Direct optimal estimation of geometric entities using algebraic projective geometry

Wolfgang Förstner
Institut für Photogrammetrie, Universität Bonn
Nussallee 15, D-53121 Bonn, wf@ipb.uni-bonn.de

Preface: Estimation theory was one of the first tools Bernhard Wrobel took into his hands when switching from his research in physical geodesy to photogrammetry, documented in his report prepared for the survey department at Bonn (Wrobel 1971). The classical Gauß-Markov model was used to solve the aerotriangulation with independent models. This setup conceptually is equivalent to the technically more demanding but in this application more intuitive Gauß-Helmert model. The so called Anblock method leads to nonlinear socalled pseudo observation equations which, however, are linear in the observations and all unknown parameters and practically do not require approximate values.

The following paper picks up recent developments in estimation theory which lead to an eigenvalue type solution which also practically requires no approximate values. However, it may be extended to problems which are described by constraints which are just linear in the unknown parameters, not necessarily in the observations. Using the classical reduction of the Gauß-Helmert model to the Gauß-Markov model using the pseudo observations yields a very concise estimation procedure.

The method is applied to the spatial forward intersection with points and lines using recent developments in projective geometry.

Abstract *The paper presents tools for optimally estimating 3D points and lines from 2D points and lines. It uses algebraic projective geometry for representing 2D and 3D geometric entities, perspective projection and its inversion. The uncertainty of the entities can easily be integrated. The direct solutions do not require approximate values.*

1 Motivation

Estimating 3D points from image points is a classical task in Photogrammetry. Automatic image processing allows to easily extract image line segments enabling to reconstruct 3D lines, besides points. In case of polyhedra, which may be used for a large class of man made objects, image line segments may also be used to determine 3D points and image points may be used to determine 3D lines.

Algebraic projective geometry allows to represent 2D points and lines, and 3D points, lines and planes in a way which simplifies geometric reasoning. More specifically, all pairwise relations between geometric entities yield bilinear expressions in their parameters. This allows simple error propagation.

Unfortunately the classical representations for 3D points and lines and their images lead to highly nonlinear expressions in the observation or condition equations, which makes approximate values for the initialization of the estimation procedure indispensible. Moreover, the relations are nonlinear in the sense, that products of observed and unknown quantities occur, which requires the use of the Gauß-Helmert model, also called mixed model or standard problem IV in adjustment theory. This model has the disadvantage, that the linearized constraints need to be linearily independent, which is difficult to guarantee in practical cases.

In case the unkown parameters occur linearly in the condition equations minimizing the contradiction, or the algebraic error allows a direct solution for the unknowns. It consists of determing the eigenvalue of a symmetric matrix, depending on the observed values. The solution has the disadvantage of being suboptimal in a statistical sense. Recently Leedan (Leedan 1997) has extended this method to yield an ML-estimate (cf. (Matei & Meer 1997)), moreover, also in the case of heteroscedastic observations, i. e. observations with different precision and possibly with correlations, assuming that each observation only takes part in one set of conditions and that correlations only occur between observation taking part in one set of condition equations.

This paper wants to show the advantage of linking algebraic projective geometry with estimating parameters using the direct solutions based on an eigenvector determination, replacing the solution of a linear equation system.

Estimating in the Gauß-Helmert model has been discussed thoroughly by Bernhard Wrobel in the context of the model block adjustment (WROBEL 1971). Especially the transformation of the Gauß-Helmert model into the Gauß-Markov model, which yields pseudo observations with an adequated covariance matrix will be essential in our development.

The paper derives an explicit estimation procedure and demonstrates the feasibility with an example from object reconstruction.

Notation: We denote coordinate vectors of planar geometric objects with small bold face letters, e. g. $\boldsymbol{x}$ in 3D space with capital bold face letters, e. g. $\boldsymbol{X}$. Vectors and matrices are denoted with slanted letters, thus $\boldsymbol{x}$ or R. Homogeneous vectors and matrices, which do represent the same object when multiplied with a scalar $\lambda \neq 0$, are denoted with upright letters, e. g. $\mathbf{x}$, $\mathbf{A}$ oder Π. Proportionality is denoted with $\cong$, e. g. $\mathbf{x} \cong \lambda\mathbf{x}$. We use the skew matrix

$$S(\boldsymbol{x}) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \tag{1}$$

of a 3-vector $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathsf{T}}$ in order to represent the cross product by:

$$\boldsymbol{a} \times \boldsymbol{b} = S(\boldsymbol{a})\boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} = -S(\boldsymbol{b})\boldsymbol{a} \tag{2}$$

2 REPRESENTATIONS

2.1 Relations and the Principle of Duality

We represent all geometric entities with homogeneous coordinates. They are elements of a projective space, as they may also be elements at infinity, e. g. a point at infinity being the intersection of two lines which are parallel.

We use two basic operations to derive one geometric entity γ from two others α and β:

1. The *join* $\gamma = \alpha \wedge \beta$ of two entities yields the linear space containing both entities. E. g. a line in 2D or 3D can be interpreted as the join of two points.

2. The *intersection* $\delta = \alpha \cap \beta$ of two entities yields the common linear subspace of both entities. E. g. a line in 3D can be seen as the intersection of two planes.

In projective space holds the principle of *duality*. Each geometric element has a dual element. In 2D points and lines are dual, in 3D points and planes are dual. The dimension of dual elements adds to $n+1$, where n is the dimension of the space of the element. Therefore to each line in 3D space having dimension 2 there exist a dual line. Moreover, join and intersection of two geomentric elements are dual operations. The incidence relation is dual to itself. The principle of duality now states: if an expression is true, its dual is also true. The dual expression is optained by exchanging all elements and relations with their dual. E. g.: The statement *'Two distinct 2D lines intersect in a unique point.'* is dual to the statement *'Two distinct 2D points are joined by a unique line.'* We will observe, that this principle is valid for all the following algebraic relations.

2.2 Points, Lines and Planes

Points $p(\mathbf{x})$ and lines $l(\mathbf{l})$ in the plane are represented with 3-vectors:

$$\mathbf{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \mathbf{l} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

resp. which can be derived from the usual representation

$$\mathbf{x} = w \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \mathbf{l} = \sqrt{a^2+b^2} \begin{pmatrix} \cos\phi \\ \sin\phi \\ -d \end{pmatrix}$$

where the factors can be chosen arbitrarily $\neq 0$.

