Projective Geometry for Photogrammetric Orientation Procedures

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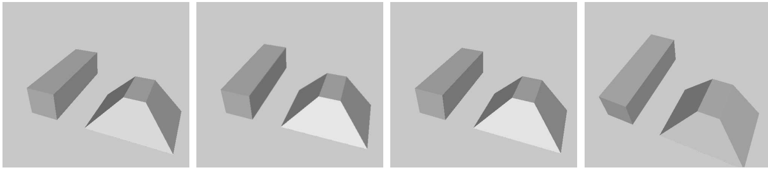
Notions

Geometry

Reconstruction of polyhedral objects



cf: file:/home/wf/slides/PCV02-Tutorial-HTML/Part-I/img3.htm

- points, lines and planes in object space
- points and lines in image space
- projecting lines and planes
- straight line preserving mappings:
  projection matrix, fundamental matrix, homographies

all easily representable within *projective geometry*

Uncertainty

inherent to mensuration

- simple representation by probabilities, very good approximation
- representation by covariance matrices
- propagation under linearity and nonlinearity

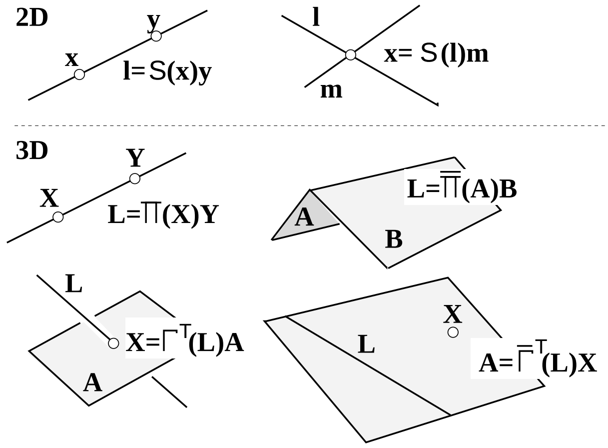
decisions depend on uncertain quantities

Multiview Geometry

- One, two, three and more images
- points and lines
- prediction and constraints

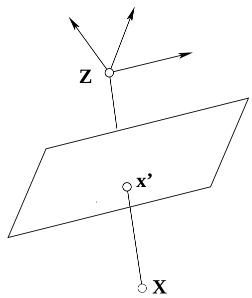
Bottom-Line

- Ease of using projective geometry



- Freedom of using camera models

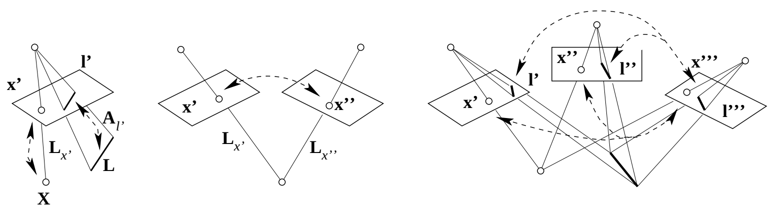
$$\mathbf{x}' = \mathsf{P}(\boldsymbol{q})\ \mathbf{X} = \mathsf{K}(\boldsymbol{q})\ \ R[I \mid -\boldsymbol{Z}]\mathbf{X}$$



equivalent to

$$\begin{aligned} x' &= c\frac{r_{11}(X-X_0)+r_{12}(Y-Y_0)+r_{13}(Z-Z_0)}{r_{31}(X-X_0)+r_{32}(Y-Y_0)+r_{33}(Z-Z_0)} + \Delta x'(\boldsymbol{q}) \\ y' &= c\frac{r_{21}(X-X_0)+r_{22}(Y-Y_0)+r_{23}(Z-Z_0)}{r_{31}(X-X_0)+r_{32}(Y-Y_0)+r_{33}(Z-Z_0)} + \Delta y(\boldsymbol{q})' \end{aligned}$$

- Simultaneous use of points and lines

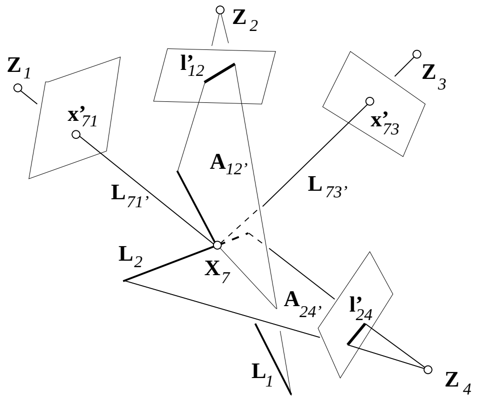


$$\begin{aligned} \mathbf{x}' &= \mathsf{P}\mathbf{X} \\ \mathbf{l}' &= \mathsf{Q}\mathbf{L} \\ \mathbf{A}_{l'} &= \mathsf{P}^{\mathsf{T}}\mathbf{l}' \\ \mathbf{L}_{x'} &= \overline{\mathsf{Q}}^{\mathsf{T}}\mathbf{x}' \end{aligned}$$

$$\mathbf{L}_{x'}^{\mathsf{T}}\overline{\mathbf{L}}_{x''} = 0$$

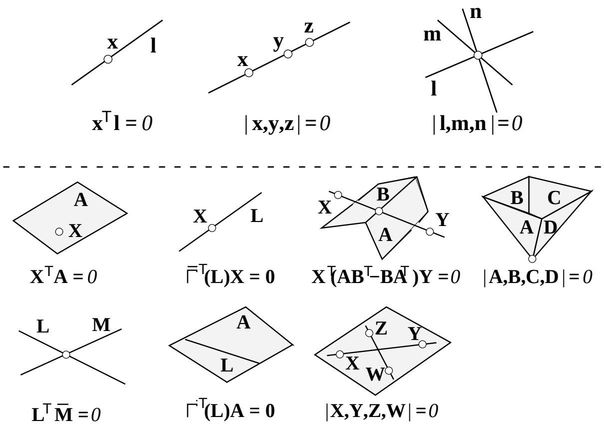
$$\begin{aligned} |\mathbf{A}_{l'}, \mathbf{A}_{l''}, \mathbf{A}_{l'''}, \mathbf{A}| &= 0 \\ |\mathbf{A}_{l_{x'}}, \mathbf{A}_{l_{y'}}, \mathbf{A}_{l_{x''}}, \mathbf{A}_{l_{x'''}}| &= 0 \end{aligned}$$

- Direct, linear approximate solutions



$$\mathsf{A}(\mathbf{x}'_{71}, \mathbf{l}'_{12}, \mathbf{x}'_{73}, \mathbf{l}'_{24})\widehat{\mathbf{X}}_7 = \mathbf{0}$$

- Integration of statistics



$$\underline{\mathbf{d}} = \Gamma^{\mathsf{T}}(\underline{\mathbf{L}})\underline{\mathbf{A}} = \Pi^{\mathsf{T}}(\underline{\mathbf{A}})\underline{\mathbf{L}} \quad \rightarrow \quad \underline{T} = \underline{\mathbf{d}}^{\mathsf{T}}\Sigma^{+}_{dd}\underline{\mathbf{d}} \sim \chi^2_2$$

References:

Hartley/Zisserman 2000: Multiple View Geometry, Cambridge Press
Faugeras/Luong 2001: Geometry of Multiple Images, MIT Press
Kanatani 1996: Statistical Optimization for Geometric Computation: Theory and Practice, Elsevier Science
Criminisy 2001: Accurate visual metrology from single and multiple uncalibated images, Springer
Matei/Meer 1997: A General Method for Errors-in-Variables Problems in Computer Vision, CVPR
Koch 1997: Parameterschätzung und Hypothesentests, Dümmler
Förstner Brunn/Heuel 2000: Statistically Testing Uncertain Geometric Relations, DAGM
Heuel 2004: Uncertain Projective Geometry - Statistical Reasoning for Polyhedral Object Reconstruction, LNCS 3008

Basics from Statistics

Uncertainty Propagation

General task:

Given:
stochastical vector $\underline{\boldsymbol{x}}$ with pdf $p_x(\boldsymbol{x})$
vector valued function $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$

unknown:
pdf $p_y(\boldsymbol{y})$ of $\underline{\boldsymbol{y}} = \boldsymbol{f}(\underline{\boldsymbol{x}})$

Discussion:

- several methods (cf. Papoulis)
- often high effort, sometimes not possible.
- Preservation of type of pdf for *linear functions* $\boldsymbol{f}$ (Uniform, Gaussian)
- Approximate uncertainty propagation if Hessian (curvature) of $\boldsymbol{f}$ and variances of $\underline{x}_i$ are small

Propagation of Mean and Variance

Theorem: Error propagation

If

$$E(\underline{\boldsymbol{x}}) = \boldsymbol{\mu}_x,\ V(\underline{\boldsymbol{x}}) = \Sigma_{xx} \quad \text{and} \quad \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$$

then in a first order approximation

$$E(\underline{\boldsymbol{y}}) = \boldsymbol{\mu}_y = \boldsymbol{f}(\boldsymbol{\mu}_x),\ V(\underline{\boldsymbol{y}}) = \Sigma_{yy} = J\Sigma_{xx}J^{\mathsf{T}}$$

with the Jacobian

$$J = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}$$

Discussion:

- pdf needs *not* be known
- if $\boldsymbol{f}$ is linear, then error propagation is rigorous
- if $\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{a}$ is linear and pdf is Gaussian, then $\underline{\boldsymbol{y}}$ is Gaussian

$$\underline{\boldsymbol{x}} \sim N(\boldsymbol{\mu}_x, \Sigma_{xx}) \quad \rightarrow \quad \underline{\boldsymbol{y}} = N(A\boldsymbol{\mu}_x + \boldsymbol{a}, A\Sigma_{xx}A^{\mathsf{T}})$$

- Error of approximation depends on
  — Hessian, curvature of $\boldsymbol{f}$
  — Covariance matrix, uncertainty of $\underline{\boldsymbol{x}}$
  (second order approximation of $\boldsymbol{\mu}_y$)

Example 1: scalar function of scalar value (arbitrary pdf)

$$E(\underline{y}) = \mu_y = f(\mu_x) + \frac{1}{2}f''(\mu_x)\sigma_x^2 + \frac{1}{24}f^{(4)}m_4 + O(f^{(n)}, m_n) \qquad n > 4$$

$$V(\underline{y}) = \sigma_y^2 = f'^2(\mu_x)\sigma_x^2 + \frac{f'(\mu_x)f'''(\mu_x)}{3}m_4 + O(f^{(n)}, m_n) \qquad n > 4$$

Example 2: scalar function of vector:

$$E(\underline{y}) = \mu_y = f(\boldsymbol{\mu}_x) + \frac{1}{2}\text{trace}(\mathsf{H}|_{x=\mu_x} \cdot \Sigma_{xx}) + O(f^{(n)}, m_n), \quad n \geq 4$$

Normalization of a vector

$$\underline{\boldsymbol{z}} = \frac{\boldsymbol{w}}{|\underline{\boldsymbol{w}}|} \qquad \Sigma_{ww} = \sigma_w^2 \, \mathsf{I} \qquad \underline{z}_i = \frac{\underline{w}_i}{|\underline{\boldsymbol{w}}|}$$

leads to

$$E(\underline{z}) = \frac{\boldsymbol{\mu}_w}{|\boldsymbol{\mu}_w|} - \frac{1}{2}\frac{\boldsymbol{\mu}_w}{|\boldsymbol{\mu}_w|^3}\,\sigma_w^2 = \frac{\boldsymbol{\mu}_w}{|\boldsymbol{\mu}_w|}\left(1 - \frac{1}{2}\frac{\sigma_w^2}{|\boldsymbol{\mu}_w|^2}\right)$$

bias(μ)/σ_μ $\sim$ directional error (usually < 1 %)
also
bias(σ)/σ_σ $\sim$ directional error (usually < 1 %)

When are covariance matrices sufficient for representing uncertainty?

