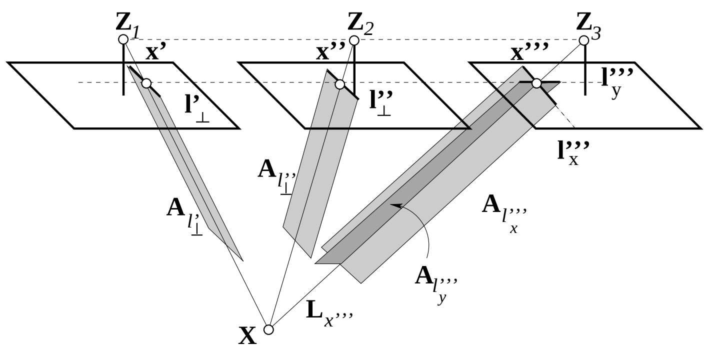
$$\mathbf{l}' = \mathcal{T}(\mathbf{l}'', \mathbf{l}''') \doteq \mathsf{Q}_1(\mathbf{A}'' \cap \mathbf{A}''') = \mathsf{Q}_1 \Pi(\mathsf{P}_2^{\mathsf{T}} \mathbf{l}'') \mathsf{P}_3^{\mathsf{T}} \mathbf{l}'''$$

- constraints for lines

$$\mathbf{l}' \times \mathcal{T}(\mathbf{l}'', \mathbf{l}''') = \mathbf{0}$$



prediction of point $\mathbf{x}'$ in the first image from $\mathbf{l}_\perp''$ and $\mathbf{x}'''$



- projection plane in second image

$$\mathbf{A}_{l''_\perp} = \mathsf{P}_2^\mathsf{T} \mathbf{l}''_\perp$$

of line

$$\mathbf{l}''_\perp = \begin{bmatrix} b''w'' \\ -a''w'' \\ a''v'' - b''u'' \end{bmatrix}$$

perpendicular to epipolar line $\mathbf{l}''(\mathbf{x}''') = (a'', b'', c'')^\mathsf{T}$ through image point $\mathbf{x}'' = (u'', v'', w'')^\mathsf{T}$

- projection line in third image

$$\mathbf{L}_{x'''} = \mathbf{A}_{l'''_x} \cap \mathbf{A}_{l'''_y}$$

possibly based on coordinate lines $\mathbf{l}'''_x = \mathbf{x}''' \times \boldsymbol{e}_1^{[3]}$ and $\mathbf{l}'''_y = \mathbf{x}''' \times \boldsymbol{e}_2^{[3]}$

$$\mathbf{x}' = \mathsf{P}_1 \mathbf{X} = \mathsf{P}_1(\mathbf{A}_{l''_\perp} \cap \mathbf{L}_{x'''}) = \mathsf{P}_1 \Pi^\mathsf{T}(\mathsf{P}_2^\mathsf{T} \mathbf{l}_{\perp''}) \mathsf{Q}_3^\mathsf{T} \mathbf{x}'''$$

or

$$\mathbf{x}' = \mathsf{T}(\mathbf{l}''_\perp, \mathbf{l}'''_x) \times \mathsf{T}(\mathbf{l}''_\perp, \mathbf{l}'''_y)$$

constraint

$$|\mathsf{P}_1^\mathsf{T} \mathbf{l}'_\perp, \mathsf{P}_2^\mathsf{T} \mathbf{l}''_\perp, \mathsf{P}_3^\mathsf{T} \mathbf{l}'''_x, \mathsf{P}_3^\mathsf{T} \mathbf{l}'''_y| = 0$$

linear in $\mathbf{x}'$, $\mathbf{x}''$, $\mathbf{x}'''$, P_1 and P_2, quadratic in P_3

linear in entries of $\mathcal{T}$ (referring to 3rd image)

Summary

- Simple relations (distances, signs)
- constructions, bilinear
- interpretation of involved matrices
- single view geometry:
  - simple, linear
  - relation to classical representation
  - simple inversion

- two image view geometry:
  - explicit expression for colinearity equation
  - relation to projection matrices
  - transfer of points and lines
- three image view geometry:
  - explicit expressions for predictions and constraints
  - inclusion of points and lines
  - explicit in the parameters of the I. O. and E. O.

——— **B r e a k** ———

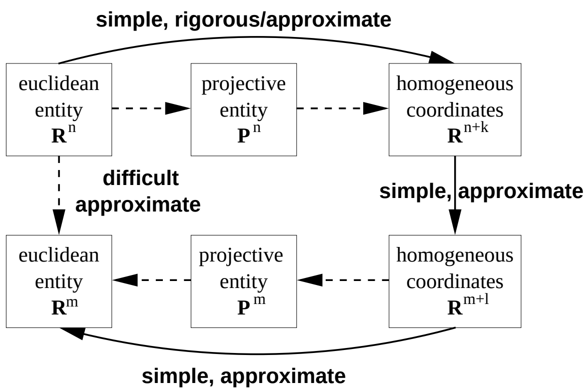
Uncertain geometric elements

Assumption:

Usefulness of homogeneous representation
Extension of representation by uncertainty

Uncertainty of Homogeneous Vectors

Principle:



Uncertainty of Geometric Entities

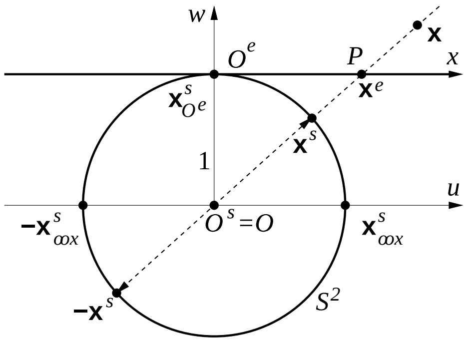
What is uncertainty of points in homogeneous coordinates?

Equivalence classes (arbitrary scaling)

$$\mathbf{x} \equiv \mathbf{y} \quad \text{iff } \mathbf{x} = \lambda \mathbf{y}$$

projective points in $\mathbb{P}^n$ are straight lines through O in $\mathbb{R}^{n+1}$

$\mathbf{x}$, $\mathbf{x}^s$ and $\mathbf{x}^e$ represent the same point



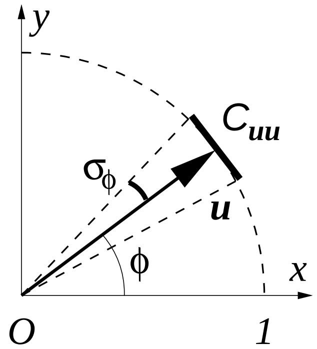
Normalization of homogeneous vector (euclidean, spherical)

$$\mathbf{x}^e = \frac{\mathbf{x}}{x_h} \quad \text{if } x_h \neq 0 \qquad\qquad \mathbf{x}^s = \frac{\mathbf{x}}{|\mathbf{x}|} = \mathsf{N}(\mathbf{x})$$

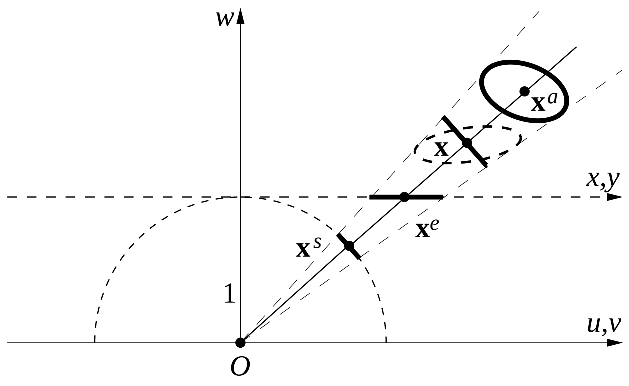
Uncertainty of a straight line?