Points $P(\mathbf{X})$ and planes $\varepsilon(\mathbf{A})$ in 3D space are represented with

$$\mathbf{X} = \begin{pmatrix} U \\ V \\ W \\ T \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

resp. which can be derived from the usual representation

$$\mathbf{X} = T \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \qquad \mathbf{A} = \sqrt{A^2 + B^2 + C^2} \begin{pmatrix} n_x \\ n_y \\ n_z \\ -d \end{pmatrix}$$

where again the factors can be chosen arbitrarily $\neq 0$.

Lines $L(\mathbf{L})$ in 3D are represented with their *Plücker coordinates*

$$\mathbf{L} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ \hline L_4 \\ L_5 \\ L_6 \end{pmatrix} = \begin{pmatrix} \boldsymbol{L} \\ \boldsymbol{L}_0 \end{pmatrix} \tag{3}$$

It can be derived from two points with Euklidean coordinates $\boldsymbol{X}$ and $\boldsymbol{Y}$

$$\mathbf{L}(\boldsymbol{X} \wedge \boldsymbol{Y}) = \begin{pmatrix} \boldsymbol{Y} - \boldsymbol{X} \\ \boldsymbol{X} \times \boldsymbol{Y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y} - \boldsymbol{X} \\ \mathsf{S}(\boldsymbol{X})\boldsymbol{Y} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{X} - \boldsymbol{Y} \\ \mathsf{S}(\boldsymbol{Y})\boldsymbol{X} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{X} - \boldsymbol{Y} \\ \boldsymbol{Y} \times \boldsymbol{X} \end{pmatrix} = -\mathbf{L}(\boldsymbol{Y} \wedge \boldsymbol{X}) \tag{4}$$

Observe, the line coordinates are linear in X_i and in Y_i and change their sign with the exchange of the points. Also, changing the points generating the line does not change the ratio of the line parameters. E. g. exchanging $\boldsymbol{X}$ by $\boldsymbol{X} + \lambda(\boldsymbol{Y} - \boldsymbol{X})$ leads to $(1 - \lambda)\mathbf{L}$. Therefore the line parameters are independent from the generating points.

The dual line $\overline{L}(\overline{\mathbf{L}})$ in 3D space is given by

$$\overline{\mathbf{L}} = \begin{pmatrix} \boldsymbol{L}_0 \\ \boldsymbol{L} \end{pmatrix} \tag{5}$$

With the matrix

$$\mathsf{C} = \begin{pmatrix} \mathbf{0} & \mathsf{I} \\ \mathsf{I} & \mathbf{0} \end{pmatrix}$$

it can be written as

$$\overline{\mathbf{L}} = \mathsf{C}\mathbf{L}$$

As $\mathsf{C}^2 = \boldsymbol{I}$ we have $\overline{\overline{\mathbf{L}}} = \mathbf{L}$.

The line parameters have to fulfill the *Plücker condition*

$$L_1 L_4 + L_2 L_5 + L_3 L_6 = \boldsymbol{L}^\mathsf{T} \boldsymbol{L}_0 = \frac{1}{2}\mathbf{L}^\mathsf{T}\overline{\mathbf{L}} = \frac{1}{2}\mathbf{L}^\mathsf{T}\mathsf{C}\mathbf{L} = 0 \tag{6}$$

which is clear, as $\boldsymbol{L} = \boldsymbol{Y} - \boldsymbol{X}$ is orthogonal to $\boldsymbol{L}_0 = \boldsymbol{X} \times \boldsymbol{Y}$.

2.2.1 Relations between Geometric Entities

All links between two geometric elements are shown to be bilinear in their homogeneous coordinates, an example being the line joining two points in 3D in (4). Thus the coordinates can be written in the form

$$\boldsymbol{\gamma} = \mathsf{A}(\boldsymbol{\alpha})\boldsymbol{\beta} = \mathsf{B}(\boldsymbol{\beta})\boldsymbol{\alpha}$$

Thus the matrices $\mathsf{A}(\boldsymbol{\alpha})$ and $\mathsf{B}(\boldsymbol{\beta})$ have entries being linear in the coordinates. At the same time they are the Jacobians of $\boldsymbol{\gamma}$:

$$\frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\beta}} = \mathsf{A}(\boldsymbol{\alpha}) \qquad \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\alpha}} = \mathsf{B}(\boldsymbol{\beta})$$

In detail we obtain the following relations:

1. The line $l(\mathbf{l})$ joining two points $p(\mathbf{x})$ and $q(\mathbf{y})$ in the plane is given by
$$\mathbf{l} = \mathbf{x} \wedge \mathbf{y} = \mathbf{x} \times \mathbf{y} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x} = -\mathbf{y} \times \mathbf{x} \tag{7}$$
The join in 2D obviously is equivalent with the cross product of the homogeneous coordinates.
This can be shown easily: A point $\mathbf{x}$ lies on the line $\mathbf{l}$ if $\mathbf{l}^\mathsf{T}\mathbf{x} = 0$ as known from the Hessian normal form. As $\mathbf{l}^\mathsf{T}\mathbf{x} = \mathbf{l}^\mathsf{T}\mathsf{S}(\mathbf{x})\mathbf{x} = \mathbf{l}^\mathsf{T}\mathbf{x} \times \mathbf{x} = \mathbf{0}$ and $\mathbf{l}^\mathsf{T}\mathbf{y} = -\mathbf{l}\mathsf{S}(\mathbf{y})\mathbf{y} = \mathbf{l}^\mathsf{T}\mathbf{y} \times \mathbf{y} = \mathbf{0}$ both points lie on the line.

2. The intersection point $p(\mathbf{x})$ of two lines $l(\mathbf{l})$ and $m(\mathbf{m})$ is given by
$$\mathbf{x} = \mathbf{l} \cap \mathbf{m} = \overline{\mathbf{l} \wedge \mathbf{m}} = \mathbf{l} \times \mathbf{m} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l} = -\mathbf{m} \cap \mathbf{l} \tag{8}$$
This is the dual situation to the previous one, as the homogeneous coordinates of dual points and lineas are identical.