- if the distribution of the original observation can be approximated by a Gaussian
  can be tested (Kolmogoroff-Smirnov-Test), not discussed
- if the functions are smooth with respect to the relative precision
- this is the case in multi view geometry,
  except for reconstruction of objects which are small in the image (say, less than 10 pixels)

Optimal Testing

Principle

Given:

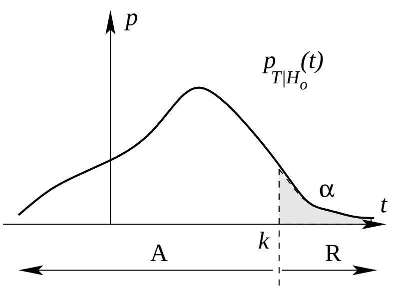
1. Null-Hypothesis H_0 to be checked
2. Alternative Hypothesis H_a, possibly parametrized;
3. Observations $\underline{\boldsymbol{x}}$

Fair argumentation:
not: Prove H_0 (never possible)
but: Try to reject H_0, $\rightarrow$

1. *either* data cannot be used to reject H_0
2. *or* data give no reason to reject H_0

Procedure: Test

1. Derive test statistic
$$T = f(\boldsymbol{x})$$
2. Derive pdf of T under the assumption that H_0 holds
$$\underline{T}|H_0 \sim p_{T|H_0}(t)$$
3. Choose acceptance region $\mathcal{A}$ and rejection region $\mathcal{R}$ for T and siginificance number α or significance level S such that
$$P(\underline{T} \in \mathcal{A}) = S = 1 - \alpha \qquad P(\underline{T} \in \mathcal{R}) = \alpha = 1 - S$$



4. Decide:
   (a) if $T \in \mathcal{R}$, then reject H_0 in favour of H_a.
   Decision may be wrong with probability α
   (b) if $T \in \mathcal{A}$, then do not reject H_0 in favour of H_a.
   Decision may be wrong with probability $\beta = f(T, H_0, H_a)$.

Example:

Given observations collected in n-vector $\boldsymbol{x}$
Hypotheses

$$H_0 : \; E(\underline{\boldsymbol{x}}) = \boldsymbol{a} \qquad \text{against} \qquad H_a : \; E(\underline{\boldsymbol{x}}) \neq \boldsymbol{a}$$

Test statistic

$$T = (\boldsymbol{x} - \boldsymbol{a})^{\mathsf{T}} \Sigma^{+} (\boldsymbol{x} - \boldsymbol{a})$$

Distribution if H_0

$$\underline{\boldsymbol{x}}|H_0 \sim N(\boldsymbol{a}, \Sigma), \;\; \text{rank}\Sigma = r \quad \rightarrow \quad \underline{T}|H_0 \sim \chi^2_r = p_T(t)$$

Siginificance number

$$\alpha$$

Acceptance region, critical value k

$$\mathcal{A} = ([0,k]| \int_0^k p_T(t)\; dt = 1 - \alpha) = [0, \chi^2_{r,1-\alpha}] \qquad r = \text{rank}(\Sigma)$$

Decision:

1. If $T > k$ then mean value is not $\boldsymbol{a}$
2. else there is no reason to assume that $\boldsymbol{\mu} \neq \boldsymbol{a}$

Discussion:

1. Pseudo inverse from

$$\Sigma = R \boldsymbol{\Lambda} R^{\mathsf{T}} \qquad \rightarrow \qquad \Sigma^{+} = R \boldsymbol{\Lambda}^{+} R^{\mathsf{T}}$$

with

$$\boldsymbol{\Lambda}^{+} = \text{Diag}(\lambda_i^{+}) \qquad \text{with} \qquad \lambda_i^{+} = \begin{cases} 1/\lambda_i, \text{ if } \lambda_i \neq 0 \\ 0, \text{ else} \end{cases}$$

2. if covariance matrix regular, then Mahalanobis distance

$$T = (\boldsymbol{x} - \boldsymbol{a})^{\mathsf{T}} \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{a}) \qquad \underline{T}|H_0 \sim \chi^2_n$$

3. if test of single value x then test statistic

$$z = \frac{x - a}{\sigma} \qquad \underline{z}|H_0 \sim N(0,1)$$

Direct Estimation with Algebraic Minimization

often constraints between
observations $\boldsymbol{l}$ and
unknown parameters $\boldsymbol{x}$
are linear in unknown parameters

$$A^{\mathsf{T}}(\boldsymbol{l})\boldsymbol{x} \stackrel{!}{=} \mathbf{0}$$

Algebraic minimization:

$$\widehat{\boldsymbol{x}}^{\mathsf{T}} A^{\mathsf{T}}(\boldsymbol{l}) \cdot A(\boldsymbol{l}) \widehat{\boldsymbol{x}} \to \min \qquad |\widehat{\boldsymbol{x}}| = 1$$

Solution

$$\mu \widehat{\boldsymbol{x}} = \left[A^{\mathsf{T}}(\boldsymbol{l}) \cdot A(\boldsymbol{l}) \right] \widehat{\boldsymbol{x}} \qquad |\widehat{\boldsymbol{x}}| = 1$$

thus using singular value decomposition (SVD)

$$\underset{N \times U}{A(\boldsymbol{l})} = \underset{N \times N}{U} \underset{N \times U}{\boldsymbol{\Lambda}} \underset{U \times U}{V^{\mathsf{T}}}$$

then optimal estimate is last column $\boldsymbol{v}_U$ of $V = (\boldsymbol{v}_1, ..., \boldsymbol{v}_U)$

$$\widehat{\boldsymbol{x}} = \boldsymbol{v}_U \qquad \text{or} \qquad \widehat{x}_u = V_{uU}$$

corresponding to smallest singular value

not optimal, arbitrary scaling of constraints → *conditioning* (below)

Main result:

Linear relations simplify error propagation.
Linear relations simplify testing.
Linear relations simplify estimation.
Projective geometry leads to linear relations!
Projective geometry simplifies multi view relations!

→

Use projective geometry
for uncertain geometric reasoning
in Photogrammetry

Representations in Projective Geometry

Views

- Vectors and matrices for geometric entities and transformations
- Algebra of join (link) and meet (intersection) of geometric entities
- Change of space: Plane to shere to capture elements at infinity

Basic concepts

- Homogeneous coordinates and matrices
- Multilinear relations

Representation of Geometric Entities

Geometric entities:
Points, lines, planes, conics, ...
transformations

Representations:
Vectors, matrices

Homogeneity:
The *algebraic* entity $\mathbf{a}$ is called *homogenous*
if $\mathbf{a}$ and $\lambda\mathbf{a}$, with $\lambda \neq 0$ represent the same *geometric* entity.

Notation:

	2D	3D	transformations
Euclidean	$\boldsymbol{x}$	$\boldsymbol{X}$	R
Homogeneous	$\mathbf{l}$, $\mathbf{x}$	$\mathbf{A}$, $\mathbf{L}$, $\mathbf{X}$	H

Euclidean and homogeneous entities

element	2D	3D
planes		$\underset{4\times1}{\mathbf{A}}, \underset{4\times1}{\mathbf{B}}, ...$
lines	$\underset{3\times1}{\mathbf{l}}, \underset{3\times1}{\mathbf{m}}, ...$	$\underset{6\times1}{\mathbf{L}}, \underset{6\times1}{\mathbf{M}}, ...$
points	$\underset{3\times1}{\mathbf{x}}, \underset{3\times1}{\mathbf{y}}, ...$	$\underset{4\times1}{\mathbf{X}}, \underset{4\times1}{\mathbf{Y}}, ...$

Basic geometric entities

Represention of points, 2D-lines and planes

Hessian normal form: Incidence relation of 2D-point and 2D-line

$$x\cos\phi + y\sin\phi - d = 0 \qquad \leftrightarrow \qquad [x, y, 1]\begin{bmatrix}\cos\phi \\ \sin\phi \\ -d\end{bmatrix} = 0$$

- point and line represented as 3-vector
- redundant representation, 2 d. o. f.
- limited: no element at infinity, e. g. intersection of parallel lines

points x in the plane
(2 d. o. f., result of observing image point)

$$\underset{3\times 1}{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ \hline w \end{bmatrix} = w \begin{bmatrix} x \\ y \\ \hline 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \hline x_h \end{bmatrix}$$

- inversion for points $w \neq 0$

$$x = \frac{u}{w} \qquad y = \frac{v}{w}$$

- partitioning: Euclidean part $\boldsymbol{x}_0$, homogeneous part x_h
- distance to origin

$$d_{xO} = \frac{|\boldsymbol{x}_0|}{|x_h|}$$

- points at infinity with $w = x_h = 0$ at horizon

$$\mathbf{x}_\infty = \begin{bmatrix} \boldsymbol{x}_0 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

  in direction (u, v) (let $\lim_{w\to 0} \mathbf{x}$)
- Euclidean normalization if $w \neq 0$ (no new information)

$$\mathbf{x}^e = \frac{\mathbf{x}}{x_h}$$

- Spherical normalization

$$\mathbf{x}^s = \mathsf{N}(\mathbf{x}) \doteq \frac{\mathbf{x}}{|\mathbf{x}|}$$

The projective space

Euclidean plane:
$\mathbb{R}^2$, represented by 2-vectors $\boldsymbol{x}$
two points $\boldsymbol{x}$ and $\boldsymbol{y}$ are identical if

$$x_i = y_i, i = 1, 2$$

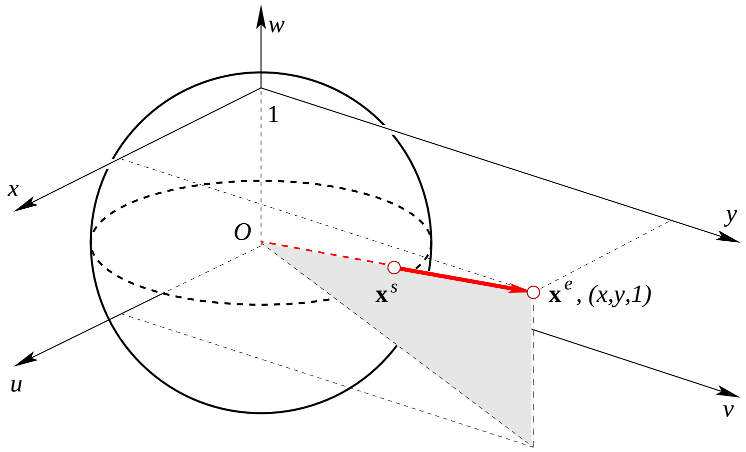
Projective plane:
$\mathbb{P}^2$, represented by 3-vectors $\mathbf{x} \neq \mathbf{0}$
two points $\mathbf{x}$ and $\mathbf{y}$ are identical if

$$x_i = \lambda y_i, i = 1, 2, 3$$

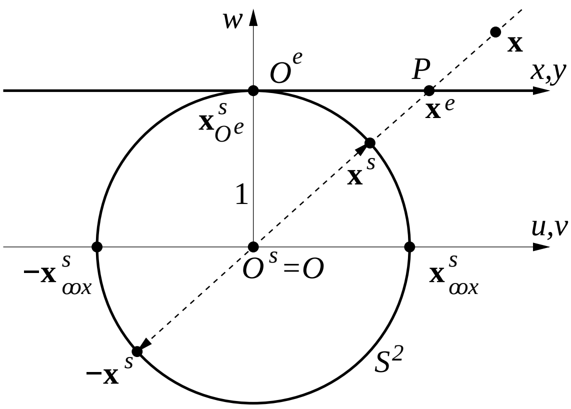
Visualization of points:

1. Lines in $\mathbb{R}^3$ through origin
   horizontal lines represent points at infinity

   Intersection with horizontal plane $w = 1$ yields Euclidean plane
2. Intersection with unit sphere S^2 yields projective plane
   Points $\mathbf{x}^s \in S^2$, $\mathbf{x}^s \cong -\mathbf{x}^s$
   points on equator represent points at infinity.
   (*to* Cinderella)
3. Points $\mathbf{x} \in \mathbb{R}^3 \setminus 0$, only direction is relevant, $\mathbf{x} \cong -\mathbf{x}$
   points in plane $w = 0$ represent points at infinity



2D-point, Euclidean and spherical homogeneous coordinates



2D-point, Euclidean and spherical homogeneous coordinates

lines l in the plane

(2 d. o. f., result of edge extraction in image)

$$\underset{3\times1}{\mathbf{l}} = \begin{bmatrix} l_1 \\ l_2 \\ \hline l_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ \hline c \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \cos\phi \\ \sin\phi \\ \hline -s \end{bmatrix} = \begin{bmatrix} \boldsymbol{l}_h \\ \hline l_0 \end{bmatrix}$$

- inversion for lines $a^2 + b^2 \neq 0$

$$s = -\frac{c}{\sqrt{a^2+b^2}} \qquad \phi = \text{atan2}\,(b, a)$$

- partitioning: Euclidean part l_0, homogeneous part $\boldsymbol{l}_h$
- distance to origin

$$d_{lO} = \frac{|l_0|}{|\boldsymbol{l}_h|}$$

- lines at infinity with $\boldsymbol{l}_h = 0$: horizon

$$\mathbf{l}_\infty = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

proof: let

$$\lim_{d\to\infty} \begin{bmatrix} -\frac{\cos\phi}{d} \\ -\frac{\sin\phi}{d} \\ 1 \end{bmatrix}$$

- Euclidean normalization if $a^2 + b^2 \neq 0$ (no new information)

$$\mathbf{l}^e = \frac{\mathbf{l}}{|\boldsymbol{l}_h|} = \frac{\mathbf{l}}{\sqrt{a^2+b^2}}$$

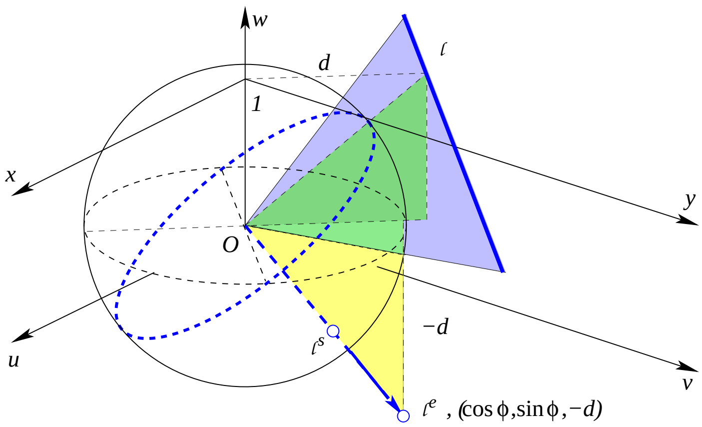
- Spherical normalization

$$\mathbf{l}^s = \mathsf{N}(\mathbf{l})$$

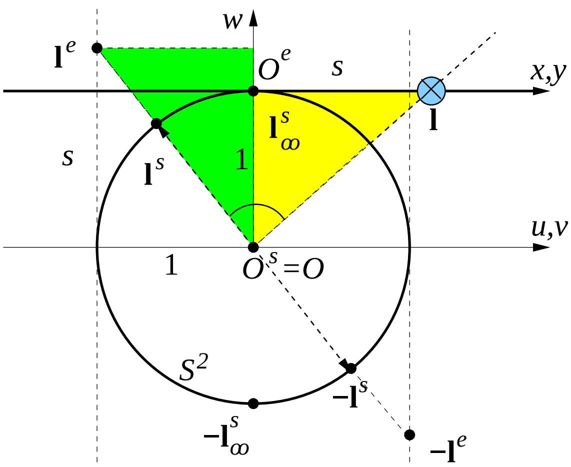
Visualization of lines in $\mathbb{P}^2$

1. Planes in $\mathbb{R}^3$ through origin.
   Intersection with horizontal plane yield lines in $\mathbb{R}^2$

   Horizontal plane represents line at infinity.
2. Great circle: intersection of plane with unit sphere s^2
   (*to* Cinderella)
3. Normal $\mathbf{l}$ to planes in $\mathbb{R}^3$ through origin,
   as $\mathbf{x}\cdot\mathbf{l} = 0$, any length
4. Normalized normal $\mathbf{l}^s \in S^2$
   normal is pole to great circle
   line is represented by pole on S^2



2D-line, Euclidean and spherical homogeneous coordinates, **plane**



2D-line, Euclidean and spherical homogeneous coordinates

Relations

- Incidence of point and line

$$\mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{x}\cdot\mathbf{l} = <\mathbf{x}, \mathbf{l}> = 0 \qquad \text{or} \qquad \boldsymbol{l}_h\cdot\boldsymbol{x}_0 + x_h l_0 = 0$$

- *Intersection* of two lines

$$\mathbf{x} = \mathbf{l} \cap \mathbf{m} \leftrightarrow (\mathbf{x}\cdot\mathbf{l} = 0 \text{ and } \mathbf{x}\cdot\mathbf{m} = 0)$$

therefore

$$\mathbf{x} = \mathbf{l} \times \mathbf{m}$$

or

$$\mathbf{x} = \mathsf{S}(\mathbf{l})\mathbf{m}$$

with skew symmetric matrix (axiator)

$$\mathsf{S}(\mathbf{l}) = \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix} = [\mathbf{l}]_\times$$

- *Join* of two points

$$\mathbf{l} = \mathbf{x} \wedge \mathbf{y} \leftrightarrow (\mathbf{l}\cdot\mathbf{x} = 0 \text{ and } \mathbf{l}\cdot\mathbf{y} = 0)$$

therefore

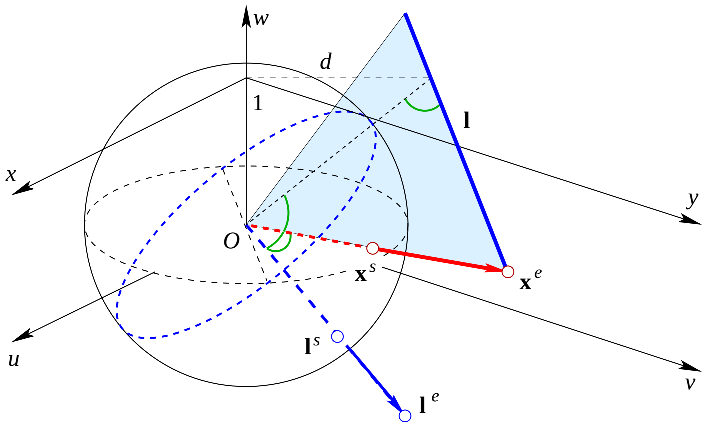
$$\mathbf{l} = \mathbf{x} \times \mathbf{y}$$

or

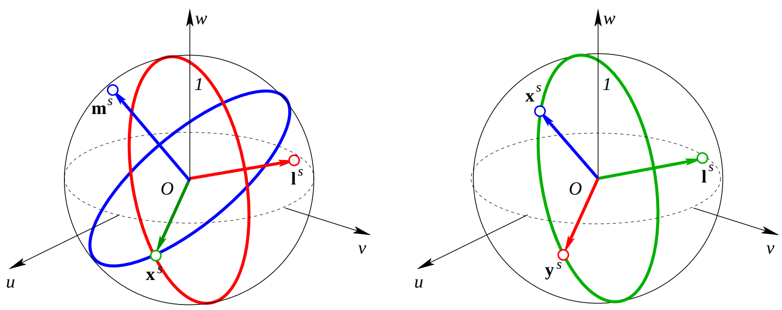
$$\mathbf{l} = \mathsf{S}(\mathbf{x})\mathbf{y}$$

with skew symmetric matrix (axiator)

$$\mathsf{S}(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} = [\mathbf{x}]_\times$$



Incidence of 2D-point and 2D-line, $\mathbf{x}^e \perp \mathbf{l}^e$ etc.



Intersection $\mathbf{x}^s = \mathsf{N}(\mathbf{l}^s \cap \mathbf{m}^s)$ and join $\mathbf{l}^s = \mathsf{N}(\mathbf{x}^s \wedge \mathbf{y}^s)$.

- Identity of points

$$\mathbf{x} = \lambda \mathbf{y} \qquad \text{or} \qquad \frac{\boldsymbol{x}_0}{x_h} = \frac{\boldsymbol{y}_0}{y_h} \qquad \text{or} \qquad \mathsf{N}(\mathbf{x}) = \mathsf{N}(\mathbf{y})$$

thus

$$\mathbf{x} \times \mathbf{y} = \mathbf{0} \qquad \text{or} \qquad x_h \boldsymbol{y}_0 - y_h \boldsymbol{x}_0 = \mathbf{0}$$

- Identity of lines

$$\mathbf{l} = \lambda \mathbf{m} \qquad \text{or} \qquad \frac{\boldsymbol{l}_h}{l_0} = \frac{\boldsymbol{m}_h}{m_0} \qquad \text{or} \qquad \mathsf{N}(\mathbf{l}) = \mathsf{N}(\mathbf{m})$$

thus

$$\mathbf{l} \times \mathbf{m} = \mathbf{0} \qquad \text{or} \qquad l_h \boldsymbol{m}_0 - m_h \boldsymbol{m}_0 = \mathbf{0}$$

- If $\mathbf{x}$ and $\mathbf{y}$ generate line $\mathbf{l}$, then all points

$$\mathbf{z} = a\mathbf{x} + b\mathbf{y} \in \mathbf{l}$$

Representation of 3D-Points and Planes

points $\mathbf{X}$ in space
(3 d. o. f., result of 3D reconstruction, grouping)

$$\underset{4\times 1}{\mathbf{X}} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \hline X_4 \end{bmatrix} = \begin{bmatrix} U \\ V \\ W \\ \hline T \end{bmatrix} = T \begin{bmatrix} X \\ Y \\ Z \\ \hline 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_0 \\ \hline X_h \end{bmatrix}$$

- inversion for points $T \neq 0$

$$X = \frac{U}{T} \qquad Y = \frac{V}{T} \qquad Z = \frac{W}{T}$$

- partitioning: Euclidean part $\boldsymbol{X}_0$, homogeneous part X_h

- distance to origin

$$d_{XO} = \frac{|\boldsymbol{X}_0|}{|X_h|}$$

- points at infinity with $T = X_h = 0$ at celestial sphere

$$\mathbf{X}_\infty = \begin{bmatrix} \boldsymbol{X}_0 \\ 0 \end{bmatrix}$$

  in direction (U, V, W) (let $\lim_{T\to 0} \mathbf{X}$)
- Euclidean normalization if $T \neq 0$ (no new information)

$$\mathbf{X}^e = \frac{\mathbf{X}}{X_h}$$

- Spherical normalization ($\mathbf{X} \in \mathbb{P}^3$)

$$\mathbf{X}^s = \mathsf{N}(\mathbf{X}) \doteq \frac{\mathbf{X}}{|\mathbf{X}|}$$

planes $\mathbf{A}$ in space
(3 d. o. f., result of 3D reconstruction, grouping)

$$\underset{4\times 1}{\mathbf{A}} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \hline A_4 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ \hline D \end{bmatrix} = \sqrt{A^2+B^2+C^2} \begin{bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \\ \hline -S \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_h \\ \hline A_0 \end{bmatrix}$$

- inversion for Planes $A^2 + B^2 + C^2 \neq 0$

$$S = -\frac{D}{\sqrt{A^2+B^2+C^2}} \qquad \alpha_i = \arccos \frac{A_i}{\sqrt{A_1^2+A_2^2+A_3^2}}$$

- partitioning: Euclidean part A_0, homogeneous part $\boldsymbol{A}_h$
- distance to origin

$$d_{AO} = \frac{|A_0|}{|\boldsymbol{A}_h|}$$

- plane at infinity with $\boldsymbol{A}_h = 0$: celestial sphere

$$\mathbf{A}_\infty = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Euclidean normalization if $T \neq 0$ (no new information)

$$\mathbf{A}^e = \frac{\mathbf{A}}{|\boldsymbol{A}_h|} = \frac{\mathbf{A}}{\sqrt{A^2 + B^2 + C^2}}$$

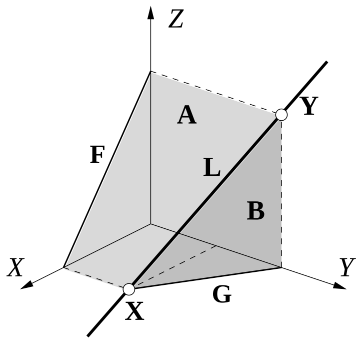
- Spherical normalization

$$\mathbf{A}^s = \mathsf{N}(\mathbf{A})$$

$\mathbf{A} \in \mathbb{P}^3$

3D-lines?