Uncertainty of a direction?

Uncertainty of direction in plane
v. Mises distribution, uncertainty of direction vector



Uncertain directions in $\mathbb{R}^3$



Equivalence of points (alternative)

$$\mathbf{x} \equiv \mathbf{y} \quad \text{iff} \quad \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{y}}{|\mathbf{y}|} \quad \text{or} \quad \mathsf{N}(\mathbf{x}) = \mathsf{N}(\mathbf{y})$$

Equivalence of uncertain points (cumulative pdf's)

$$\underline{\mathbf{x}} \equiv \underline{\mathbf{y}} \quad \text{iff} \quad cpdf[\mathsf{N}(\underline{\mathbf{x}})] = cpdf[\mathsf{N}(\underline{\mathbf{y}})]$$

Equivalence of uncertain points with *Gaussian distribution* of $\mathbf{x}$
(rigorous, too narrow for practical applications)

$$\underline{\mathbf{x}} \equiv \underline{\mathbf{y}} \quad \text{iff} \quad \boldsymbol{\mu}_x = \lambda \boldsymbol{\mu}_y \quad \text{and} \quad \Sigma_{xx} = \lambda^2 \Sigma_{yy}$$

Equivalence of uncertain points with Gaussian distribution of $\mathbf{x}$
(approximate)

$$\underline{\mathbf{x}} \equiv \underline{\mathbf{y}} \quad \text{iff} \quad \boldsymbol{\mu}_{x^s}(\boldsymbol{\mu}_x) = \boldsymbol{\mu}_{y^s}(\boldsymbol{\mu}_y) \quad \text{and} \quad J_x \Sigma_{xx} J_x^{\mathsf{T}} = J_y \Sigma_{yy} J_y$$

with

$$J_x = \frac{1}{\sqrt{\mathbf{x}^{\mathsf{T}}\mathbf{x}}}(I - \mathbf{x}(\mathbf{x}^{\mathsf{T}}\mathbf{x})^{-1}\mathbf{x}^{\mathsf{T}})$$

Proof: One can show $\Sigma_{x^s x^s} = J_x \Sigma_{xx} J_x^{\mathsf{T}}$ up to second order terms
ideal rank(covariance matrix) = d. o. f.

Representation of uncertain geometric entities

uncertain points $\mathbf{x}$ and lines $\mathbf{l}$ in the plane (2 d. o. f.) $\rightarrow$

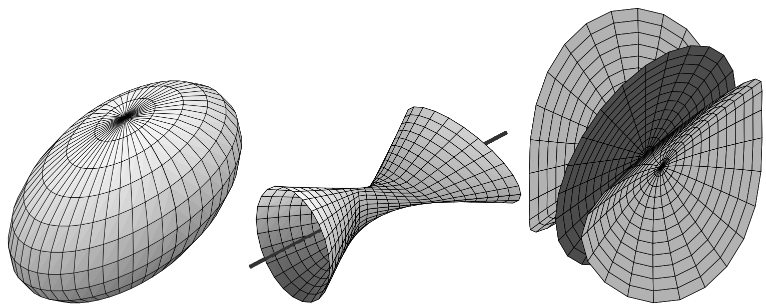
$$[\underset{3\times 1}{\mathbf{x}}, \underset{3\times 3}{\Sigma_{xx}}] \qquad [\underset{3\times 1}{\mathbf{l}}, \underset{3\times 3}{\Sigma_{ll}}]$$

uncertain points $\mathbf{X}$, lines $\mathbf{L}$ and planes $\mathbf{A}$ in space (3, 4, and 3 d. o. f.) $\rightarrow$

$$[\underset{4\times 1}{\mathbf{X}}, \underset{4\times 4}{\Sigma_{XX}}] \qquad [\underset{6\times 1}{\mathbf{L}}, \underset{6\times 6}{\Sigma_{LL}}] \qquad [\underset{4\times 1}{\mathbf{A}}, \underset{4\times 4}{\Sigma_{AA}}]$$

uncertain projection parameters (11 d. o. f.)

$$[\underset{12\times 1}{\mathbf{p}}, \underset{12\times 12}{\Sigma_{pp}}] \qquad [\underset{18\times 1}{\mathbf{q}}, \underset{18\times 18}{\Sigma_{qq}}]$$



Conditioning and Normalization

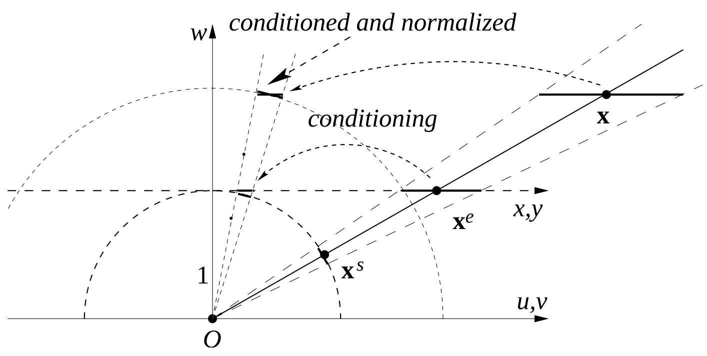
Not meaningful homogeneous vectors:

$$\mathbf{x} = \begin{bmatrix} 5325395.10 \\ 4238023.34 \\ 1 \end{bmatrix} \qquad \mathbf{l} = \begin{bmatrix} 0.3609 \\ 0.9326 \\ 407358.35 \end{bmatrix}$$

Effects:

- numerical instability
- bias in statistical reasoning

Means: Conditioning (sometimes also called normalization, cf. Hartley 1995)



Conditioning:

Example:

$$\begin{bmatrix} \dot{\boldsymbol{x}}_0 \\ \dot{x}_h \end{bmatrix} = \dot{\mathbf{x}} = W(f)\,\mathbf{x} \qquad \text{with} \qquad W(f) = \begin{bmatrix} f\,\mathsf{I}_{2\times 2} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\Sigma_{\dot{x}\dot{x}} = \mathsf{W}(f)\ \Sigma_{xx}\ \mathsf{W}(f)$$

- increases numerical stability
- decreases bias

If $|\dot{\boldsymbol{x}}_0|/|\dot{x}_h| \leq 0.1$ bias is negligible.