3. The line $L(\mathbf{L})$ joining two points $P(\mathbf{X})$ and $Q(\mathbf{Y})$ in 3D space is given by (cf. (3))
$$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X} \tag{9}$$
with the matrix
$$\Pi(\mathbf{X}) = \left(\begin{array}{ccc|c} X_4 & 0 & 0 & -X_1 \\ 0 & X_4 & 0 & -X_2 \\ 0 & 0 & X_4 & -X_3 \\ \hline 0 & -X_3 & X_2 & 0 \\ X_3 & 0 & -X_1 & 0 \\ -X_2 & X_1 & 0 & 0 \end{array}\right) \tag{10}$$

4. The line $l(\mathbf{L})$ being the intersection of two planes $\varepsilon_1(\mathbf{A})$ and $\varepsilon_2(\mathbf{B})$ is given by
$$\mathbf{L} = \mathbf{A} \cap \mathbf{B} = \overline{\mathbf{A} \wedge \mathbf{B}} = \overline{\Pi(\mathbf{A})\mathbf{B}} = C\Pi(\mathbf{A})\mathbf{B} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A} \tag{11}$$
with the matrix
$$\overline{\Pi}(\mathbf{X}) = \left(\begin{array}{ccc|c} 0 & -X_3 & X_2 & 0 \\ X_3 & 0 & -X_1 & 0 \\ -X_2 & X_1 & 0 & 0 \\ \hline X_4 & 0 & 0 & -X_1 \\ 0 & X_4 & 0 & -X_2 \\ 0 & 0 & X_4 & -X_3 \end{array}\right) = C\Pi(\mathbf{X}) \tag{12}$$
being dual to $\Pi(\mathbf{X})$. Thus we could have written $\overline{\mathbf{L}} = C\mathbf{L} = \mathbf{A} \wedge \mathbf{B} = \Pi(\mathbf{A})\mathbf{B}$. Multiplying with $C^{-1} = C$ would have lead to (11). Observe that
$$\Pi^\mathsf{T}(\mathbf{X})\overline{\Pi}(\mathbf{X}) = \mathbf{0}$$
This is the dual situation to the the previous one, as analogeously, points and planes which are mutually dual, have the same coordinates.

5. The plane $\varepsilon(\mathbf{A})$ joining a point $P(\mathbf{X})$ and a line $L(\mathbf{L})$ is given by
$$\mathbf{A} = \mathbf{X} \wedge \mathbf{L} = \overline{\Pi}^\mathsf{T}(\mathbf{X})\mathbf{L} = \Gamma(\mathbf{L})\mathbf{X} = -\Gamma^\mathsf{T}(\mathbf{L})\mathbf{X} = \mathbf{L} \wedge \mathbf{X} \tag{13}$$
with the matrix
$$\Gamma(\mathbf{L}) = \left(\begin{array}{ccc|c} 0 & -L_3 & L_2 & L_4 \\ L_3 & 0 & -L_1 & L_5 \\ -L_2 & L_1 & 0 & L_6 \\ \hline -L_4 & -L_5 & -L_6 & 0 \end{array}\right) = -\Gamma^\mathsf{T}(\mathbf{L}) \tag{14}$$
This can easily be seen if the line is the join of $\mathbf{X} = (\boldsymbol{X}^\mathsf{T}, 1)^\mathsf{T}$ and $\mathbf{Y} = (\boldsymbol{Y}^\mathsf{T}, 1)^\mathsf{T}$, thus $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ as then
$$\Gamma(\mathbf{L}) = \begin{pmatrix} \mathsf{S}(\boldsymbol{Y} - \boldsymbol{X}) & \boldsymbol{X} \times \boldsymbol{Y} \\ -\boldsymbol{X}^\mathsf{T} \times \boldsymbol{Y}^\mathsf{T} & 0 \end{pmatrix}$$
and therefore $\mathbf{X} \wedge (\mathbf{X} \wedge \mathbf{L}) = \mathbf{X}^\mathsf{T}\mathbf{A} = 0$ and $\mathbf{Y} \wedge (\mathbf{X} \wedge \mathbf{L}) = \mathbf{Y}^\mathsf{T}\mathbf{A} = 0$.

6. The point $P(\mathbf{X})$ being the intersection of the plane $\varepsilon(\mathbf{A})$ and the line $L(\mathbf{L})$

$$\mathbf{X} = \mathbf{A} \cap \mathbf{L} = \overline{\mathbf{A} \wedge \mathbf{L}} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L} = \overline{\Gamma}(\mathbf{L})\mathbf{A} = -\overline{\Gamma}^{\mathsf{T}}(\mathbf{L})\mathbf{A} = \mathbf{L} \cap \mathbf{A} \tag{15}$$

with the rank 2 matrix

$$\overline{\Gamma}(\mathbf{L}) = \Gamma(\overline{\mathbf{L}}) \tag{16}$$

This is the dual situation to the the previous one. Observe that

$$\Gamma^{\mathsf{T}}(\mathbf{L})\overline{\Gamma}(\mathbf{L}) = \mathbf{0}$$

Therefore the nullspace of $\Gamma(\mathbf{L})$, which we need later, is the column space of $\overline{\Gamma}(\mathbf{L})$, thus can be spanned by the two columns of $\overline{\Gamma}(\mathbf{L})$ with largest absolute value. The same holds vice versa.

Incidence of two objects can use the inner products, namely for points and lines in the plane, for points and planes in 3D-space and for pairs of lines

$$<\mathbf{x}, \mathbf{l}> = \mathbf{x}^{\mathsf{T}}\mathbf{l} = 0, \quad <\mathbf{X}, \mathbf{A}> = \mathbf{X}^{\mathsf{T}}\mathbf{A} = 0, \quad <\mathbf{L}, \mathbf{M}> = \mathbf{L}^{\mathsf{T}}\overline{\mathbf{M}} = \mathbf{L}^{\mathsf{T}}C\mathbf{M} = 0 \tag{17}$$

The first two relations directly follow from the Hessian form of the 2D line and the plane. The last relation can be derived from the coplanarity condition of four points. Observe the selfduality of these relations. Also, the Plücker condition can be expressed as

$$<\mathbf{L}, \mathbf{L}> = \mathbf{L}^{\mathsf{T}}C\mathbf{L} = 0$$

The incidence of a line $\mathbf{L}$ with a point $\mathbf{X}$ or a plane $\mathbf{A}$ can be expressed as

$$\mathbf{X} \wedge \mathbf{L} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L} = \Gamma(\mathbf{L})\mathbf{X} = \mathbf{0} \qquad \mathbf{A} \cap \mathbf{L} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L} = \overline{\Gamma}(\mathbf{L})\mathbf{A} = \mathbf{0} \tag{18}$$

Both constraints result from the fact that the plane $\mathbf{X} \wedge \mathbf{L}$ and the point $\mathbf{A} \cap \mathbf{L}$ generated from two incident entities are indefinite, thus their homogeneous vectors are $\mathbf{0}$. Observe, both constraints only represent two geometric constaints (FÖRSTNER et al. 2000), e. g. the line needs to be parallel to the plane and have distance zero. They are linear in the coordinates of both entities which will be intensively used in the following.