3D-Lines (I)



3D-lines have 4 degrees of freedom:
two points in given planes or two planes, e. g. parallel to Y- and Z-axis.

Julius Plücker
1801–1868

Mathematician
Discoveries in electronics

1829 Bonn
1834 Halle
1847 Bonn

Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement (1868-69)

PLÜCKER matrix

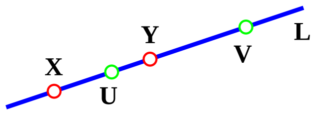
$$\underset{4\times 4}{\Gamma} = \mathbf{X}\mathbf{Y}^\mathsf{T} - \mathbf{Y}\mathbf{X}^\mathsf{T}$$

or

$$\gamma_{ij} = X_i Y_j - X_j Y_i = \begin{vmatrix} X_i & X_j \\ Y_i & Y_j \end{vmatrix}$$

- Diagonal of Plücker matrix=0: $\gamma_{ii} = 0$
- antisymmetric: $\gamma_{ij} = -\gamma_{ji}$
- six different values $\gamma_{ij}, i \neq j$
- rank 2-matrix

- Change of generating points



$$(\mathbf{X}, \mathbf{Y}) \rightarrow (\mathbf{U}, \mathbf{V}) = (a\mathbf{X} + b\mathbf{Y}, c\mathbf{X} + d\mathbf{Y})$$

$$\begin{bmatrix} U_i & U_j \\ V_i & V_j \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X_i & X_j \\ Y_i & Y_j \end{bmatrix}$$

yields

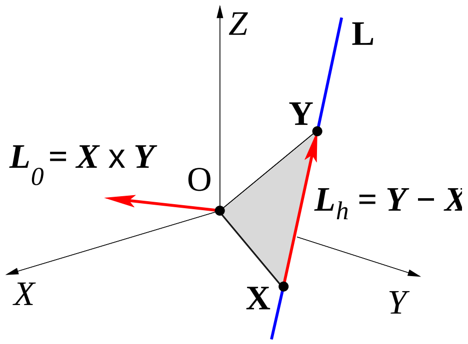
$$\gamma_{ij} \quad \rightarrow \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \gamma_{ij}$$

same matrix, homogeneity

Choice on sequence of elements

from $\Gamma = \begin{bmatrix} . & L_6 & . & . \\ . & . & L_4 & . \\ L_5 & . & . & . \\ L_1 & L_2 & L_3 & . \end{bmatrix}$: line $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ joining two points

$$\underset{6\times 1}{\mathbf{L}} = \mathbf{X} \wedge \mathbf{Y} = \begin{bmatrix} \gamma_{41} \\ \gamma_{42} \\ \gamma_{43} \\ \hline \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} X_4 Y_1 - Y_4 X_1 \\ X_4 Y_2 - Y_4 X_2 \\ X_4 Y_3 - Y_4 X_3 \\ \hline X_2 Y_3 - Y_2 X_3 \\ X_3 Y_1 - Y_3 X_1 \\ X_1 Y_2 - Y_1 X_2 \end{bmatrix} = \begin{bmatrix} X_h \boldsymbol{Y}_0 - Y_h \boldsymbol{X}_0 \\ \boldsymbol{X}_0 \times \boldsymbol{Y}_0 \end{bmatrix}$$



3D-lines can be represented with 6 parameters: $\{\boldsymbol{L}_h, \boldsymbol{L}_0\}$

PLÜCKER-coordinates

$$\underset{6\times 1}{\mathbf{L}} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \hline L_4 \\ L_5 \\ L_6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_h \\ \hline \boldsymbol{L}_0 \end{bmatrix}$$

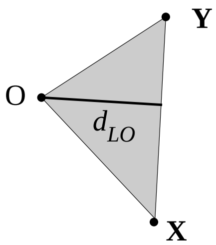
- direction vector $\boldsymbol{L}_h$, homogeneous part
- normal vector $\boldsymbol{L}_0$, Euclidean part
- homogeneous vector
- Plücker constraint

$$\boldsymbol{L}_0 \perp \boldsymbol{L}_h \qquad \boldsymbol{L}_0 \cdot \boldsymbol{L}_h = \boldsymbol{L}_0^{\mathsf{T}} \boldsymbol{L}_h = 0$$

- Distance from origin

$$d_{LO} = \frac{|\boldsymbol{L}_0|}{|\boldsymbol{L}_h|}$$

from area A of triangle (OXY)



$$2A = \left| \frac{\boldsymbol{X}_0}{X_h} \times \frac{\boldsymbol{Y}_0}{Y_h} \right| = \left| \frac{\boldsymbol{Y}_0}{Y_h} - \frac{\boldsymbol{X}_0}{X_h} \right| d_{LO}$$

- line at infinity

$$\mathbf{L}_\infty = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{L}_0 \end{bmatrix}$$

great circle at celestial sphere with normal $\boldsymbol{L}_0$

- Euclidean normalization

$$^e\mathbf{L} = \frac{\mathbf{L}}{|\boldsymbol{L}_h|}$$

with unit direction vector

$$^e\boldsymbol{L}_h = \frac{\boldsymbol{L}_h}{|\boldsymbol{L}_h|}$$

and

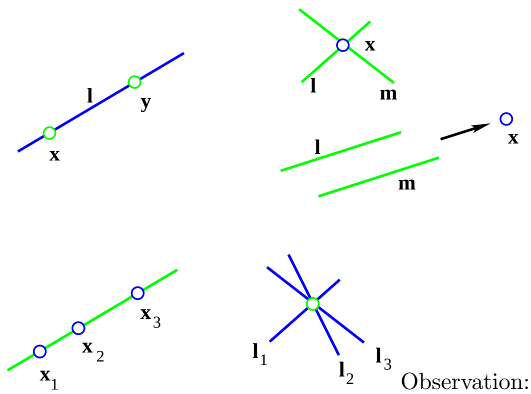
$$d_{LO} = |\, ^e\mathbf{L}_0|$$

- spherical normalization

$$^s\mathbf{L} = \frac{\mathbf{L}}{|\mathbf{L}|}$$

$(\mathbf{L} \in \mathbb{P}^5)$

Duality



Observation:
all statements appear in pairs with points and lines exchanged

Duality in 2D

Duality:

Correspondence of relations due to construction.

point $\mathbf{x}$	line $\mathbf{l}$
join $\mathbf{x} \wedge \mathbf{y}$ of points	intersection $\mathbf{l} \cap \mathbf{m}$ of lines
point $\mathbf{x}$ on line $\mathbf{l}$	line $\mathbf{l}$ through point $\mathbf{x}$
collinearity of points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$	concurrency of lines $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$

Duality relations in 2D

We use notation

$$\mathbf{l} = \mathbf{x} \wedge \mathbf{y} \qquad \circ\!\!-\!\!\bullet \qquad \mathbf{x} = \mathbf{l} \cap \mathbf{m}$$

Formally:

$$\mathbf{l} = \overline{\mathbf{x}} = D_3\mathbf{x} \qquad \text{with} \qquad D_3 = I_3$$

point $\mathbf{x}$ and dual line $\mathbf{l} = \overline{\mathbf{x}}$ have identical coordinates.

Inner product of two vectors

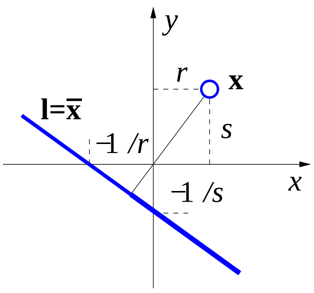
$$< \mathbf{x}, \mathbf{y} >_{M} = \mathbf{x}^{\mathsf{T}}\mathsf{M}\mathbf{y}$$

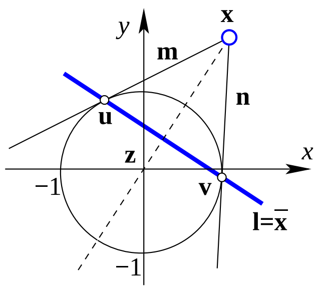
Determinant

$$D = |\mathbf{x};\ \underbrace{\mathbf{y}, \mathbf{z}}_{\mathbf{l}=\mathbf{y}\wedge\mathbf{z}}\ | = \mathbf{x}\cdot\overline{\mathbf{l}} = < \mathbf{x}, \mathbf{l} >_I = \mathbf{x}\cdot\mathbf{l} = \mathbf{x}^{\mathsf{T}}\mathbf{l}$$

and

$$D = |\mathbf{l};\ \underbrace{\mathbf{m}, \mathbf{n}}_{\mathbf{x}=\mathbf{m}\wedge\mathbf{n}}\ | = \mathbf{l}\cdot\overline{\mathbf{x}} = < \mathbf{l}, \mathbf{x} >_I = \mathbf{l}\cdot\mathbf{x} = \mathbf{l}^{\mathsf{T}}\mathbf{x}$$





Point $\mathbf{x}$ and dual line $\mathbf{l} = \overline{\mathbf{x}}$. Pole $\mathbf{x}$ and polar line $\mathbf{l} = \overline{\mathbf{x}}$.

Duality in 3D: Points and Planes

- points and planes by analogy
- lines?