General rule for geometric entity with

homogeneous part $\mathbf{x}_h$
inhomogeneous part $\mathbf{x}_0$

- Center the data
- Condition the data and guarantee

$$\frac{|\dot{\mathbf{x}}_0|}{|\dot{\mathbf{x}}_h|} \leq 0.1$$

For *transformations*:

$$\mathbf{x}' = \mathsf{H}\mathbf{x}$$

$$\dot{\mathsf{H}} = \mathsf{W}(f')\ \mathsf{H}\ \mathsf{W}^{-1}(f)$$

Normalization

transform to spherically normalized coordinates

$$\underline{\mathbf{x}}^n = |\mathbf{x}| \, \frac{\underline{\mathbf{x}}}{|\underline{\mathbf{x}}|}$$

with Jacobian matrix

$$P_x = I - \frac{\mathbf{x}\mathbf{x}^\mathsf{T}}{\mathbf{x}^\mathsf{T}\mathbf{x}}$$

thus

$$E(\underline{\mathbf{x}}^n) = E(\underline{\mathbf{x}})$$

and

$$\Sigma_{x^n x^n} = P_x \Sigma_{xx} P_x$$

Construction of Uncertain Elements

Constructions cf. slide **??**

uncertain construction (bilinear)

$$\underline{\mathbf{c}} = \mathsf{U}(\underline{\mathbf{a}})\underline{\mathbf{b}} = \mathsf{V}(\underline{\mathbf{b}})\underline{\mathbf{a}}$$

then

$$\left[\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}\right] \rightarrow [\mathbf{c}, \Sigma_{cc}]$$

$$\Sigma_{cc} = \mathsf{U}(\mathbf{a})\Sigma_{bb}\mathsf{U}^\mathsf{T}(\mathbf{a}) + \mathsf{V}(\mathbf{b})\Sigma_{ab}\mathsf{U}^\mathsf{T}(\mathbf{a}) + \mathsf{U}(\mathbf{a})\Sigma_{ba}\mathsf{V}^\mathsf{T}(\mathbf{b}) + \mathsf{V}(\mathbf{b})\Sigma_{aa}\mathsf{V}^\mathsf{T}(\mathbf{b})$$

simple error propagation independent on distribution

Degree of approximation: relative bias in $\boldsymbol{\mu}$ and σ^2 = directional uncertainty (cf. slide **??**)

Example:

Given

$$(\boldsymbol{x}, \Sigma_{xx}), (\boldsymbol{y}, \Sigma_{yy})$$

Joining line

$$\mathbf{l} = \mathbf{x} \times \mathbf{y} = \mathsf{S}_x\mathbf{y} = -\mathsf{S}_y\mathbf{x}$$

Covariance matrix

$$\Sigma_{\mathrm{ll}} = \mathsf{S}_x \Sigma_{\mathrm{yy}} \mathsf{S}_x^\mathsf{T} + \mathsf{S}_y \Sigma_{\mathrm{xx}} \mathsf{S}_y^\mathsf{T}$$

with

$$\Sigma_{\mathrm{xx}} = \begin{bmatrix} \Sigma_{xx} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \qquad \Sigma_{\mathrm{yy}} = \begin{bmatrix} \Sigma_{yy} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}$$

Assumption

$$\mathbf{p} = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \qquad \Sigma_{pp} = \Sigma_{qq} = \sigma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$\Sigma_{ll} = \sigma^2 \begin{pmatrix} 2 & 0 & -x_1 - x_2 \\ 0 & 2 & -y_1 - y_2 \\ -x_1 - x_2 & -y_1 - y_2 & {y_1}^2 + {x_1}^2 + {y_2}^2 + {x_2}^2 \end{pmatrix}$$

with determinant

$$|\Sigma_{ll}| = 2\,\sigma^6\,[(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

if $\mathbf{x} \not\equiv \mathbf{y}$ then $|\Sigma_{ll}| \neq 0$,
thus homogeneous vector with full rank covariance matrix

Linear Estimation from Constraints

many types of constraints:

- incidence, identity
- parallelity, orthogonality
- distance, ...

can be used for determination/estimation

of *geometric entities*

and

of *transformations*

Gauß-Helmert model

- N observed entities $\boldsymbol{l} = (l_n)$
- U unknown geometric entities $\boldsymbol{x} = (x_u)$
- G geometric constraints
  $$\boldsymbol{g}(\widehat{\boldsymbol{l}}, \widehat{\boldsymbol{x}}) = \mathbf{0}$$
  for fitted/corrected observations$\widehat{\boldsymbol{l}}$ and unknowns $\widehat{\boldsymbol{x}}$
- H additional constraints on parameters
  $$\boldsymbol{h}(\widehat{\boldsymbol{x}}) = \mathbf{0}$$

... *in our context:*

constraints almost always linear in unknowns, thus

$$\boldsymbol{g}(\widehat{\boldsymbol{l}}, \widehat{\boldsymbol{x}}) = \mathsf{A}(\widehat{\boldsymbol{l}})\widehat{\boldsymbol{x}} = \mathbf{0}$$

additional constraints mostly

$$h_1(\widehat{\boldsymbol{x}}) \equiv |\widehat{\boldsymbol{x}}|^2 - 1 = 0$$

possibly Plücker constraint $< \mathbf{L}, \mathbf{L} >= \mathbf{L}^{\mathsf{T}} \mathsf{D}_6 \mathbf{L} = 0$

$$h_2(\widehat{\boldsymbol{x}}) \equiv \widehat{\boldsymbol{x}}^{\mathsf{T}} \mathsf{D} \widehat{\boldsymbol{x}} = 0$$

Example:

- Intersection point $\mathbf{x}$ of I lines $\mathbf{l}_i$ (cf. slide **??**)

$$\mathbf{l}_i^\mathsf{T}\mathbf{x} = 0 \qquad i = 1, ..., I$$

with

$$|\mathbf{x}|^2 = \mathbf{x}^\mathsf{T}\mathbf{x} = 1$$

- Mapping of I 2D-points $\mathbf{x}_i$, $\mathbf{x}'_i = \mathsf{H}\mathbf{x}_i$ (cf. slide p. **??**)

$$\mathbf{x}'_i = (I_3 \otimes \mathbf{x}_i^\mathsf{T})\text{vec}(\mathsf{H}^\mathsf{T}) \qquad \text{or} \qquad (\mathsf{S}_{x'_i} \otimes \mathbf{x}_i^\mathsf{T})\mathbf{h} = \mathbf{0} \qquad i = 1, ..., I$$

with

$$|\mathbf{h}|^2 = \mathbf{h}^\mathsf{T}\mathbf{h} = 1$$

Direct Estimation with Algebraic Minimization

- constraints linear in unknown parameters
- redundant, selection of constraints cf. below

Algebraic minimization:

$$\widehat{\boldsymbol{x}}^\mathsf{T} A^\mathsf{T}(\boldsymbol{l}) \cdot A(\boldsymbol{l})\widehat{\boldsymbol{x}} \rightarrow \min \qquad |\widehat{\boldsymbol{x}}| = 1$$

Solution

$$\mu\widehat{\boldsymbol{x}} = \left[A^\mathsf{T}(\boldsymbol{l}) \cdot A(\boldsymbol{l})\right]\widehat{\boldsymbol{x}} \qquad |\widehat{\boldsymbol{x}}| = 1$$

thus using singular value decomposition (SVD)

$$\underset{N \times U}{A(\boldsymbol{l})} = \underset{N \times N}{U} \; \underset{N \times U}{\boldsymbol{\Lambda}} \; \underset{U \times U}{V^\mathsf{T}}$$

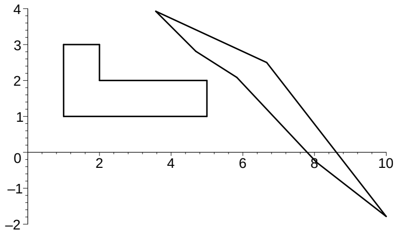
then optimal estimate is last column $\boldsymbol{v}_U$ of $V = (\boldsymbol{v}_1, ..., \boldsymbol{v}_U)$

$$\widehat{\boldsymbol{x}} = \boldsymbol{v}_U \qquad \text{or} \qquad \widehat{x}_u = V_{uU}$$

corresponding to smallest singular value

not optimal, arbitrary scaling of constraints → *conditioning* (below)

Numerical Example



Rectangular polygon and its projective image.