Table 1 summarizes the expressions for constructing new geometric entities.

Table 1: *Construction of new geometric entities. The matrices* S, Π, $\overline{\Pi}$, Γ *and* $\overline{\Gamma}$ *are given in eqs. 1, 10, 12, 14 and 16 resp. All forms are linear in the coordinates of the given entities allowing rigorous error propagation.*

entity	link	expression	eq.
$p(\mathbf{x})$, $q(\mathbf{y})$	$l = p \wedge q$	$\mathbf{l} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$	(7)
$l(\mathbf{l})$, $m(\mathbf{m})$	$p = l \cap m$	$\mathbf{x} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$	(8)
$P_1(\mathbf{X})$, $P_2(\mathbf{Y})$	$L = P_1 \wedge P_2$	$\mathbf{L} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$	(9)
$\varepsilon_1(\mathbf{A})$, $\varepsilon_2(\mathbf{B})$	$L = \varepsilon_1 \cap \varepsilon_2$	$\mathbf{L} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A}$	(11),(5)
$P(\mathbf{X})$, $L(\mathbf{L})$	$\varepsilon = P \wedge L$	$\mathbf{A} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L} = \Gamma(\mathbf{L})\mathbf{X}$	(13)
$\varepsilon(\mathbf{A})$, $L(\mathbf{L})$	$P = \varepsilon \cap L$	$\mathbf{X} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L} = \overline{\Gamma}(\mathbf{L})\mathbf{A}$	(15)

2.2.2 Uncertainty of Geometric Entities

The uncertainty of the geometric entities can be represented by their covariance matrix. Due to the homogeneity of the representation the covariance matrices do not have full rank. All entities, except 3D lines, have covariances with rank deficiency 1, due to the unknown scaling; 3D lines have a covariance matrix with rank deficiency 2, due to the additional Plücker condition (6).

For testing and inversion we need the nullspaces of the covariance matrices. In case the given covariance matrix already has the correct rank, we only need to know its null space H. The pseudo inverse can be determined from

$$\begin{pmatrix} \mathbf{\Sigma}^{+} & -H(H^{\mathsf{T}}H)^{-1} \\ -H^{\mathsf{T}}(H^{\mathsf{T}}H)^{-1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{\Sigma} & H \\ H^{\mathsf{T}} & \mathbf{0} \end{pmatrix}^{-1} \tag{19}$$

As the product of both matrices is I we find $\mathbf{\Sigma} H = \mathbf{0}$ and $\mathbf{\Sigma}^+ H = \mathbf{0}$. The calculation using the inversion of an extended matrix is more efficient than the use of the explicit expression $\mathbf{\Sigma}^+ = (\mathbf{\Sigma} + H(H^\mathsf{T} H)^{-1} H^\mathsf{T})^{-1} - H(H^\mathsf{T} H)^{-1} H^\mathsf{T}$.

In case we fix the length of the geometric entities and take the Plücker condition for 3D lines into accout we have the following null spaces of the covariance matrices:

$$\mathcal{N}(\mathbf{\Sigma}_{xx}) = \mathbf{x},\ \mathcal{N}(\mathbf{\Sigma}_{ll}) = \mathbf{l},\ \mathcal{N}(\mathbf{\Sigma}_{XX}) = \mathbf{X},\ \mathcal{N}(\mathbf{\Sigma}_{AA}) = \mathbf{A},\ \mathcal{N}(\mathbf{\Sigma}_{LL}) = (\mathbf{L}, \overline{\mathbf{L}}) \tag{20}$$

These null spaces can be used to enforce the covariance matrix to have the correct rank.

In case the given covariance matrix of an entitiy is $\mathbf{\Sigma}^{(0)}$ and does not have the correct rank, the covariance matrix we will use can be determined by

$$\mathbf{\Sigma}_{xx} = P_g \mathbf{\Sigma}_{xx}^{(0)} P_g, \text{ with } \quad P_g = I - H(H^\mathsf{T} H)^{-1} H^\mathsf{T} \tag{21}$$

where H is the null space of the geometric entity.

We finally may use error propagation or the propagation of covariances of linear $\boldsymbol{y} = A\boldsymbol{x}$ functions of $\boldsymbol{x}$ with covariance matrix $\mathbf{\Sigma}_{xx}$ leading to $\mathbf{\Sigma}_{yy} = A\mathbf{\Sigma}_{xx}A^\mathsf{T}$ to obtain rigorous expressions for the covariance matrices of constructed entities $\boldsymbol{\gamma} = \mathsf{A}(\boldsymbol{\alpha})\boldsymbol{\beta} = \mathsf{B}(\boldsymbol{\beta})\boldsymbol{\alpha}$:

$$\mathbf{\Sigma}_{\gamma\gamma} = \mathsf{A}(\boldsymbol{\alpha})\mathbf{\Sigma}_{\beta\beta}\mathsf{A}^\mathsf{T}(\boldsymbol{\alpha}) + \mathsf{B}(\boldsymbol{\beta})\mathbf{\Sigma}_{\alpha\alpha}\mathsf{B}^\mathsf{T}(\boldsymbol{\beta})$$

in case of stochastic independence.

2.3 Projection and Inverse Projection

2.3.1 Points

The projection of a 3D point $P(\mathbf{X})$ onto the image plane yields the image point $p'(\mathbf{x}')$ via a direct linear transformation (DLT, cf. Fig. 1).

$$\mathbf{x}' = \mathsf{P}\mathbf{X} \qquad \text{or} \qquad \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} \mathbf{1}^\mathsf{T} \\ \mathbf{2}^\mathsf{T} \\ \mathbf{3}^\mathsf{T} \end{pmatrix} \mathbf{X} = \begin{pmatrix} <\mathbf{1}, \mathbf{X}> \\ <\mathbf{2}, \mathbf{X}> \\ <\mathbf{3}, \mathbf{X}> \end{pmatrix}$$

with the projection matrix

$$\mathsf{P} = \mathsf{K}R(I | -\boldsymbol{X}_o)$$

where $(\cdot|\cdot)$ denotes concatenation.