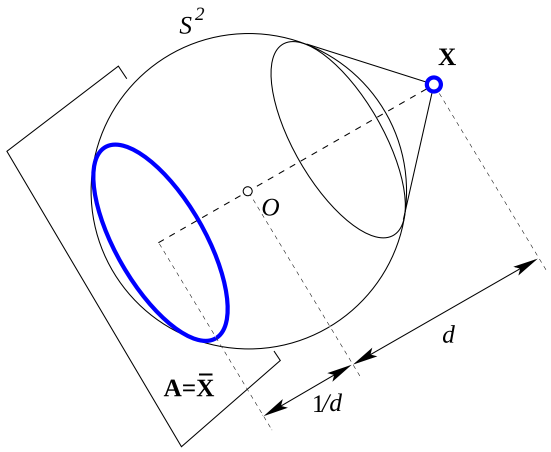
Duality points and planes

Formal:

$$\mathbf{A} = \overline{\mathbf{X}} = D_4\mathbf{X} \qquad \text{with} \qquad D_4 = I_4$$

point $\mathbf{X}$	plane $\mathbf{A}$
join $\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z}$ of points	intersection $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$ of planes
point $\mathbf{X}$ on plane $\mathbf{A}$	plane $\mathbf{A}$ through point $\mathbf{X}$
coplanarity of points $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ and $\mathbf{U}$	concurrency of planes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$

Duality relations in 3D; points and planes



Duality of 3D-points and planes

——— **B r e a k** ———

Duality with 3-Lines

Line from two planes, dually to line from points

Plücker matrix from planes

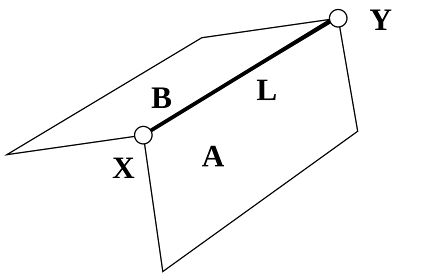
$$\underset{4\times 4}{\mathsf{G}} = \mathbf{A}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{A}^{\mathsf{T}}$$

or

$$g_{ij} = A_i B_j - A_j B_i = \begin{vmatrix} A_i & B_i \\ A_j & B_j \end{vmatrix}$$

relation between Plücker matrices $\mathsf{\Gamma}$ from points and G from planes

$$\mathsf{\Gamma} = \mathbf{X}\mathbf{Y}^{\mathsf{T}} - \mathbf{Y}\mathbf{X}^{\mathsf{T}} \qquad \circ\!\!-\!\!\bullet \qquad \mathsf{G} = \mathbf{A}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{A}$$



If line $\mathbf{X} \wedge \mathbf{Y}$ identical with $\mathbf{A} \cap \mathbf{B}$ then

$$\mathbf{X}^{\mathsf{T}}\mathbf{A} = 0 \qquad \mathbf{X}^{\mathsf{T}}\mathbf{B} = 0 \qquad \mathbf{Y}^{\mathsf{T}}\mathbf{A} = 0 \qquad \mathbf{Y}^{\mathsf{T}}\mathbf{B} = 0$$

We find

$$\begin{bmatrix} g_{23} \\ g_{31} \\ g_{12} \\ g_{41} \\ g_{42} \\ g_{43} \end{bmatrix} = \lambda \begin{bmatrix} \gamma_{41} \\ \gamma_{42} \\ \gamma_{43} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}$$

- line γ_{ij} is dual to line g_{ij} with same indices
- scaled, first and second 3-vector exchanged
- $\mathsf{\Gamma}$ and G equivalent, same information
- representation of 3D-line:
  Choice $\mathsf{\Gamma} = (\gamma_{ij})$ or $\mathsf{G} = (g_{ij})$?

We choose *Plücker coordinates*

$$\mathbf{L} = \begin{bmatrix} \boldsymbol{L}_h \\ \hline \boldsymbol{L}_0 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \hline L_4 \\ L_5 \\ L_6 \end{bmatrix} = \begin{bmatrix} L_{41} \\ L_{42} \\ L_{43} \\ \hline L_{23} \\ L_{31} \\ L_{12} \end{bmatrix}$$

and *Plücker matrix*

$$\Gamma(\mathbf{L}) = \left[\begin{array}{ccc|c} 0 & L_{12} & -L_{31} & -L_{41} \\ -L_{12} & 0 & L_{23} & -L_{42} \\ L_{31} & -L_{23} & 0 & -L_{43} \\ \hline L_{41} & L_{42} & L_{43} & 0 \end{array}\right] = \left[\begin{array}{c|c} -S_{L_0} & -\boldsymbol{L}_h \\ \hline \boldsymbol{L}_h^{\mathsf{T}} & 0 \end{array}\right]$$

Value of λ for dualling a 3D-line?

- Cannot be decided within our framework
- we choose $\lambda = 1$

Duality of 3D-lines

The dual $\overline{\mathbf{L}}$ of a 3D-line $\mathbf{L}$ is a 3D-line.

Formally

$$\overline{\mathbf{L}} = D_6 \mathbf{L} = \begin{bmatrix} \mathbf{0} & I_3 \\ I_3 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{L}_h \\ \boldsymbol{L}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_0 \\ \boldsymbol{L}_h \end{bmatrix}$$

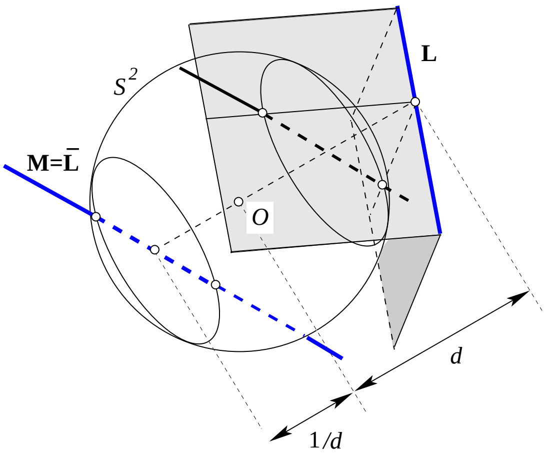
direction $\boldsymbol{L}_h$ and normal $\boldsymbol{L}_0$ are exchanged.

Dual Plücker matrix = Plücker matrix of dual line

$$\overline{\Gamma}(\mathbf{L}) \doteq \Gamma(\overline{\mathbf{L}}) = \left[\begin{array}{ccc|c} 0 & L_{43} & -L_{42} & -L_{23} \\ -L_{43} & 0 & L_{41} & -L_{31} \\ L_{42} & -L_{41} & 0 & -L_{12} \\ \hline L_{23} & L_{31} & L_{12} & 0 \end{array}\right] = \left[\begin{array}{c|c} -S_{L_h} & -\boldsymbol{L}_0 \\ \hline \boldsymbol{L}_0^{\mathsf{T}} & 0 \end{array}\right]$$

line $\mathbf{L}$	line $\overline{\mathbf{L}}$
coplanarity of lines $\mathbf{L}$ and $\mathbf{M}$	coplanarity of lines $\overline{\mathbf{L}}$ and $\overline{\mathbf{M}}$

Duality relations in 3D; lines



Duality of 3D-lines

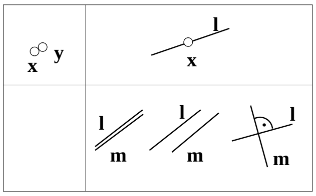
two partitionings

$$
\begin{aligned}
D &= |\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U}| \\
&= <\mathbf{X}, \mathbf{A}>= \mathbf{X} \cdot \overline{\mathbf{A}} = \mathbf{X}^{\mathsf{T}} \mathbf{A} \\
&= -<\mathbf{L}, \mathbf{M}>= -\mathbf{L} \cdot \overline{\mathbf{M}} = -\mathbf{L}^{\mathsf{T}} D_6 \mathbf{M}
\end{aligned}
$$

——— **B r e a k** ———

Constraints

Constraints in 2D



Elements	relation	eq.	d. o. f.
points $\mathbf{x}$, $\mathbf{y}$	$\mathbf{x} = \mathbf{y}$	$\mathbf{x} \times \mathbf{y} = \mathbf{0}$	2
lines $\mathbf{l}$, $\mathbf{m}$	$\mathbf{l} = \mathbf{m}$	$\mathbf{l} \times \mathbf{m} = \mathbf{0}$	2
point $\mathbf{x}$, line $\mathbf{l}$	$\mathbf{x} \in \mathbf{l}$	$\mathbf{x}^\mathsf{T}\mathbf{l} = 0$	1
lines $\mathbf{l}$, $\mathbf{m}$	$\mathbf{l} \perp \mathbf{m}$	$\mathbf{l}_h^\mathsf{T}\mathbf{m}_h = 0$	1
lines $\mathbf{l}$, $\mathbf{m}$	$\mathbf{l} \vert\vert \mathbf{m}$	$\vert\mathbf{l}_h, \mathbf{m}_h\vert = 0$	1

Observe: constraints (point on line in 2D)
but 3 algebraic constraints
thus: only two linearily independent constraints

Problem: selection of two linearily independent constraints:
explicit:

$$\mathsf{S}(\mathbf{x})\mathbf{y} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:
Select maximum absolute value in $\mathsf{S}(\mathbf{x})$ with index (jk)
Choose rows j and k of $\mathsf{S}(\mathbf{x})$
Thus independent constraints:

$$\begin{bmatrix} \boldsymbol{e}_j^{\mathsf{T}} \\ \boldsymbol{e}_k^{\mathsf{T}} \end{bmatrix} \mathsf{S}(\mathbf{x})\mathbf{y} = \mathbf{0}$$

will be generalized

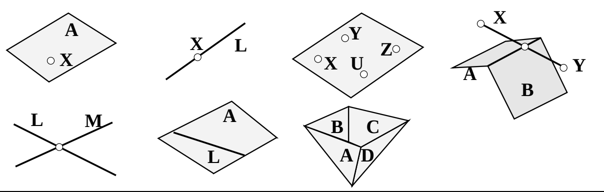
Constraints in 3D

Basis

$$D = |\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U}| = < \mathbf{X}, \mathbf{Y} \wedge \mathbf{Z} \wedge \mathbf{U} > = < \mathbf{X} \wedge \mathbf{Y}, \mathbf{Z} \wedge \mathbf{U} >$$

and

$$D = |\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}| = < \mathbf{A}, \mathbf{B} \cap \mathbf{C} \cap \mathbf{D} > = < \mathbf{A} \cap \mathbf{B}, \mathbf{C} \cap \mathbf{D} >$$



3D-elements	incidence	constraint	d. o. f.
point $\mathbf{X}$, plane $\mathbf{A}$	$\mathbf{X} \in \mathbf{A}$	$\mathbf{X}^{\mathsf{T}}\mathbf{A} = 0$	1
two lines $\mathbf{L}$, $\mathbf{M}$	$\mathbf{L} \cap \mathbf{M} \neq \emptyset$	$\mathbf{L}^{\mathsf{T}}\overline{\mathbf{M}} = 0$	1
point $\mathbf{X}$, line $\mathbf{L}$	$\mathbf{X} \in \mathbf{L}$	$\overline{\Gamma}(\mathbf{L})\mathbf{X} = \mathbf{0}$	2
line $\mathbf{L}$, plane $\mathbf{A}$	$\mathbf{L} \in \mathbf{A}$	$\Gamma(\mathbf{L})\mathbf{A} = \mathbf{0}$	2
lines $\mathbf{X} \wedge \mathbf{Y}$, $\mathbf{A} \cap \mathbf{B}$	$(\mathbf{X} \wedge \mathbf{Y}) \cap (\mathbf{A} \cap \mathbf{B}) \neq \emptyset$	$\mathbf{X}^{\mathsf{T}}(\mathbf{A}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{A}^{\mathsf{T}})\mathbf{Y} = 0$	1
four points	$\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z} \wedge \mathbf{W} \neq U$	$\lvert\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}\rvert = 0$	1
four planes	$\mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \cap \mathbf{D} \neq \emptyset$	$\lvert\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\rvert = 0$	1

Proof of 4. line:
point $\mathbf{X}$ on line $\mathbf{L} = \mathbf{C} \cap \mathbf{D}$, say, we prove: $\mathbf{X} \in \mathbf{L} = \mathbf{C} \cap \mathbf{D} \;\rightarrow\; \overline{\mathsf{\Gamma}}(\mathbf{L})\mathbf{X} = \mathbf{0}$

$$\mathbf{X} \in \mathbf{C},\ \mathbf{X} \in \mathbf{D} \rightarrow \mathbf{X}^{\mathsf{T}}\mathbf{C} = 0,\ \mathbf{X}^{\mathsf{T}}\mathbf{D} = 0 \rightarrow -(\mathbf{C}\mathbf{D}^{\mathsf{T}} - \mathbf{D}\mathbf{C}^{\mathsf{T}})\mathbf{X} = \mathbf{0}$$

Proof of 3. line, dual proof

Proof of 5.
coplanarity of lines $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ and $\mathbf{M} = \mathbf{A} \cap \mathbf{B}$ the we should have
$D = \mathbf{X}^{\mathsf{T}}(\mathbf{A}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{A}^{\mathsf{T}})\mathbf{Y} = \mathbf{X}^{\mathsf{T}}\ \mathsf{G}(\mathbf{A} \cap \mathbf{B})\ \mathbf{Y} = 0$

$$D = \sum_i \sum_j g_{ij} X_i Y_j = \sum_i \sum_{j>i} g_{ij}(X_i Y_j - X_j Y_i) = \overline{\mathbf{M}}^{\mathsf{T}}\mathbf{L} = < \mathbf{X} \wedge \mathbf{Y}, \mathbf{A} \cap \mathbf{B} >$$