Given homography

$$\widetilde{\mathsf{H}} = \begin{bmatrix} 5 & 1 & 2 \\ -2 & 4 & 1 \\ 0.4 & 0.8 & 0 \end{bmatrix}$$

estimated parameters, normalized, such that $|\widehat{\mathsf{H}}| = 1$

$$\widehat{\mathsf{H}} = \begin{bmatrix} 0.69408 & 0.13884 & 0.27949 \\ 0.27798 & 0.55555 & 0.13975 \\ 0.055488 & 0.11111 & 0.00026795 \end{bmatrix}$$

estimated homography, normalized such that $H_{12} = 1$.

$$\widehat{\mathsf{H}} = \begin{bmatrix} 4.9991 & \mathbf{1.0000} & 2.0130 \\ -2.0022 & 4.0013 & 1.0065 \\ 0.39965 & 0.80027 & 0.0019299 \end{bmatrix}$$

for comparison with

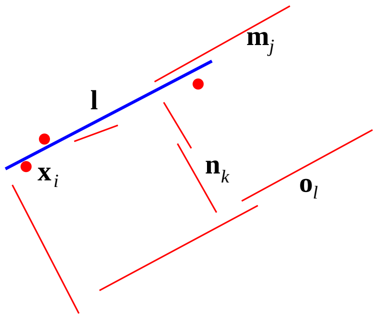
$$\widetilde{\mathsf{H}} = \begin{bmatrix} 5 & \mathbf{1} & 2 \\ -2 & 4 & 1 \\ 0.4 & 0.8 & 0 \end{bmatrix}$$

Example on Versatility:

Unknown: 2D-line $\mathbf{l}$

Given:

- I points $\mathbf{x}_i$ on line $\mathbf{l}$
- J lines $\mathbf{m}_j$ collinear to $\mathbf{l}$
- K lines $\mathbf{n}_k$ normal to $\mathbf{l}$
- L lines $\mathbf{o}_l$ parallel to $\mathbf{l}$



Constraints:

- I points $\mathbf{x}_i$ on line $\mathbf{l}$

$$\mathbf{x}_i^\mathsf{T}\mathbf{l} = 0$$

- J lines $\mathbf{m}_j$ collinear to $\mathbf{l}$

$$\mathsf{S}(\mathbf{m}_j)\mathbf{l} = \mathbf{0}$$

- K lines $\mathbf{n}_k$ normal to $\mathbf{l}$

$$\boldsymbol{n}_{hk}^\mathsf{T}\boldsymbol{l}_h = \mathbf{n}_k^\mathsf{T} Y \mathbf{l} = 0$$

with

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- L lines $\mathbf{o}_l$ parallel to $\mathbf{l}$

$$|\boldsymbol{o}_{hl}, \boldsymbol{l}_h|_l = \mathbf{o}_l^\mathsf{T} Z \mathbf{l} = 0$$

with

$$Z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: homogeneous equation system

$$\begin{bmatrix} (\mathbf{x}_i^\mathsf{T}) \\ (\mathsf{S}(\mathbf{m}_j)) \\ (\mathbf{n}_k^\mathsf{T} Y) \\ (\mathbf{o}_l^\mathsf{T} Z) \end{bmatrix} \mathbf{l} = \mathbf{0}$$

$\widehat{\mathbf{l}}$ is right eigenvector of $(I + 3J + 3K + 3L) \times 3$-matrix

Orientation procedures

Orientation of a single image

given: 3D-points $\mathbf{X}_i, i = 1, ..., I$; 3D-lines $\mathbf{L}_j, j = 1, ..., J$
observed: image points $\mathbf{x}_i, i = 1, ..., I$; image lines $\mathbf{l}_j, j = 1, ..., J$
unknown: projection matrix P

constraint for *corresponding points* (2 d. o. f.)

$$\mathsf{S}(\mathbf{x}'_i)\mathsf{P}\ \ \mathbf{X}_i = -\mathsf{S}(\mathsf{P}\mathbf{X}_i)\ \ \mathbf{x}'_i = (\mathsf{S}(\mathbf{x}'_i) \otimes \mathbf{X}_i^\mathsf{T})\ \ \mathbf{p} = \mathbf{0}$$

from $\mathbf{x}'_i = \mathsf{P}\mathbf{X}_i$
linear in $\mathbf{X}_i$, $\mathbf{x}'_i$, and $\mathbf{p} = \text{vec}(\mathsf{P}^\mathsf{T})$

explicitely

$$\underbrace{\begin{bmatrix} \mathbf{0}^\mathsf{T} & -\mathbf{X}_i^\mathsf{T} & y_i\mathbf{X}_i^\mathsf{T} \\ -\mathbf{X}_i^\mathsf{T} & \mathbf{0}^\mathsf{T} & -x_i\mathbf{X}_i^\mathsf{T} \\ -y_i\mathbf{X}_i^\mathsf{T} & x_i\mathbf{X}_i^\mathsf{T} & \mathbf{0}^\mathsf{T} \end{bmatrix}}_{3\times 12} \mathbf{p} = \mathbf{0}$$

with 12-vector

$$\mathbf{p} = \text{vec}(\mathsf{P}^\mathsf{T}) = [p_{11}, p_{12}, ..., p_{34}]^\mathsf{T}$$

only two constraints per point are necessary

constraint for *corresponding lines* (2 d.o.f.)

$$\mathsf{S}(\mathbf{l}'_j)\mathsf{Q}\mathbf{L}_j = \mathbf{0} \qquad \text{from} \qquad \mathbf{l}'_j = \mathsf{Q}\mathbf{L}_j$$

linear in $\mathbf{l}'_j$, Q and $\mathbf{L}_j$, however *quadratic* in P

better

$$\Gamma^\mathsf{T}(\mathbf{L}_j)\mathsf{P}^\mathsf{T}\ \ \mathbf{l}'_j = \Pi^\mathsf{T}(\mathsf{P}^\mathsf{T}\mathbf{l}'_j)\ \ \mathbf{L}_j = (\Gamma^\mathsf{T}(\mathbf{L}_j) \otimes {\mathbf{l}_j}'^\mathsf{T})\ \ \mathbf{p} = \mathbf{0}$$

from $\mathbf{L} \in \mathbf{A}_{l'} \equiv \mathbf{L} \in \mathsf{P}^\mathsf{T}\mathbf{l}'$
linear in $\mathbf{l}'_j$, $\mathbf{L}_j$ and $\mathbf{p} = \text{vec}(\mathsf{P}^\mathsf{T})$

explicitely

$$\underbrace{\begin{bmatrix} \mathbf{0}^\mathsf{T} & -L_{6j}\mathbf{l}_j^\mathsf{T} & L_{5j}{\mathbf{l}_j}'^\mathsf{T} & L_{1j}{\mathbf{l}_j}'^\mathsf{T} \\ L_{6j}{\mathbf{l}_j}'^\mathsf{T} & \mathbf{0}^\mathsf{T} & -L_{4j}{\mathbf{l}_j}'^\mathsf{T} & L_{2j}{\mathbf{l}_j}'^\mathsf{T} \\ -L_{5j}{\mathbf{l}_j}'^\mathsf{T} & L_{4j}{\mathbf{l}_j}'^\mathsf{T} & \mathbf{0}^\mathsf{T} & L_{3j}{\mathbf{l}_j}'^\mathsf{T} \\ -L_{1j}{\mathbf{l}_j}'^\mathsf{T} & -L_{2j}{\mathbf{l}_j}'^\mathsf{T} & -L_{3j}{\mathbf{l}_j}'^\mathsf{T} & \mathbf{0}^\mathsf{T} \end{bmatrix}}_{4\times 12} \mathbf{p} = \mathbf{0}$$