The 3×4 projection matrix P can be explicitely related to the 6 parameters of the exterior orientation and 5 parameters of the interior orientation namely the Euclidean coordinates $\boldsymbol{X}_o$ of the projection centre $O(\boldsymbol{X}_o)$, the rotation matrix R, the principle distance c, the coordinates (x'_H, y'_H) of the principle point, the shear s and the scale difference of the x'- and the y'-coordinates. The parameters of the interior orientation are collected in the 3×3 *calibration matrix*

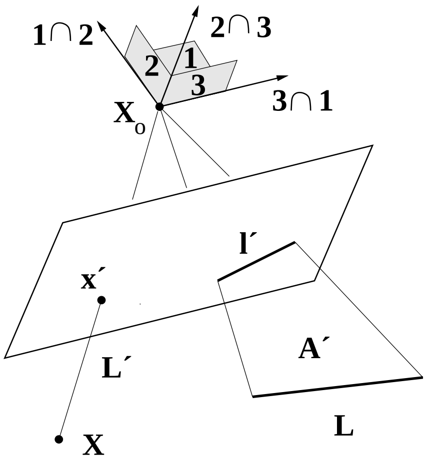
$$K \doteq \begin{pmatrix} c & cs & x'_H \\ 0 & c(1+m) & y'_H \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s & x'_H \\ 0 & 1+m & y'_H \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

It is an upper diagonal matrix and can be arbitrarily scaled, if no interpretation of its elements is required. Observe the part $R(\boldsymbol{I}| - \boldsymbol{X}_o)$ transforms the object coordinates into the camera system, the second factor $Diag(c, c, 1)$ of the calibration matrix performs the projection and the first factor the calibration.

The projection matrix in general has rank 3 and the null space of its transpose is the homogeneous vector $\mathbf{X}_o = (\boldsymbol{X}_o^\mathsf{T}, 0)^\mathsf{T}$ of the projection centre as $\mathsf{P}\mathbf{X}_o = \mathbf{0}$. Therefore the three row vectors $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ of the projection matrix P can be interpreted as the parameters of planes. The vector $\mathbf{1}$ is a plane through the line $u' = 0$ as $u' = \mathbf{1} \cdot \mathbf{X} = <\mathbf{1}, \mathbf{X}> = 0$, for all $\mathbf{X}$ and passes through the projection centre. Similarily $\mathbf{2}$ is a plane through $v' = 0$, and $\mathbf{3}$ is the focal plane parallel to the image plane, as then $w' = \mathbf{3} \cdot \mathbf{X} = 0$. The three planes intersect in the projection centre:

$$\mathbf{X}_o = \mathbf{1} \cap \mathbf{2} \cap \mathbf{3}$$

Figure 1: *shows the geometric situation for the projection of a 3D point* $\mathbf{X}$ *and a 3D line* $\mathbf{L}$ *into one image, yielding the image point* $\mathbf{x}' = \mathsf{P}\mathbf{X}$ *and the image line* $\mathbf{l} = \tilde{\mathsf{P}}\overline{\mathbf{L}}$. *The projection ray* $\mathbf{L}' = \tilde{\mathsf{P}}^{\mathsf{T}}\mathbf{x}'$ *and the projection plane* $\mathbf{A}' = \mathsf{P}^{\mathsf{T}}\mathbf{l}'$ *can easily be determined using the projection matrices for points and lines.*



2.3.2 Lines

A similar projection relation holds for 3D lines. We obtain the direct linear transformation of 3D lines (Faugeras & Papadopoulo 1998, Förstner 2000)

$$\mathbf{l}' = \tilde{\mathsf{P}}\overline{\mathbf{L}} \qquad \text{or} \qquad \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{1}}^{\mathsf{T}} \\ \tilde{\mathbf{2}}^{\mathsf{T}} \\ \tilde{\mathbf{3}}^{\mathsf{T}} \end{pmatrix} \overline{\mathbf{L}} = \begin{pmatrix} < \tilde{\mathbf{1}}, \mathbf{L} > \\ < \tilde{\mathbf{2}}, \mathbf{L} > \\ < \tilde{\mathbf{3}}, \mathbf{L} > \end{pmatrix}$$

with the 3×6 projection matrix $\tilde{\mathsf{P}}$

$$\tilde{\mathsf{P}} = \begin{pmatrix} (\mathbf{2} \cap \mathbf{3})^{\mathsf{T}} \\ (\mathbf{3} \cap \mathbf{1})^{\mathsf{T}} \\ (\mathbf{1} \cap \mathbf{2})^{\mathsf{T}} \end{pmatrix} \tag{23}$$

Its three rows are 6-vectors representing 3D lines, namely the intersections of the principle planes, thus the three coordinate axes of the camera system.

2.3.3 Inversion

Inversion of the projection leads to projection rays $\mathbf{L}'$ for image points $\mathbf{x}'$

$$\mathbf{L}' = \tilde{\mathsf{P}}^{\mathsf{T}}\mathbf{x}' = u'\mathbf{2} \cap \mathbf{3} + v'\mathbf{3} \cap \mathbf{1} + w'\mathbf{1} \cap \mathbf{2} \tag{24}$$

The expression for $\mathbf{L}'$ results from the incidence relation $\mathbf{x}'^{\mathsf{T}}\mathbf{l}' = 0$ for all lines $\mathbf{l}' = \tilde{\mathsf{P}}\overline{\mathbf{L}}$ passing through $\mathbf{x}'$, leading to $(\mathbf{x}'^{\mathsf{T}}\tilde{\mathsf{P}})\ \overline{\mathbf{L}} = < \mathbf{L}', \mathbf{L} >= 0$.

A similar expression can be given for the projection planes $\mathbf{A}'$ for image lines $\mathbf{l}'$

$$\mathbf{A}' = \mathsf{P}^{\mathsf{T}}\mathbf{l}' = a'\mathbf{1} + b'\mathbf{2} + c'\mathbf{3} \tag{25}$$

The expression results from the incidence relation $\mathbf{l}'^{\mathsf{T}}\mathbf{x}' = 0$ for all points $\mathbf{x}' = \mathsf{P}\mathbf{X}$ on the line $\mathbf{l}'$, leading to $(\mathbf{l}'^{\mathsf{T}}\mathsf{P})\ \mathbf{X} =< \mathbf{A}', \mathbf{X} >= 0$.

3 ESTIMATION

3.1 Best Estimates

We assume our estimation problem to have a special structure:

1. We want to determine U unknown parameters $\beta_u, u = 1, ..., U$, collected in the vector $\boldsymbol{\beta}$. We have I groups of observations $\boldsymbol{y}_i, i = 1, ..., I$ having n_i observations each. They may be collected in the vector $\boldsymbol{y}$ with length $N = \sum_i n_i$.