Alternative expression for Plücker constraint (quadratic!)

$$\boldsymbol{L}_h \cdot \boldsymbol{L}_0 = 0 \qquad \leftrightarrow \qquad < \mathbf{L}, \mathbf{L} > = \mathbf{L}^{\mathsf{T}}\overline{\mathbf{L}} = \mathbf{L}^{\mathsf{T}} D_6 \mathbf{L} = 0$$

Selection of independent constraints:

$\mathsf{\Gamma}(\mathbf{L})\mathbf{A} = \mathbf{0}$ or $\overline{\mathsf{\Gamma}}(\mathbf{L})\mathbf{X} = \mathbf{0}$: rank $= 2$,
choose two rows i and j with largest element

Transformations

Transformations in 2D

Homography: plane to plane

$$\mathbf{x}' = \mathsf{H}\mathbf{x} \qquad \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

or

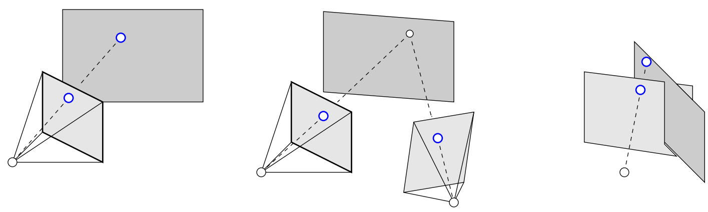
$$x' = \frac{au + bv + cw}{gu + hv + iw} \qquad y' = \frac{du + ev + fw}{gu + hv + iw}$$

rational functions:
linear in numerator and denominator, same denominator

Issues:

- Use
- Straight line preservation
- Number of parameters
- Jacobian w.r.t. transformation parameters
- Constraints
- Hierarchy of transformations
- Concatenation and inversion
- Transformation of lines

- **Use**



Projection of plane. Point transfer via plane Camera rotation for mosaiking

- **Straight line preserving**
  If three points are collinear, their images are also collinear:

  homography is straight line preserving

$$|\mathbf{x}', \mathbf{y}', \mathbf{z}'| = |\mathsf{H}\mathbf{x}, \mathsf{H}\mathbf{y}, \mathsf{H}\mathbf{z}| = |\mathsf{H}|^3 \; |\mathbf{x}, \mathbf{y}, \mathbf{z}| = 0$$

- **Number of parameters**
  8, homogeneous matrix

$$\mathsf{H} \cong \lambda \mathsf{H}$$

normalize (Frobenius norm)

$$||\mathsf{H}|| = \sum_{ij} h_{ij}^2 = 1$$

or

$$\max_{ij} |h_{ij}| = 1$$

$\rightarrow$
at least four points or at least four lines
are necessary for determining H

or 1 and 3 points/lines or vice versa

method cf. below

- **Jacobian w.r.t. transformation parameters**
  General

$$\underset{3\times1}{\mathbf{x}}{}' = \underset{3\times3}{\mathsf{H}} \; \underset{3\times1}{\mathbf{x}}$$

With the Kronecker-product this can be written as

$$\mathbf{x}' = \begin{bmatrix} \boldsymbol{h}_1^\mathsf{T} \\ \boldsymbol{h}_2^\mathsf{T} \\ \boldsymbol{h}_3^\mathsf{T} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \boldsymbol{h}_1^\mathsf{T}\mathbf{x} \\ \boldsymbol{h}_2^\mathsf{T}\mathbf{x} \\ \boldsymbol{h}_3^\mathsf{T}\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^\mathsf{T} & & \\ & \mathbf{x}^\mathsf{T} & \\ & & \mathbf{x}^\mathsf{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \\ \boldsymbol{h}_3 \end{bmatrix} = (I \otimes \mathbf{x}^\mathsf{T})\mathbf{h}$$

where elements of $\mathbf{H}$ row wise are collected in

$$\mathbf{h} = [h_{11}, h_{12}, ..., h_{33}]^\mathsf{T} = \mathrm{vec}(\mathsf{H}^\mathsf{T})$$

vec-operator, stacks columns of matrix

$$\text{vec}\mathsf{A} = \text{vec}(\boldsymbol{a}_1, ..., \boldsymbol{a}_N) = \begin{bmatrix} \boldsymbol{a}_1 \\ ... \\ \boldsymbol{a}_N \end{bmatrix}$$

Kronecker product takes all products

$$\mathsf{A} \otimes \mathsf{B} = \{a_{ij}\mathsf{B}\}$$

leads to

$$\text{vec}(\mathsf{ABC}) = (\mathsf{C}^\mathsf{T} \otimes \mathsf{A})\text{vec}\mathsf{B} = (\mathsf{A} \otimes \mathsf{C}^\mathsf{T})\text{vec}(\mathsf{B}^\mathsf{T})$$

especially

$$\boldsymbol{c} = \mathsf{A}\boldsymbol{b} = \mathsf{IA}\boldsymbol{b} \quad \rightarrow \quad \boldsymbol{c} = (\mathsf{I} \otimes \boldsymbol{b}^\mathsf{T})\text{vec}(\mathsf{A}^\mathsf{T}) = (\boldsymbol{b}^\mathsf{T} \otimes \mathsf{I})\text{vec}\mathsf{A}$$

$$d = \boldsymbol{a}^\mathsf{T}\mathsf{B}\boldsymbol{c} \quad \rightarrow \quad d = (\boldsymbol{a}^\mathsf{T} \otimes \boldsymbol{c}^\mathsf{T})\text{vec}(\mathsf{B}^\mathsf{T}) = (\boldsymbol{c}^\mathsf{T} \otimes \boldsymbol{a}^\mathsf{T})\text{vec}\mathsf{B}$$

- **Constraints**
  between points: 2 constraints, one redundant

$$\mathbf{x}' \times \mathsf{H}\mathbf{x} = \mathbf{0} \qquad \mathsf{S}(\mathbf{x}')\mathsf{H}\mathbf{x} = \mathbf{0}$$

cannot be solved for H

$\rightarrow$

use of *Kronecker*-product and vec-operator:

with colums of H

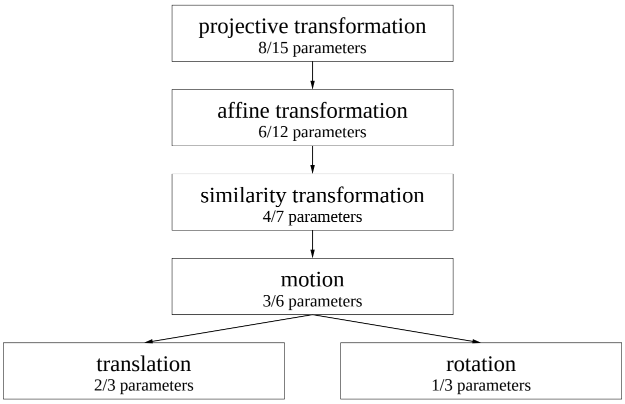
$$\mathbf{h} = \text{vec}(\mathsf{H}^\mathsf{T}) = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} \quad \rightarrow \quad (\mathsf{S}(\mathbf{x}') \otimes \mathbf{x}^\mathsf{T})\mathbf{h} = \mathbf{0}$$

or (choose 2)

$$\begin{bmatrix} 0 & 0 & 0 & -w'u & -w'v & -w'w & v'u & v'v & v'w \\ w'u & w'v & w'w & 0 & 0 & 0 & -u'u & -u'v & -u'w \\ -v'u & -v'v & -v'w & u'u & u'v & u'w & 0 & 0 & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- **Hierarchy of transformations**



Hierarchy of transformations with number of parameters in 2D and 3D

– Translation

$$\mathsf{H} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

– Rotation

$$\mathsf{H} = \begin{bmatrix} \boldsymbol{R} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

– Motion: rotation + translation or translation + rotation

$$\mathsf{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \quad \text{or} \quad \mathsf{H} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{Rt} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

– Similarity transformation

$$\mathsf{H} = \begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ -b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

– Individual scaling

$$\mathsf{H} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– Shears: symmetric or assymetric

$$\mathsf{H} = \begin{bmatrix} 1 & s & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{H} = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– Affine transformation:

$$\mathsf{H} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

Invariant: ideal points $\mathbf{x} = (u, v, 0)^\mathsf{T}$ stay

$$\begin{bmatrix} u' \\ v' \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

$\rightarrow$ Invariant: **parallelity of lines**
(interection of parallel lines stay at infinity)

- **Concatenation**

$${}^k\mathsf{H}_i = {}^k\mathsf{H}_j \, {}^j\mathsf{H}_i$$

Example: translation + rotation

$$\begin{bmatrix} R & R\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

Inversion

$${}^k\mathsf{H}_j = \left[{}^j\mathsf{H}_k \right]^{-1}$$

Example: Translation

$$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} I & -\boldsymbol{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

- **Transformation of lines**

$$\mathbf{l}' = \mathsf{H}^{-\mathsf{T}}\mathbf{l} \qquad \text{or} \qquad \mathsf{H}^{\mathsf{T}}\mathbf{l}' = \mathbf{l}$$

Proof: If $\mathbf{x} \in \mathbf{l}$ then $\mathbf{x}' \in \mathbf{l}'$:

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = \mathbf{x}'^{\mathsf{T}}\underbrace{\mathsf{H}^{-\mathsf{T}}\mathbf{l}}_{\mathbf{l}'} = \mathbf{x}'^{\mathsf{T}}\mathbf{l}' =$$

Observe duality: multiplication with transposed matrix from left

$$\mathbf{x}' = \mathsf{H}\mathbf{x} \quad \circ\!\!-\!\!\bullet \quad \mathsf{H}^{\mathsf{T}}\mathbf{l}' = \mathbf{l}$$

Transformations of 3D-Points and Planes

Homography: space to space

$$\mathbf{X}' = \mathsf{H}\mathbf{X} \qquad \begin{bmatrix} U' \\ V' \\ W' \\ T' \end{bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{bmatrix} \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix}$$

or

$$\begin{aligned} X' &= \frac{AU + BV + CW + DT}{MU + NV + OW + PT} \\ Y' &= \frac{EU + FV + GW + HT}{MU + NV + OW + PT} \\ Z' &= \frac{IU + JV + KW + LT}{MU + NV + OW + PT} \end{aligned}$$

- Use:
  - optical projection with lens (cf. Cinderella)
  - absolute orientation after relative orientation with uncalibrated straight line preserving cameras
- 15 parameters
- at least 5 3D-points, 5 planes or 4 3D-lines are necessary
- Transformation of planes

$$\mathbf{A}' = \mathsf{H}^{-\mathsf{T}}\mathbf{A} \qquad \text{or} \qquad \mathsf{H}^{\mathsf{T}}\mathbf{A}' = \mathbf{A}$$

Transformations of 3D-lines

If 3D-points are transformed according $\mathbf{X}' = \mathsf{H}\mathbf{X}$ then

$$\Gamma(\mathbf{L}') = \mathsf{H}\Gamma(\mathbf{L})\mathsf{H}^{\mathsf{T}}$$

Proof:

$$\Gamma(\mathbf{L}') = \mathbf{X}'\mathbf{Y}'^{\mathsf{T}} - \mathbf{Y}'\mathbf{X}'^{\mathsf{T}} = \mathsf{H}\mathbf{X}\mathbf{Y}^{\mathsf{T}}\mathsf{H}^{\mathsf{T}} - \mathsf{H}\mathbf{Y}\mathbf{X}^{\mathsf{T}}\mathsf{H}^{\mathsf{T}}$$

Transformation of Plücker coordinates?