only two constraints per point are necessary

DLT with points and lines

! points and lines must not sit in plane, small deviations suffice

direct solution for calibrated cameras exist:

for points, and for lines

Relative orientation of two images

Estimation of Essential Matrix

given: calibration matrices $\mathsf{K}_k, k = 1, 2$

observed: corresponding image points $(\mathbf{x}'_i, \mathbf{x}''_i), i = 1, ..., I$

unknown: relative orientation and pair of projection matrices

Coplanarity for each pair $(\mathbf{x}', \mathbf{x}'')_i, I = 1, ..., I$ of corresponing points

$$\mathbf{m}_i'^{\mathsf{T}} \mathsf{E} \mathbf{m}_i'' = 0 \qquad \text{or} \qquad (\mathbf{m}'^{\mathsf{T}}_i \otimes \mathbf{m}''^{\mathsf{T}}_i)\mathbf{e} = 0$$

with directions in camera systems

$$\mathbf{m}_i = \mathsf{K}_1^{-1}\mathbf{x}_i \qquad \mathbf{m}_i'' = \mathsf{K}_2^{-1}\mathbf{x}'' \qquad \mathbf{e} = \text{vec}(\mathsf{E}^{\mathsf{T}})$$

$I \times 9$ homogeneous equation system

$$\begin{bmatrix} \mathbf{m}'^{\mathsf{T}}_1 \otimes \mathbf{m}''^{\mathsf{T}}_1 \\ ... \\ \mathbf{m}'^{\mathsf{T}}_i \otimes \mathbf{m}''^{\mathsf{T}}_i \\ ... \\ \mathbf{m}'^{\mathsf{T}}_I \otimes \mathbf{m}''^{\mathsf{T}}_I \end{bmatrix} \mathbf{e} = \begin{bmatrix} 0 \\ ... \\ 0 \\ ... \\ 0 \end{bmatrix}$$

If $I \geq 8$ and poins $\mathbf{X}_i$ not coplanar then unique solution (SVD)

Determination of basis and rotation

$$\mathsf{E} = \mathsf{S}_{b'} R_2^{\mathsf{T}}$$

Basis:

due to

$$\mathsf{E}\mathsf{E}^{\mathsf{T}} = \mathsf{S}_{b'}\mathsf{S}_{b'}^{\mathsf{T}} = |\mathbf{b}'|^2 I - \mathbf{b}'\mathbf{b}'^{\mathsf{T}}$$

E ideally has eigenvalues $|\mathbf{b}'|^2(1, 1, 0)$

SVD of E

$$\mathsf{E} = U \mathbf{\Lambda} V^{\mathsf{T}}$$

ideally with

$$\mathbf{\Lambda} = |\mathbf{b}'|^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rotation:

From SVD

$$\mathsf{E} = U \mathbf{\Lambda} V^{\mathsf{T}}$$

skew symmetric matrix

$$\mathsf{S}_{b'} = \pm\, U Z U^{\mathsf{T}}$$

rotation matrix

$$R_2^{\mathsf{T}} = U W V^{\mathsf{T}} \quad \text{or} \quad R_2^{\mathsf{T}} = U W^{\mathsf{T}} V^{\mathsf{T}}$$

with

$$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad ZW = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4 solutions: choose the one with points in front of camera
(forward intersection cf. below)

D. Nister proposed a **direct solution for 5 corresponding points**

using the constraints for E:

$$\mathsf{E}\mathsf{E}^{\mathsf{T}}\mathsf{E} - \frac{1}{2}\mathrm{tr}(\mathsf{E}\mathsf{E}^{\mathsf{T}})\,\mathsf{E} \stackrel{!}{=} \underset{3\times 3}{\mathbf{0}}$$

up to 10 solutions, mostly 2 or 4.

D. Nister (2004): An efficient solution to the five-point relative pose problem, T-PAMI, 2004.

Projection matrices:

$$\mathsf{P}_1 = \mathsf{K}_1[\mathsf{I}|\mathbf{0}] \qquad \mathsf{P}_2 = \mathsf{K}_2\mathsf{R}[\mathsf{I}| - \boldsymbol{B}]$$

Estimation of Fundamental Matrix

given: straight line preserving images

observed: corresponding image points $(\mathbf{x}'_i, \mathbf{x}''_i), i = 1, ..., I$

unknown: relative orientation and pair of projection matrices

Coplanarity for each pair $(\mathbf{x}', \mathbf{x}'')_i, I = 1, ..., I$ of corresponing points

$$\mathbf{x}_i{'}^{\mathsf{T}} \mathsf{F} \mathbf{x}''_i = 0 \qquad \text{or} \qquad (\mathbf{x}'^{\mathsf{T}}_i \otimes \mathbf{x}''^{\mathsf{T}}_i)\mathbf{f} = 0$$

with

$$\mathbf{f} = \text{vec}(\mathsf{f}^{\mathsf{T}})$$

$I \times 9$ homogeneous equation system

$$\mathsf{A}\mathbf{f} = \begin{bmatrix} \mathbf{x}'^{\mathsf{T}}_1 \otimes \mathbf{x}''^{\mathsf{T}}_1 \\ ... \\ \mathbf{x}'^{\mathsf{T}}_i \otimes \mathbf{x}''^{\mathsf{T}}_i \\ ... \\ \mathbf{x}'^{\mathsf{T}}_I \otimes \mathbf{x}''^{\mathsf{T}}_I \end{bmatrix} \mathbf{f} = \begin{bmatrix} 0 \\ ... \\ 0 \\ ... \\ 0 \end{bmatrix}$$

If $I \geq 8$ and points $\mathbf{X}_i$ not coplanar then unique solution (SVD)

If $I = 7$ and points $\mathbf{X}_i$ not coplanar then up to three solutions:

now we have rank$\mathsf{A} = 7$
nullspace of A is 2-dimensional

Let $\mathbf{f}_1$ and $\mathbf{f}_2$ are last two columns of V of SVD
Then

$$\mathbf{f} = a\mathbf{f}_1 + (1 - a)\mathbf{f}_2 \qquad \text{or} \qquad \mathsf{F} = a\mathsf{F}_1 + (1 - a)\mathsf{F}_2$$

is space of all solutions

Singularity of F yields constraint $|a\mathsf{F}_1 + (1 - a)\mathsf{F}_2|$ on a: up to three solutions.