2. In our application we assume each of the I groups of observations to be linked with the unknown parameters $\boldsymbol{\beta}$ by a set of m_i constraints $w_{ij}, j = 1, ..., m_i$ collected in the vector

$$\mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta}) = \mathsf{A}_i^\mathsf{T}(\boldsymbol{y}_i)\, \boldsymbol{\beta} = \mathbf{0} \tag{26}$$

leading to the $M = \sum_i m_i$ constraints

$$\mathbf{w}(\boldsymbol{y}, \boldsymbol{\beta}) = \mathsf{A}^\mathsf{T}(\boldsymbol{y})\, \boldsymbol{\beta} = \mathbf{0} \tag{27}$$

The essential part is the *linearity* of these constraints in the *unknown parameters* and their *homogeneity*[1].

We later assume the constraints also to depend linearly on the observations, thus being of the structure

$$\mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta}) = \mathsf{A}_i(\boldsymbol{y}_i)\boldsymbol{\beta} = \mathsf{B}_i(\boldsymbol{\beta})\boldsymbol{y}_i = \mathbf{0} \tag{28}$$

The constraints are supposed to be valid for the true values of the unknown parameters and the observations. They should also hold for the fitted values $\widehat{\boldsymbol{y}}$ and $\widehat{\boldsymbol{\beta}}$.

With the matrices

$$\mathsf{A}(\boldsymbol{y}) = \begin{pmatrix} \mathsf{A}_1(\boldsymbol{y}_1) \\ \dots \\ \mathsf{A}_I(\boldsymbol{y}_I) \end{pmatrix} \qquad \mathsf{B}(\boldsymbol{\beta}) = \mathrm{Diag}(\mathsf{B}_i(\boldsymbol{\beta}))$$

the bilinear constraints can be written as

$$\mathbf{w}(\boldsymbol{y}, \boldsymbol{\beta}) = \mathsf{A}(\boldsymbol{y})\boldsymbol{\beta} = \mathsf{B}(\boldsymbol{\beta})\boldsymbol{y} = \mathbf{0}$$

3. Due to the homogeneity of the constraints (26) we need the additional constraint between the unknown parameters only

$$\boldsymbol{\beta}^\mathsf{T}\boldsymbol{\beta} = 1 \tag{29}$$

4. In case of estimated 3D lines we in addition have the Plücker condition $\mathbf{L}^\mathsf{T}\overline{\mathbf{L}} = 0$ being a constraint of the form

$$\frac{1}{2}\boldsymbol{\beta}^\mathsf{T} C \boldsymbol{\beta} = 0 \tag{30}$$

5. The observed values are uncertain, their uncertainty is given by

$$\underline{\boldsymbol{y}} \sim N(\tilde{\boldsymbol{y}}, \boldsymbol{\Sigma}_{yy}) = N(\tilde{\boldsymbol{y}}, \mathrm{Diag}(\boldsymbol{\Sigma}_{y_i y_i}))$$

stating the groups to be mutually independent, however allow for full covariance matrices within the groups, the tilde ˜ indicating the true value.

The optimal estimate $\widehat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$ is given by finding the minimum

$$\Omega = \frac{1}{2}(\widehat{\boldsymbol{y}} - \boldsymbol{y})^\mathsf{T}\boldsymbol{\Sigma}_{yy}^{+}(\widehat{\boldsymbol{y}} - \boldsymbol{y}) = \frac{1}{2}\sum_{i=1}^{I}(\widehat{\boldsymbol{y}}_i - \boldsymbol{y}_i)^\mathsf{T}\boldsymbol{\Sigma}_{y_i y_i}^{+}(\widehat{\boldsymbol{y}}_i - \boldsymbol{y}_i) \tag{31}$$

under the given constraints. Observe, we allow the observations to be correlated with a covariance matrix not having full rank. The effective number of observations then will be lower than N namely $\sum_i \mathrm{rk}\,(\boldsymbol{\Sigma}_{y_i y_i})$.

We will split the estimation problem into two parts: The first one takes only the basic constraints (27) and the normalization constraint (29) into account. Instead of iteratively solving a set of normal equations we iteratively solve an eigenvalue problem, which practically needs no approximate values. The solution is a direct generalization of the classical procedure for directly solving a problem of type (26) with constraint groups of size 1.

The second step then updates this estimate based on the additional constraints (30).

This two step procedure is due to Matei et al. (Matei & Meer 1997, Leedan 1997). In our case the basic constraints are all bilinear which leads to a simplified procedure.

[1] In case they are not homogeneous, the following expressions become more involved (Matei & Meer 1997)

3.2 Minimizing the Algebraic Distance

We first give a solution which does not take the uncertainty of the observed values and the additional constraint into account. Thus we give a solution to the problem

$$\mathsf{A}_i^{\mathsf{T}}(\widehat{\boldsymbol{y}}_i)\,\widehat{\boldsymbol{\beta}} = 0,\ I = 1, ..., I \qquad \widehat{\boldsymbol{\beta}}^{\mathsf{T}}\widehat{\boldsymbol{\beta}} = 1$$

The first constraint will not be fulfilled by the given observations $\boldsymbol{y}_i$ leading to the residuals of the constraints

$$\mathbf{w}_i = \mathbf{w}(\boldsymbol{y}_i, \widehat{\boldsymbol{\beta}}) = \mathsf{A}_i^{\mathsf{T}}(\boldsymbol{y}_i)\,\widehat{\boldsymbol{\beta}},\ i = 1, ..., I$$

Therefore we minimize the algebraic distance

$$\Omega_1 = \sum_{i=1}^{I} \mathbf{w}_i^{\mathsf{T}}\,\mathbf{w}_i = \boldsymbol{\beta}^{\mathsf{T}} \left(\sum_{i=1}^{I} \mathsf{A}_i(\boldsymbol{y}_i)\,\mathsf{A}_i^{\mathsf{T}}(\boldsymbol{y}_i) \right) \boldsymbol{\beta} \qquad \text{under} \qquad \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{\beta} = 1$$

This minimization problem leads to the solution:

$$\widehat{\boldsymbol{\beta}} = \mathbf{e}_i | \lambda_i(\mathsf{M}) = \min$$

stating the optimal estimate $\widehat{\boldsymbol{\beta}}$ to be the smallest normalized eigenvector of the matrix

$$\mathsf{M} = \sum_{i=1}^{I} \mathsf{A}_i(\boldsymbol{y}_i)\,\mathsf{A}_i^{\mathsf{T}}(\boldsymbol{y}_i) = \mathsf{A}(\boldsymbol{y})\,\mathsf{A}^{\mathsf{T}}(\boldsymbol{y})$$

This is a *direct* solution, as no approximate values are necessary. This solution can also be used in case the constraints are linear only in the unknown parameters but possibly nonlinear in the observations (cf. (Duda & Hart 1973), pp. 332, pp. 377 and (Taubin 1993)).