3D-Motion of 3D-lines:

$$\begin{bmatrix} \boldsymbol{X}'_0 \\ X'_h \end{bmatrix} = \begin{bmatrix} R & \boldsymbol{T} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_0 \\ X_h \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \boldsymbol{X}'_0 \\ X'_h \end{bmatrix} = \begin{bmatrix} R\boldsymbol{X}_0 + X_h\boldsymbol{T} \\ X_h \end{bmatrix}$$

then

$$\begin{bmatrix} \boldsymbol{L}'_h \\ \boldsymbol{L}'_0 \end{bmatrix} = \begin{bmatrix} X'_h\boldsymbol{Y}'_0 - Y'_h\boldsymbol{X}'_0 \\ \boldsymbol{X}'_0 \times \boldsymbol{Y}'_0 \end{bmatrix} = \ldots = \begin{bmatrix} R\boldsymbol{L}_h \\ \mathsf{S}(\boldsymbol{T})R\,\boldsymbol{L}_h + R\,\boldsymbol{L}_0 \end{bmatrix}$$

or

$$\mathbf{L}' = \mathsf{H}_L\mathbf{L}$$

with

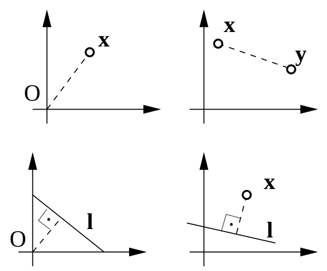
$$\mathsf{H}_L = \begin{bmatrix} R & \mathbf{0} \\ \mathsf{S}(\boldsymbol{T})R & R \end{bmatrix}$$

Summary:

- homogeneous coordinates for points, lines and planes
- duality: symmetry in representation
- easy representable constraints
- transformations homogeneous
- simple direct estimation of geometric entities and transformations

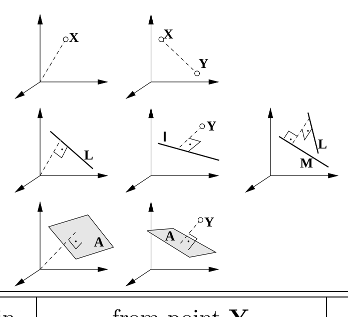
Distances and Signs

Distances in 2D



distances of	from origin	from point $\mathbf{y}$
2D-point $\mathbf{x}$	$d_{xO} = \dfrac{\lvert \boldsymbol{x}_0 \rvert}{\lvert x_h \rvert}$	$d_{xy} = \dfrac{\lvert x_h \boldsymbol{y}_0 - y_h \boldsymbol{x}_0 \rvert}{\lvert x_h y_h \rvert}$
2D-line $\mathbf{l}$	$d_{lO} = \dfrac{\lvert l_0 \rvert}{\lvert \boldsymbol{l}_h \rvert}$	$d_{ly} = \dfrac{\lvert \mathbf{y}^{\mathsf{T}} \mathbf{l} \rvert}{\lvert y_h \boldsymbol{l}_h \rvert}$

Distances in 3D



distance of	from origin	from point $\mathbf{Y}$	from line $\mathbf{M}$
3D point $\mathbf{X}$	$d_{XO} = \dfrac{\lvert \boldsymbol{X}_0 \rvert}{\lvert X_h \rvert}$	$d_{XY} = \dfrac{\lvert X_h \boldsymbol{Y}_0 - X_h \boldsymbol{X}_0 \rvert}{\lvert X_h Y_h \rvert}$	see below ↙
3D line $\mathbf{L}$	$d_{LO} = \dfrac{\lvert \boldsymbol{L}_0 \rvert}{\lvert \boldsymbol{L}_h \rvert}$	$d_{LY} = \dfrac{\lvert \boldsymbol{Y}_0 \times \boldsymbol{L}_h - Y_h \boldsymbol{L}_0 \rvert}{\lvert Y_h \boldsymbol{L}_h \rvert}$	$d_{LM} = \dfrac{\lvert \overline{\mathbf{L}}^{\mathsf{T}} \mathbf{M} \rvert}{\lvert \boldsymbol{L}_h \times \boldsymbol{M}_h \rvert}$
3D plane $\mathbf{A}$	$d_{AO} = \dfrac{\lvert A_0 \rvert}{\lvert \boldsymbol{A}_\mathbf{0} \rvert}$	$d_{AY} = \dfrac{\lvert \mathbf{A}^{\mathsf{T}} \mathbf{Y} \rvert}{\lvert Y_h \boldsymbol{A}_h \rvert}$	–

Proof of d_{LY}:
shift point $\mathbf{Y}$ and line $\mathbf{L}$ by $\boldsymbol{T} = -\boldsymbol{Y}_0/Y_h \rightarrow \mathbf{Y}$ becomes origin.

Proof of d_{LM} based on $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ and $\mathbf{M} = \mathbf{Z} \wedge \mathbf{T}$:

$$\begin{aligned}
\overline{\mathbf{L}}^{\mathsf{T}}\mathbf{M} &= \boldsymbol{L}_h^{\mathsf{T}}\boldsymbol{M}_0 + \boldsymbol{M}_h^{\mathsf{T}}\boldsymbol{L}_0 \\
&= (X_h\boldsymbol{Y}_0 - Y_h\boldsymbol{X}_0)^{\mathsf{T}}(\boldsymbol{Z}_0 \times \boldsymbol{T}_0) + (Z_h\boldsymbol{T}_0 - T_h\boldsymbol{Z}_0)^{\mathsf{T}}(\boldsymbol{X}_0 \times \boldsymbol{Y}_0) \\
&= [X_h\boldsymbol{Y}_0, \boldsymbol{Z}_0, \boldsymbol{T}_0] - [Y_h\boldsymbol{X}_0, \boldsymbol{Z}_0, \boldsymbol{T}_0] + \\
&\quad +[Z_h\boldsymbol{T}_0, \boldsymbol{X}_0, \boldsymbol{Y}_0] - [T_h\boldsymbol{Z}_0, \boldsymbol{X}_0, \boldsymbol{Y}_0] \\
&= [(X_h\boldsymbol{Y}_0 - Y_h\boldsymbol{X}_0), (Z_h\boldsymbol{T}_0 - T_h\boldsymbol{Z}_0), (Z_h\boldsymbol{X}_0 - X_h\boldsymbol{Z}_0)]/(X_h Z_h) \\
&= (\boldsymbol{L}_h \times \boldsymbol{M}_h) \cdot \left(\frac{\boldsymbol{X}_0}{X_h} - \frac{\boldsymbol{Z}_0}{Z_h}\right)
\end{aligned}$$

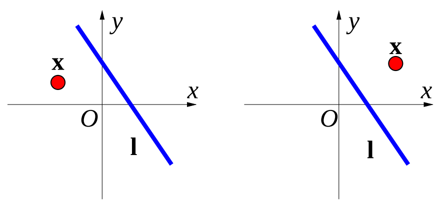
projection of $\overline{\mathbf{XZ}}$ onto normal $\boldsymbol{L}_h \times \boldsymbol{M}_h$ to $\mathbf{L}$ and $\mathbf{M}$
except for scale of $\boldsymbol{L}_h \times \boldsymbol{M}_h$

Signs in 2D

Relative position of point $\mathbf{x}$ and line $\mathbf{l}$

$$\frac{<\mathbf{x}, \mathbf{l}>}{\text{sign}(x_h l_0)}$$

> 0: point $\mathbf{x}$ on same side of $\mathbf{l}$ as origin



$\mathbf{x} \cdot \mathbf{l}/sign(x_h\ l_0) > 0$ $\mathbf{x} \cdot \mathbf{l}/sign(x_h\ l_0) < 0$

Point $\mathbf{y}$ in tringle $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$

Solve for $\mathbf{z}$

$$\left(\frac{\mathbf{x}_1}{x_{1h}}, \frac{\mathbf{x}_2}{x_{1h}}, \frac{\mathbf{x}_3}{x_{1h}}\right) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

If

$$\text{sign} z_1 = \text{sign} z_2 = \text{sign} z_3$$

then point in triangle

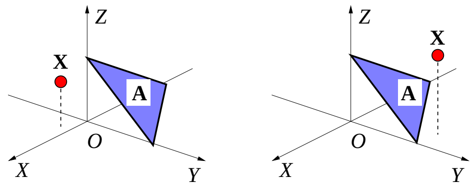
(z_i are Barycentric coordinates of $\mathbf{y}$ if normalized with $y_h = y_3$
as $z_1 + z_2 + z_3 = 1$)

Signs in 3D

Relative position of point **X** and line **A**

$$\frac{<\mathbf{X},\mathbf{A}>}{\text{sign}(X_h A_0)}$$

>0: point **X** on same side of **A** as origin

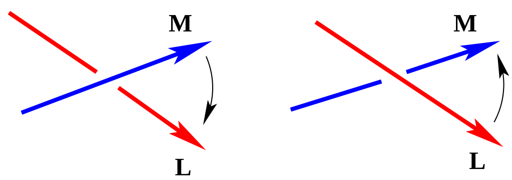


$\mathbf{X}\cdot\mathbf{A}\ /sign(X_h\ A_0) > 0$ $\quad$ $\mathbf{X}\cdot\mathbf{A}\ /sign(X_h\ A_0) < 0$

Relative position of two lines **L** and **M**

$$<\mathbf{L},\mathbf{M}>$$

>0: line **L** can be turned into **M** by right screw



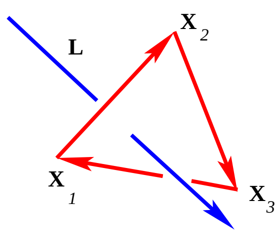
$\mathbf{L}^\mathsf{T}\ \overline{\mathbf{M}}\ > 0$ $\quad$ $\mathbf{L}^\mathsf{T}\ \overline{\mathbf{M}}\ < 0$

3D-line **L** through 3D-triangle $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$

If

$$\text{sign} <\mathbf{L}, \mathbf{X}_1 \wedge \mathbf{X}_2> = \text{sign} <\mathbf{L}, \mathbf{X}_2 \wedge \mathbf{X}_3> = \text{sign} <\mathbf{L}, \mathbf{X}_3 \wedge \mathbf{X}_1>$$

then line through triangle



useful for terrain (TIN) visualization

Constructions

- Bilinear or trilinear forms
- Symmetric or antisymmetric
- Matrix-vector products

$$\mathbf{a} = \mathsf{U}(\mathbf{b})\mathbf{c} = \mathsf{V}(\mathbf{c})\mathbf{b}$$

- Matrix representation of geometric entities

$$\mathsf{U}(\mathbf{b}) = \frac{\partial \mathbf{a}}{\partial \mathbf{c}} \qquad \mathsf{V}(\mathbf{c}) = \frac{\partial \mathbf{a}}{\partial \mathbf{b}}$$

linear in $\mathbf{b}$ and $\mathbf{c}$

- Interpretation of rows and columns
- Basis for simple error propagation

Constructions in 2D

2D-Line

$$\mathbf{l} = \mathbf{x} \wedge \mathbf{y} = \mathbf{x} \times \mathbf{y}$$

- antisymmetric $\mathbf{l} = \mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$
- Matrix vector product $\mathbf{l} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$
- Matrix representation, Jacobian:

$$\mathsf{S}(\mathbf{x}) \doteq \frac{\partial(\mathbf{x} \times \mathbf{y})}{\partial \mathbf{y}} = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix}$$