Projection matrices:

not unique

F only has 7 d.o.f., we need 11 for P
$\rightarrow$
4 d.o.f may be fixed arbitrary

general result:

$$\mathsf{P}_1 = [I|\mathbf{0}] \qquad \mathsf{P}_2 = [\mathsf{S}_{e''}\mathsf{F}^{\mathsf{T}} + \mathbf{e}''\mathbf{d}^{\mathsf{T}}|\alpha\mathbf{e}''] = [\mathsf{A}|\mathbf{a}']$$

with arbitrary scalar α and arbitrary vector $\mathbf{d}$

As in general $\mathsf{P} = [\mathsf{K}R|\mathbf{a}']$ and we cannot fix K:
choose $\mathbf{d}'$ such that A is close to rotation

result:

$$\mathsf{P}_2 = \left[\left(2 \, ||\mathbf{e}''\mathbf{e}'^{\mathsf{T}}|| \, \mathsf{S}_{e''}\mathsf{F}^{\mathsf{T}} + ||\mathsf{S}_{e''}\mathsf{F}^{\mathsf{T}}|| \, \mathbf{e}''\mathbf{e}'^{\mathsf{T}} \right) \mid \alpha\mathbf{e}'' \right]$$

where $\mathbf{e}'$ and $\mathbf{e}''$ are the epipoles (left and right eigenvectors of F) and $\alpha \neq 0$

Forward intersection of points and lines

given:
projection matrices $\mathsf{P}_k, k = 1, ..., K$
observed:
corresponding image points $(\mathbf{x}'_{ik}), i = 1, ..., I; k = 1, ..., K$
corresponding image lines$(\mathbf{l}'_{jk}), j = 1, ..., J; k = 1, ..., K$
unknown:
3D-points $\mathbf{X}_i, i = 1, ..., I$, 3D-Lines $\mathbf{L}_j, j = 1, ..., J$

observe: $\mathbf{l}_{j1} \equiv \mathbf{l}'_j$, $\mathbf{l}_{j2} \equiv \mathbf{l}''_j$, $\mathbf{l}_{j3} \equiv \mathbf{l}'''_j$, etc.

constraints for unknown 3D-points $\mathbf{X}_i$

$$\underbrace{\begin{bmatrix} \mathsf{S}(\mathbf{x}'_{i1})\mathsf{P}_1 \\ ... \\ \mathsf{S}(\mathbf{x}'_{ik})\mathsf{P}_k \\ ... \\ \mathsf{S}(\mathbf{x}'_{iK})\mathsf{P}_K \end{bmatrix}}_{3K \times 4} \mathbf{X}_i = \mathbf{0}$$

for $K = 2$ a simpler solution exists.

constraints for unknwon 3D-lines $\mathbf{L}_j$:

$$\underbrace{\begin{bmatrix} \Pi^\mathsf{T}(\mathsf{P}_1^\mathsf{T}\mathbf{l}'_{j1}) \\ ... \\ \Pi^\mathsf{T}(\mathsf{P}_k^\mathsf{T}\mathbf{l}'_{jk}) \\ ... \\ \Pi^\mathsf{T}(\mathsf{P}_K^\mathsf{T}\mathbf{l}'_{jK}) \end{bmatrix}}_{4K \times 6} \mathbf{L}_j = \mathbf{0}$$

unique solution for $K = 2$

$$\mathbf{L}_j = (\mathsf{P}_1^\mathsf{T}\mathbf{l}'_j) \cap (\mathsf{P}_2^\mathsf{T}\mathbf{l}''_j) = \overline{\Pi}(\mathsf{P}_1^\mathsf{T}\mathbf{l}'_j)\mathsf{P}_2^\mathsf{T}\mathbf{l}''_j$$

Relative orientation of three images

line correspondecies (two constraints)

$$\mathbf{l}' \times \mathcal{T}(\mathbf{l}'', \mathbf{l}''') = \mathbf{0}$$

point correspondencies:
two lines through $\mathbf{x}''$ (two constraints)

$$x_k''' = T_{ijk} x_i' l^{a}{}_j''$$

$$x_k''' = T_{ijk} x_i' l^{b}{}_j''$$

one additional constraint with $\mathbf{l}'''$ perpendicular to epipolar line
(coplanarity of $\mathbf{x}'$ and $\mathbf{x}''$, one contraint)

$$x_j'' = T_{ijk} x_i' l_k'''$$

Condition for simultaneous estimation of tensor coefficients

$$4n_x + 2n_l \geq 26$$

e. g.: 4 points and 5 lines (no relative oriention of two images possible)

Derivation of F_{ij} and P_i from $\mathcal{T}$ possible
Result: up to 3D-homography (absolute orientation with ≥ 5 control points)

Absolute orientation

$$\mathbf{X}' = \mathsf{H}\mathbf{X} \qquad \begin{bmatrix} \boldsymbol{X}' \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda R & \boldsymbol{T} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ 1 \end{bmatrix}$$

linear algorithm for H

Constraints for 3D-homography

constraints for *corresponding 3D-points* (3 d. o. f.)

$$\Pi(\mathbf{X}')\mathsf{H}\ \mathbf{X} = -\Pi(\mathsf{H}\mathbf{X})\ \mathbf{X}' = (\Pi(\mathbf{X}') \otimes \mathbf{X}^{\mathsf{T}})\ \mathbf{h} = \mathbf{0}$$

linear in $\mathbf{X}$, $\mathbf{X}'$ and $\mathbf{h} = \text{vec}(\mathsf{H}^{\mathsf{T}})$
for estimating 3D-homography

at least 5 corresponding points

direct solution for rotation, scale and translation exist ($\geq$ 3 points)

constraints for *corresponding planes* (3 d. o. f.)

$$\overline{\Pi}(\mathbf{A})\mathsf{H}^{\mathsf{T}}\ \mathbf{A}' = -\overline{\Pi}(\mathsf{H}^{\mathsf{T}}\mathbf{A}')\ \mathbf{A} = ({\mathbf{A}'}^{\mathsf{T}} \otimes \overline{\Pi}(\mathbf{A}))\ \mathbf{h} = \mathbf{0}$$

linear in $\mathbf{A}'$, $\mathbf{A}$ and $\mathbf{h} = \text{vec}(\mathsf{H}^{\mathsf{T}})$
for estimating 3D-homography from points and planes

at least 5 corresponding planes

direct solution for rotation, scale and translation exist ($\geq$ 3 planes)

constraints for *corresponding 3D-lines* (4 d. o. f.)

$$\overline{\Gamma}(\mathbf{L}')\mathsf{H}\ \Gamma(\mathbf{L}) = \Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}}\ \overline{\Gamma}(\mathbf{L}') = -(\Gamma(\mathbf{L}) \otimes \overline{\Gamma}(\mathbf{L}'))\ \mathbf{h} = \mathit{0}$$

from interpretation of columns of $\Gamma(\mathbf{L})$ as points and of $\overline{\Gamma}(\mathbf{L})$ as planes
linear in $\mathbf{L}$, $\mathbf{L}'$, $\mathbf{h} = \text{vec}(\mathsf{H}^{\mathsf{T}})$
for estimating homography from 3D-points, planes and 3D-lines

at least 4 corresponding lines

direct solution for rotation, scale and translation exist ($\geq$ 2 3D-lines) explicitely

$$\left[\begin{array}{c} \Pi(\mathbf{X}'_1) \otimes \mathbf{X}_1^{\mathsf{T}} \\ ... \\ \Pi(\mathbf{X}'_I) \otimes \mathbf{X}_I^{\mathsf{T}} \\ \hline \Gamma(\mathbf{L}_1) \otimes \overline{\Gamma}(\mathbf{L}'_1) \\ ... \\ \Gamma(\mathbf{L}_J) \otimes \overline{\Gamma}(\mathbf{L}'_J) \\ \hline \mathbf{A'}_1^{\mathsf{T}} \otimes \overline{\Pi}(\mathbf{A}_1) \\ ... \\ \mathbf{A'}_M^{\mathsf{T}} \otimes \overline{\Pi}(\mathbf{A}_M) \end{array}\right] \mathbf{h} = \mathbf{0}$$

at least $3I + 4J + 3M \geq 16$

Bundle adjustment with points and lines

observed: image points $\mathbf{x}'_{ik}$ in image $k = 1, ..., K$
observed: image lines $\mathbf{l}'_{jk}$ in image $k = 1, ..., K$