The solution obviously is suboptimal as it depends on the mutual scaling of the observations and does not take into account their uncertainty. Taubin gives a solution which is invariant to the scaling of the variables (Taubin 1993), which is identical to the optimal solution for uncorrelated observations with the same variance. Matei and Meer give a solution to the case of observations of different weight.

3.3 Minimizing the Weighted Algebraic Distance

The following solution is equivalent to the one of Matei and Meer [2], but it is much simpler.

We just need to take the uncertainty of the residuals $\mathbf{w}_i(\boldsymbol{y}_i, \widehat{\boldsymbol{\beta}})$ of the constraints into account. This follows from (26)

$$\boldsymbol{\Sigma}_{w_i w_i} = \left(\left. \frac{\partial \mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta})}{\partial \boldsymbol{y}_i} \right|_{y=\widehat{y}, \beta=\widehat{\beta}} \right) \boldsymbol{\Sigma}_{y_i y_i} \left(\left. \frac{\partial \mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta})}{\partial \boldsymbol{y}_i} \right|_{y=\widehat{y}, \beta=\widehat{\beta}} \right)^{\mathsf{T}} \tag{32}$$

This leads to the following optimization problem which in case of normally distributed observations yields the ML-estimate: Minimize the form

$$\Omega = \mathbf{w}^{\mathsf{T}} \boldsymbol{\Sigma}_{ww}^{+} \mathbf{w} = \sum_{i=1}^{I} \mathbf{w}_i^{\mathsf{T}} \boldsymbol{\Sigma}_{w_i w_i}^{+} \mathbf{w}_i \rightarrow \min \tag{33}$$

under the given constraints. The pseudo inverse is to be taken in case the constraints are linearily dependent.

It is known from adjustment theory, that the minimum of

$$\Omega = \widehat{\boldsymbol{e}}^{\mathsf{T}} \boldsymbol{\Sigma}_{yy}^{+} \widehat{\boldsymbol{e}} = \mathbf{w}^{\mathsf{T}} \boldsymbol{\Sigma}_{ww}^{+} \mathbf{w}$$

from (31) is identical to the minimum of Ω in (33) in case the same constraints are used.

The reason is that the two models, the Gauß-Helmert model

$$\boldsymbol{c}_w + A\boldsymbol{\Delta\beta} - B\boldsymbol{e} = \mathbf{0} \qquad D(\boldsymbol{e}) = \boldsymbol{\Sigma}_{yy} \tag{34}$$

with

$$\boldsymbol{c}_w = \boldsymbol{w}(\boldsymbol{y}^{(0)}, \boldsymbol{\beta}^{(0)}) + B(\boldsymbol{y} - \boldsymbol{y}^{(0)}) \tag{35}$$

[2] except for the bias, which is taken into account in theirs.

and the Gauß-Markov model

$$-\boldsymbol{c}_w + \boldsymbol{v} = A\boldsymbol{\Delta}\beta \qquad D(\boldsymbol{v}) = B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}} \tag{36}$$

with

$$\boldsymbol{v} = B\boldsymbol{e}$$

under the same constraints lead to the same normal equation system, thus to the same estimates and therefore to the same sum of the weigthed squared residuals (cf. Appendix). Here $\mathsf{B}(\widehat{\boldsymbol{\beta}})$ is identical to the Jacobian in (32). This reduction of the Gauß-Helmert model (34) to the Gauß-Markov model, also called the *Vermittelnde Ausgleichung* (36) here leads to the *Quasi vermittelnde Ausgleichung* useful for modeling the block adjustment with independent photogrammetric models (WROBEL 1971).

We now specialize to the structure of our constraints. Together with the factorization of $\mathbf{w}$ we therefore need to find the optimal value for $\widehat{\boldsymbol{\beta}}$ minimizing

$$\Omega = \boldsymbol{\beta}^{\mathsf{T}}\mathsf{A}^{\mathsf{T}}(\boldsymbol{y})\left(\mathsf{B}(\boldsymbol{\beta})\boldsymbol{\Sigma}_{yy}\mathsf{B}^{\mathsf{T}}(\boldsymbol{\beta})\right)^{+}\mathsf{A}(\boldsymbol{y})\,\boldsymbol{\beta}$$

under the constraint

$$\widehat{\boldsymbol{\beta}}^{\mathsf{T}}\widehat{\boldsymbol{\beta}} = 1$$

This leads to the optimal estimate $\widehat{\boldsymbol{\beta}}$ being the normalized eigenvector corresponding to the smallest eigenvalue of the matrix[3] (see appendix eq. (55))

$$\mathsf{M} = \mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}})\left(\mathsf{B}(\widehat{\boldsymbol{\beta}})\boldsymbol{\Sigma}_{yy}\mathsf{B}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}})\right)^{+}\mathsf{A}(\boldsymbol{y})$$

Observe, we need the estimated values $\widehat{\boldsymbol{\beta}}$ for the error propagation of $\mathbf{w}(\boldsymbol{y},\widehat{\boldsymbol{\beta}})$ in $\mathsf{B}(\widehat{\boldsymbol{\beta}})\boldsymbol{\Sigma}_{yy}\mathsf{B}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}})$ and the fitted observations $\widehat{\boldsymbol{y}}$ in the left factor $\mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}})$. Therefore we need to iterate. This is done by using the unknown parameters $\widehat{\boldsymbol{\beta}}^{(\nu-1)}$ and the fitted observations $\widehat{\boldsymbol{y}}^{(\nu-1)}$ from the previous iteration and determine the minimum eigenvector $\widehat{\boldsymbol{\beta}}^{(\nu)}$ of

$$\mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}}^{(\nu-1)})\left(\mathsf{B}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\boldsymbol{\Sigma}_{yy}\mathsf{B}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\right)^{+}\mathsf{A}(\boldsymbol{y})\,\widehat{\boldsymbol{\beta}}^{(\nu)} = \lambda\,\widehat{\boldsymbol{\beta}}^{(\nu)} \tag{37}$$