2D-Point

$$\mathbf{x} = \mathbf{l} \wedge \mathbf{m} = \mathbf{l} \times \mathbf{m}$$

- antisymmetric $\mathbf{x} = \mathbf{l} \wedge \mathbf{m} = -\mathbf{m} \wedge \mathbf{l}$
- Matrix vector product $\mathbf{x} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$
- Matrix representation , Jacobian:

$$\mathsf{S}(\mathbf{l}) = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

Constructions of 3D-Lines

3D-Line

from two points

$$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}: \ \mathbf{L} = \begin{bmatrix} X_h \boldsymbol{Y}_0 - Y_h \boldsymbol{X}_0 \\ \boldsymbol{X}_0 \times \boldsymbol{Y}_0 \end{bmatrix}$$

from two planes

$$\mathbf{L} = \mathbf{A} \cap \mathbf{B}: \ \mathbf{L} = \begin{bmatrix} \boldsymbol{A}_h \times \boldsymbol{B}_h \\ A_0 \boldsymbol{B}_h - B_0 \boldsymbol{A}_h \end{bmatrix}$$

Matrix representation of point (Π for *p*oints and *p*lanes)

$$\frac{\partial(\mathbf{X} \wedge \mathbf{Y})}{\partial \mathbf{Y}} \doteq \Pi(\mathbf{X}) = \begin{bmatrix} \boldsymbol{X}_h I & -\boldsymbol{X}_0 \\ S_{X_0} & \mathbf{0} \end{bmatrix} = \left[\begin{array}{ccc|c} T & 0 & 0 & -U \\ 0 & T & 0 & -V \\ 0 & 0 & T & -W \\ \hline 0 & -W & V & 0 \\ W & 0 & -U & 0 \\ -V & U & 0 & 0 \end{array}\right]$$

Thus

$$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = \Pi(\mathbf{X})\mathbf{Y}$$

- join is antisymmetric

$$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X} = -\mathbf{Y} \wedge \mathbf{X}$$

- rank and null space

$$\text{rank}[\Pi(\mathbf{X})] = 3 \qquad \Pi(\mathbf{X})\mathbf{X} = \mathbf{0}$$

Matrix representation of plane

$$\frac{\partial(\mathbf{A} \cap \mathbf{B})}{\partial \mathbf{B}} \doteq \overline{\Pi}(\mathbf{A}) = D_6 \Pi(\mathbf{A}) = \begin{bmatrix} S_{A_0} & \mathbf{0} \\ \boldsymbol{A}_0 \boldsymbol{I} & -\boldsymbol{A}_h \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & -C & B & 0 \\ C & 0 & -A & 0 \\ -B & A & 0 & 0 \\ \hline D & 0 & 0 & -A \\ 0 & D & 0 & -B \\ 0 & 0 & D & -C \end{array}\right]$$

Again antisymmetry

$$\mathbf{L} = \mathbf{A} \cap \mathbf{B} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A} = -\mathbf{B} \cap \mathbf{A}$$

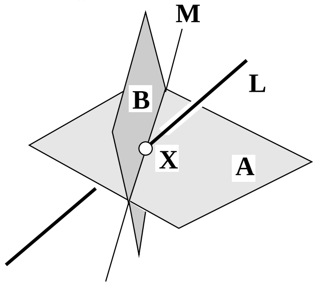
Observe duality relation

$$\mathbf{L} = \Pi(\mathbf{X})\mathbf{Y} \qquad \circ\!\!-\!\!\bullet \qquad \overline{\mathbf{L}} = \Pi(\mathbf{A})\mathbf{B})$$

Join and Intersection with 3D-Line

Intersection point $\mathbf{X}$ of 3D-line $\mathbf{L}$ and plane $\mathbf{A}$

$$\mathbf{X} = \mathbf{L} \cap \mathbf{A} = \mathbf{A} \cap \mathbf{L}$$



any plane $\mathbf{B} \neq \mathbf{A}$ through $\mathbf{X}$ intersects $\mathbf{A}$ in line $\mathbf{M} = \mathbf{A} \cap \mathbf{B}$ passing through $\mathbf{L}$

$$< \mathbf{L}, \mathbf{M} >= \mathbf{B}^\mathsf{T} \Gamma(\mathbf{L}) \mathbf{A} = -\mathbf{B}^\mathsf{T} \Gamma^\mathsf{T}(\mathbf{L}) \mathbf{A} = -\mathbf{B}^\mathsf{T} \mathbf{X} = \mathbf{0}$$

thus (taking the transposed Plücker matrix for symmetry reasons)

$$\mathbf{X} = \Gamma^\mathsf{T}(\mathbf{L})\mathbf{A} = \begin{bmatrix} \mathsf{S}(\boldsymbol{L}_0) & \boldsymbol{L}_h \\ -\boldsymbol{L}_h^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_h \\ A_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_0 \times \boldsymbol{A}_h + \boldsymbol{L}_h A_0 \\ -\boldsymbol{L}_h^\mathsf{T} \boldsymbol{A}_h \end{bmatrix}$$

but also (by comparison)

$$\mathbf{X} = \Pi^\mathsf{T}(\mathbf{A})\mathbf{L} = \begin{bmatrix} A_0 \boldsymbol{I} & -\mathsf{S}(\boldsymbol{A}_h) \\ -\boldsymbol{A}_h^\mathsf{T} & \mathbf{0}^\mathsf{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{L}_h \\ \boldsymbol{L}_0 \end{bmatrix} = \begin{bmatrix} A_0 \boldsymbol{L}_h - \boldsymbol{A}_h \times \boldsymbol{L}_0 \\ -\boldsymbol{A}_h^\mathsf{T} \boldsymbol{L}_h \end{bmatrix}$$

Finally

$$\mathbf{X} = \mathbf{L} \cap \mathbf{A} = \Gamma^\mathsf{T}(\mathbf{L})\mathbf{A} = \Pi^\mathsf{T}(\mathbf{A})\mathbf{L}$$

Plane $\mathbf{A}$ joining 3D-line $\mathbf{L}$ and point $\mathbf{X}$

$$\mathbf{A} = \mathbf{L} \wedge \mathbf{X} = \mathbf{X} \wedge \mathbf{L}$$

by duality

$$\mathbf{A} = \mathbf{L} \wedge \mathbf{X} = \overline{\Gamma}^\mathsf{T}(\mathbf{L})\mathbf{X} = \overline{\Pi}^\mathsf{T}(\mathbf{X})\mathbf{L}$$

as

$$\mathbf{X} = \Gamma^\mathsf{T}(\mathbf{L})\mathbf{A} = \Pi^\mathsf{T}(\mathbf{A})\mathbf{L} \qquad \circ\!\!-\!\!\bullet \qquad A = \Gamma^\mathsf{T}(\overline{\mathbf{L}})\mathbf{X} = \Pi^\mathsf{T}(\mathbf{X})\overline{\mathbf{L}}$$

Constructions

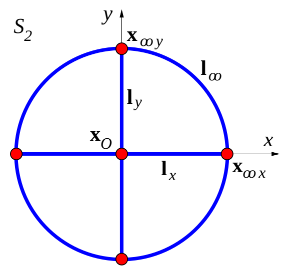
link	expression
$\mathbf{l} = \mathbf{x} \wedge \mathbf{y}$	$\mathbf{l} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$
$\mathbf{x} = \mathbf{l} \cap \mathbf{m}$	$\mathbf{x} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$
$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$	$\mathbf{L} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$
$\mathbf{L} = \mathbf{A} \cap \mathbf{B}$	$\mathbf{L} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A}$
$\mathbf{A} = \mathbf{L} \wedge \mathbf{X}$	$\mathbf{A} = \overline{\Gamma}^{\mathsf{T}}(\mathbf{L})\mathbf{X} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L}$
$\mathbf{X} = \mathbf{L} \cap \mathbf{A}$	$\mathbf{X} = \Gamma^{\mathsf{T}}(\mathbf{L})\mathbf{A} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L}$

Interpretation of Matrix Representations

- matrices $\mathsf{S}(\mathbf{x})$ and $\mathsf{S}(\mathbf{l})$ for 2D-points and 2D-lines
- matrices $\Pi(\mathbf{X})$ and $\Pi(\mathbf{A})$ for 3D-points and planes
- matrices $\Gamma(\mathbf{L})$ and $\overline{\Gamma}(\mathbf{L})$ for 3D-lines

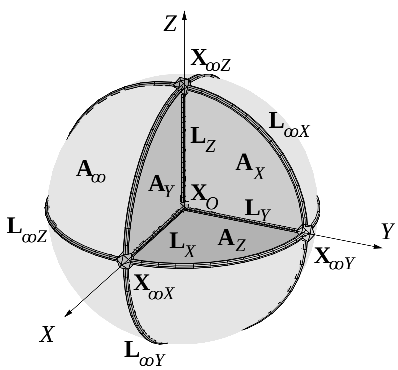
only 2D

Coordinate-Points, -Lines and -Planes



Coordinate-points and coordinate-lines in 2D using projective space P^2 seen from outside
(stereographic projection of sphere S^2 from south-pole onto plane $w = 0$, cf. part I, slide 24)

$$\mathbf{x}_{\infty x} = \mathbf{l}_y = \boldsymbol{e}_1 \qquad \mathbf{x}_{\infty y} = \mathbf{l}_x = \boldsymbol{e}_1 \qquad \mathbf{x}_O = \mathbf{l}_\infty = \boldsymbol{e}_3$$



Coordinate-points, coordinate-lines and coordinate-planes in 3D (S^3)

$$\mathbf{X}_{\infty X} = \mathbf{A}_X = \boldsymbol{e}_1,\ \mathbf{X}_{\infty Y} = \mathbf{A}_y = \boldsymbol{e}_2,\ \mathbf{X}_{\infty Z} = \mathbf{A}_Z = \boldsymbol{e}_3 \quad \mathbf{X}_O = \mathbf{A}_\infty = \boldsymbol{e}_4$$

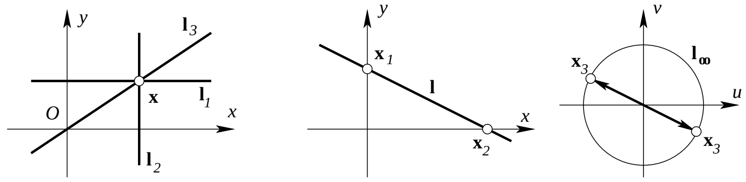
$$\mathbf{L}_X = \boldsymbol{e}_1 \quad \mathbf{L}_Y = \boldsymbol{e}_2 \quad \mathbf{L}_Z = \boldsymbol{e}_3 \quad \mathbf{L}_{\infty X} = \boldsymbol{e}_4 \quad \mathbf{L}_{\infty Y} = \boldsymbol{e}_5 \quad \mathbf{L}_{\infty Z} = \boldsymbol{e}_6$$

Interpretation of Matrix Representation of 2D-Point

$$\mathsf{S}(\mathbf{x}) \doteq \frac{\partial(\mathbf{x}\times\mathbf{y})}{\partial\mathbf{y}} = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} = [\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3]$$

Columns/rows: lines $\mathbf{l}_i$ through $\mathbf{x}$

$$\mathbf{l}_i = \mathsf{S}(\mathbf{x})\boldsymbol{e}_i^{[3]} = \mathbf{x}\times\boldsymbol{e}_i^{[3]}$$

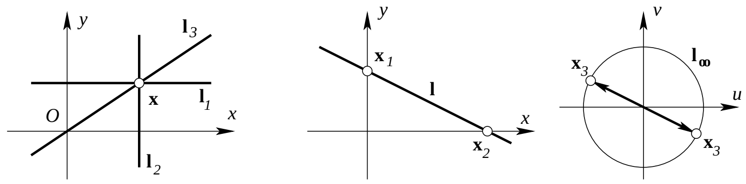


Interpretation of Matrix Representation of 2D-Line

$$\mathsf{S}(\mathbf{l}) = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$$

Columns/rows: points $\mathbf{x}_i$ on $\mathbf{l}$

$$\mathbf{x}_i = \mathsf{S}(\mathbf{l})\boldsymbol{e}_i^{[3]} = \mathbf{l}\times\boldsymbol{e}_i^{[3]}$$



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