given/observed: control points $\mathbf{X}_{CP,i}$, for some i
given/observed: control lines $\mathbf{L}_{CP,j}$, for some j

unknwown: orientation parameters $\boldsymbol{t}_k$ and calibration parameters $\boldsymbol{p}_k$ in

$$\mathbf{x}'_{ik} = \mathsf{P}_k(\boldsymbol{t}_k, \boldsymbol{p}_k)\mathbf{X}_i$$

unknwown: 3D-points $\mathbf{X}_i$ and 3D-lines $\mathbf{L}_j$

solution: Gauß-Helmert model (general constraints)

image points of unknown object points

$$\mathsf{S}(\mathbf{x}'_{ik})\mathsf{P}_k(\widehat{\boldsymbol{t}}_k, \widehat{\boldsymbol{p}}_k)\,\widehat{\mathbf{X}}_i = \mathbf{0} \qquad i = 1, ..., I; k = 1, ..., K$$

image lines of unknown object lines

$$\mathsf{S}(\mathbf{l}'_{jk}) = \mathsf{Q}_k(\widehat{\boldsymbol{t}}_k, \widehat{\boldsymbol{p}}_k)\,\widehat{\mathbf{L}}_j = \mathbf{0} \qquad j = 1, ..., J; k = 1, ..., K$$

reminder: $\mathsf{Q}_k = f(\mathsf{P}_k)$

image points of control points

$$\mathsf{S}(\mathbf{x}'_{ik})\mathsf{P}_k(\widehat{\boldsymbol{t}}_k, \widehat{\boldsymbol{p}}_k)\ \mathbf{X}_{CP,i} = \mathbf{0} \qquad i = 1, ..., I_{CP}; k = 1, ..., K$$

image lines of control lines

$$\mathsf{S}(\mathbf{l}'_{jk})\mathsf{Q}_k(\widehat{\boldsymbol{t}}_k, \widehat{\boldsymbol{p}}_k)\ \mathbf{L}_{CL,j} = \mathbf{0} \qquad j = 1, ..., J_{CL}; k = 1, ..., K$$

Comments:

- requires approximate values
- any model can be used: specify unknown parameters $\boldsymbol{t}_k, \boldsymbol{p}_k$
- differentiation with finite differences, e.g.
  use expected standard deviations for Δx_i
- if unknown 3D-points are not at infinity
  use $\mathbf{X}^\mathsf{T} = [X, Y, Z, 1]$, thus only 3 unknown parameters

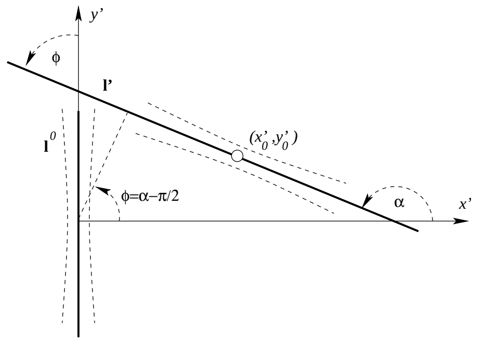
- apply length and Plücker constraint for 3D-lines

  $$\mathbf{L}_j^\mathsf{T}\mathbf{L}_j = 1 \qquad \mathbf{L}_j^\mathsf{T}\overline{\mathbf{L}_j} = 0$$

  adds 2 Lagrangian multipliers per 3D-line,
  thus 4 additional parameters in normal equation matrix
- alternative: use approximate 3D-line $\mathbf{L}_j^{(0)}$ and estimate 4 parameters for transforming it into optimal line (similar to estimating differential rotation 3-vector Δr $R \approx R^{(0)}(I + \mathsf{S}_{\Delta r})$
- covariance matrix of image points

  $$[\Sigma_{x'x'}] = \begin{bmatrix} \sigma^2_{x'} & \sigma_{x'y'} & 0 \\ \sigma_{x'y'} & \sigma^2_{x'} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- covariance matrix of image lines, centred at (x'_0, y'_0), with uncertain direction α and position d across the line



$\sigma_\phi = \sigma_\alpha$ and σ_d from feature extraction
fitting line through extracted edge pixels

Rotation of uncertain line on y'-axis into line

$$\mathbf{l}' = \mathsf{M}\mathbf{l}^0 = \begin{bmatrix} \cos\phi & -\sin\phi & x'_0 \\ \sin\phi & \cos\phi & y'_0 \\ 0 & 0 & 1 \end{bmatrix}^{-\mathsf{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with normal direction $\phi = \alpha - \pi/2$

Thus

$$\Sigma_{l'l'} = \mathsf{M}\Sigma_{l^0l^0}\mathsf{M}^\mathsf{T}$$

with covariance matrix of uncertain line $\mathbf{l}^0$ on y'-axis

$$\underline{\mathbf{l}}^0 \sim N(\boldsymbol{\mu}_{l^0}, \Sigma_{l^0l^0}) \qquad \boldsymbol{\mu}_{l^0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Sigma_{l^0l^0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_\alpha^2 & 0 \\ 0 & 0 & \sigma_d^2 \end{bmatrix}$$

Testing Geometric Relations

Example: Testing Identity of Two 2D-points

Test of $\boldsymbol{x} = \boldsymbol{y}$

Classical procedure
Difference:

$$\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x} \sim N(\boldsymbol{\mu}_d, \Sigma_{dd}) = N(\boldsymbol{\mu}_y - \boldsymbol{\mu}_x, \Sigma_{xx} + \Sigma_{yy})$$

Test of

$$H_0 : \ \boldsymbol{\mu}_d = \mathbf{0} \qquad H_a : \ \boldsymbol{\mu}_d \neq \mathbf{0}$$

Test statistic

$$T = \boldsymbol{d}^{\mathsf{T}} \Sigma_{dd}^{-1} \boldsymbol{d} \sim \chi_2^2$$

Discussion:
+ choose proper vector with $E(\underline{\boldsymbol{d}}|H_0) = \mathbf{0}$
+ choose Mahalanobis distance as test statistic
+ d. o. f. = size of vector $\boldsymbol{d}$
– complex for other relations

Transfer to homogenous representation? (cf. below)

Alternative solution
Estimate mean point $\boldsymbol{m}$

$$\widehat{\boldsymbol{m}} = (\Sigma_{xx}^{-1} + \Sigma_{yy}^{-1})^{-1}(\Sigma_{xx}^{-1}\boldsymbol{x} + \Sigma_{yy}^{-1}\boldsymbol{y})$$

Determine weigthed squares of residuals

$$\Omega = (\boldsymbol{x} - \widehat{\boldsymbol{m}})^{\mathsf{T}}\Sigma_{xx}^{-1}(\boldsymbol{x} - \widehat{\boldsymbol{m}}) + (\boldsymbol{y} - \widehat{\boldsymbol{m}})^{\mathsf{T}}\Sigma_{yy}^{-1}(\boldsymbol{y} - \widehat{\boldsymbol{m}})$$

Test statistic

$$T = \Omega \sim \chi_2^2$$

Discussion:
+ same result as previous
+ use weighted squares of residual as test statistic
o requires optimally correct geometric entities, based on constraints
– requires estimation process
– slower (at least factor 5)

Transfer to homogenous representation?
→ above (cf. KANATANI)

New procedure (FÖRSTNER ET. AL 2000)
'Difference': line $\mathbf{l}$ generated by $\mathbf{x}$ and $\mathbf{y}$ is not defined, thus $\mathbf{l} = \mathbf{0}$

$$\mathbf{d}|H_0 = \mathbf{x} \times \mathbf{y}|H_0 \sim N(\mathbf{0}, \Sigma_{\mathrm{dd}})$$

$$\Sigma_{\mathrm{dd}} = \mathsf{S}(\boldsymbol{\mu}_x)\Sigma_{\mathrm{yy}}\mathsf{S}^{\mathsf{T}}(\boldsymbol{\mu}_x) + \mathsf{S}(\boldsymbol{\mu}_y)\Sigma_{\mathrm{xx}}\mathsf{S}^{\mathsf{T}}(\boldsymbol{\mu}_y)$$

Problems:
– $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$ not known
– number of elements in $\mathbf{d}$ too large, depending on constraints

Solution:
+ Use $\widehat{\boldsymbol{\mu}}_x = \mathbf{x}$ and $\widehat{\boldsymbol{\mu}}_y = \mathbf{y}$ as *approximations*
+ Select independent constraints (cf. above)

Discussion:
+ simple
+ fast
+ very good approximation if test is not rejected
+ approximate test statistic increases monotonically with rigorous one
0 Conditioning and Normalization necessary to reduce bias
— only approximation if test is rejected

Normalization only of covariance matrix, no scaling necessary

Procedure for Testing Geometric Entities

1. determine the difference d, $\mathbf{d}$, $\mathbf{D}$ or D (cf. tables 3, 4).
2. select r independent constraints
3. determine the covariance matrix Σ_{dd} of the r selected elements $\boldsymbol{d}$ of differences
4. determine the test statistic T

$$T = \boldsymbol{d}^{\mathsf{T}} \Sigma_{dd}^{+} \boldsymbol{d} \sim \chi_r^2$$

5. choose a significance number α
   compare T with the critical value $\chi^2_{r,\alpha}$.
   If $T > \chi^2_{r,\alpha}$ then reject hypothesis on relation

1	2	3	4	5
No.	2D-entities	relation	dof	test
1	$\mathbf{x}$, $\mathbf{y}$	$\mathbf{x} \equiv \mathbf{y}$	2	$\mathbf{d} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$
2	$\mathbf{x}$, $\mathbf{l}$	$\mathbf{x} \in \mathbf{l}$	1	$d = \mathbf{x}^{\mathsf{T}}\mathbf{l} = \mathbf{l}^{\mathsf{T}}\mathbf{x}$
3	$\mathbf{l}$, $\mathbf{m}$	$\mathbf{l} \equiv \mathbf{m}$	2	$\mathbf{d} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$

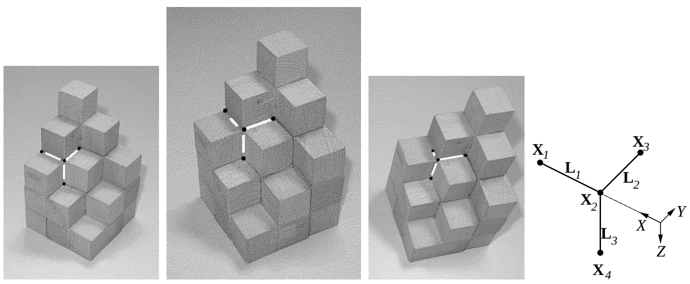
Tabelle 3: *shows 3 relationships between points and lines useful for 2D grouping, together with the degree of freedom and the essential part of the test statistic.*

1	2	3	4	5
No.	3D-entities	relation	dof	test
4	$\mathbf{X}$, $\mathbf{Y}$	$\mathbf{X} \equiv \mathbf{Y}$	3	$\mathbf{D} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$
5	$\mathbf{X}$, $\mathbf{L}$	$\mathbf{X} \in \mathbf{L}$	2	$\mathbf{D} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L} = \overline{\Gamma}^{\mathsf{T}}(\mathbf{L})\mathbf{X}$
6	$\mathbf{X}$, $\mathbf{A}$	$\mathbf{X} \in \mathbf{A}$	1	$d = \mathbf{X}^{\mathsf{T}}\mathbf{A} = \mathbf{A}^{\mathsf{T}}\mathbf{X}$
7	$\mathbf{L}$, $\mathbf{M}$	$\mathbf{L} \equiv \mathbf{M}$	4	$\mathsf{D} = \overline{\Gamma}(\mathbf{L})\Gamma(\mathbf{M})$
8		$\mathbf{L} \cap \mathbf{M} \neq \emptyset$	1	$d = \overline{\mathbf{L}}^{\mathsf{T}}\mathbf{M} = \overline{\mathbf{M}}^{\mathsf{T}}\mathbf{L}$
9	$\mathbf{L}$, $\mathbf{A}$	$\mathbf{L} \in \mathbf{A}$	2	$\mathbf{D} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L} = \Gamma^{\mathsf{T}}(\mathbf{L})\mathbf{A}$
10	$\mathbf{A}$, $\mathbf{B}$	$\mathbf{A} \equiv \mathbf{B}$	3	$\mathbf{D} = \Pi(\mathbf{A})\mathbf{B} = -\Pi(\mathbf{B})\mathbf{A}$

Tabelle 4: *shows 7 relationships between points, lines and planes useful for 3D grouping, together with the degree of freedom and the essential part of the test statistic.*

Conclusions

Example



Result for 3D points and 3D lines

point	type	X [mm]	Y [mm]	Z [mm]	red.	$\hat{\sigma}_0$[1]	$\sigma_{\hat{X}}$ [mm]	$\sigma_{\hat{Y}}$ [mm]	$\sigma_{\hat{Z}}$ [mm]
1	alg.	3.90	-0.16	-0.14	4				
	opt.	4.20	0.03	0.09	4	1.56	0.35	0.27	0.31
2	alg.	1.95	-0.09	-0.06	12				
	opt.	1.99	-0.01	-0.07	12	2.08	0.88	0.70	0.96
3	alg.	2.04	1.89	-0.07	6				
	opt.	2.13	1.95	0.03	6	3.92	0.57	0.46	0.62
4	alg.	2.25	0.14	2.21	6				
	opt.	2.08	0.04	1.04	6	4.41	0.49	0.41	0.58

line	type	L_1[1]	L_2 [1]	L_3 [1]	L_4 [mm]	L_5 [mm]	L_6 [mm]	red.	$\hat{\sigma}_o$[1]
1	alg.	0.999	0.009	0.008	0.000	-0.157	0.187	8	
	opt.	0.993	0.041	0.105	-0.007	0.443	0.238	8	2.10
2	alg.	-0.024	-1.000	0.019	-0.042	-0.378	-1.981	8	
	opt.	-0.081	-0.995	-0.062	-0.146	0.131	-1.908	8	1.02
3	alg.	0.102	0.067	0.992	-0.011	-2.018	0.138	8	
	opt.	0.072	0.047	0.996	0.054	-2.085	0.095	8	2.00

Summary

- transparent representation of projective geometry
  homogeneous coordinates, Plücker coordinates
- useful for spatial reasoning and multiview analysis
  direct linear solutions
- linked with statistics
  error propagation, optimal testing, estimation
- broad application in geometric analysis of intensity and range images
  grouping; planes as independent observations)
- easy to use procedures

Further material

- Software SUGR in PEARL available
  (Statistically Uncertain Geometric Reasoning)
- SUGR in JAVA is available soon,
  (construction and test available now)
- Stephan Heuel's thesis available, LNCS 3008
- 5th Edition of ASPRS Manual of Photogrammetry available
  (Ed.: Ch. McGlone)