The fitted values of the observations can be determined individually from

$$\widehat{\boldsymbol{y}}_i^{(\nu-1)} = \left(I - \boldsymbol{\Sigma}_{y_iy_i}\mathsf{B}_i^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\left(\mathsf{B}_i(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\boldsymbol{\Sigma}_{y_iy_i}\mathsf{B}_i^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\right)^{+}\mathsf{B}_i^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\right)\boldsymbol{y}_i \tag{38}$$

Taking the constraint $\widehat{\boldsymbol{\beta}}^{\mathsf{T}}\widehat{\boldsymbol{\beta}} = 1$ into account we also can determine the covariance matrix $\boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}}$ of the estimated value from

$$\boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} = [\mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}})\left(\mathsf{B}(\widehat{\boldsymbol{\beta}})\boldsymbol{\Sigma}_{yy}\mathsf{B}(\widehat{\boldsymbol{\beta}})\right)^{+}\mathsf{A}(\widehat{\boldsymbol{y}})]^{+} \tag{39}$$

using its null space $\widehat{\boldsymbol{\beta}}$ in (19).

The estimated variance factor is given by:

$$\widehat{\sigma}_0^2 = \frac{\Omega}{R} \tag{40}$$

where Ω is taken from (33) and the redundancy R is the number of effective constraints G_{eff} reduced by the number of effective unknowns $U - 1$, i. e.

$$R = \sum_i \text{rk}\,(\boldsymbol{\Sigma}_{w_iw_i}) \;-\; (U-1)$$

the number of unknown parameters β_i being U. In case the redundancy is large enough, say > 30, this can be used to determine the estimated covariance matrix of the unknown parameters

$$\widehat{\boldsymbol{\Sigma}}_{\widehat{\beta}\widehat{\beta}} = \widehat{\sigma}_0^2\boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}}$$

[3]Matai and Meer first take the Jacobian of Ω with respect to the unknowns $\boldsymbol{\beta}$ and from that derive a generalized eigenvalue problem to solve for $\boldsymbol{\beta}$. Here we obtain an ordinary eigenvalue problem.

3.4 Further Constraints

In cased further constraints are to be fullfilled we need to update the estimate. This is performed using the method of constraints between the observations alone by taking the estimates $\widehat{\boldsymbol{\beta}}$ as observations and impose the desired constraint.

In case these further constraints are of the form

$$\boldsymbol{h}(\widehat{\widehat{\boldsymbol{\beta}}}) = 0$$

the optimal estimate is given by

$$\widehat{\widehat{\boldsymbol{\beta}}} = \widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{e}}$$

with

$$\widehat{\boldsymbol{e}} = \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} J (J^{\mathsf{T}} \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} J)^{-1} h(\widehat{\boldsymbol{\beta}})$$

where

$$J = \left. \frac{\partial \boldsymbol{h}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \right|^{\mathsf{T}}_{\beta = \widehat{\beta}^{(0)}}$$

is the Jacobian of the constraints evaluated at an initial value for the final estimate. This induces an iteration procedure, which however, will converge quickly in our case, as the initial values are very close to the final estimate.

In our case we have the constraint $h(\boldsymbol{\beta}) = \frac{1}{2}\boldsymbol{\beta}^{\mathsf{T}} C \boldsymbol{\beta} = 0$ with the Jacobian

$$J = C\boldsymbol{\beta}^{(\nu)}$$

with

$$\boldsymbol{\beta}^{(0)} = \widehat{\boldsymbol{\beta}}$$

in the first iteration. Therfore the estimate can be obtained iteratively from

$$\widehat{\widehat{\boldsymbol{\beta}}} = \widehat{\boldsymbol{\beta}} - \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} C \widehat{\boldsymbol{\beta}} (\widehat{\boldsymbol{\beta}}^{\mathsf{T}} C^{\mathsf{T}} \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} C \widehat{\boldsymbol{\beta}})^{-1} h(\boldsymbol{\beta})$$

This can be simplified using

$$\sigma_h^2 = \widehat{\boldsymbol{\beta}}^{\mathsf{T}} C^{\mathsf{T}} \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} C \widehat{\boldsymbol{\beta}} \tag{41}$$

to

$$\widehat{\widehat{\boldsymbol{\beta}}} = \widehat{\boldsymbol{\beta}} - \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} C \widehat{\boldsymbol{\beta}} \, \frac{h(\boldsymbol{\beta})}{\sigma_h^2} \tag{42}$$

The covariance of the final estimate is given by

$$\boldsymbol{\Sigma}_{\widehat{\widehat{\beta}}\widehat{\widehat{\beta}}} = \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} - \frac{1}{\sigma_h^2} \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} C \widehat{\boldsymbol{\beta}} \widehat{\boldsymbol{\beta}}^{\mathsf{T}} C^{\mathsf{T}} \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} \tag{43}$$

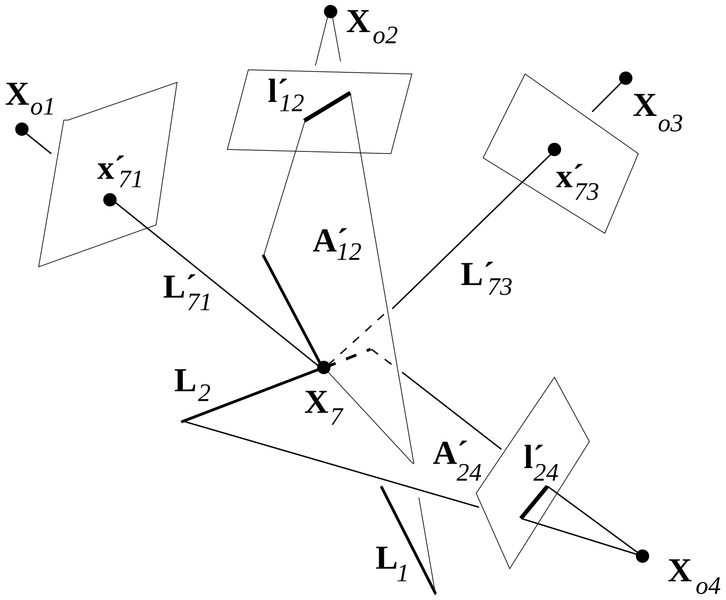
The estimated variance factor of the complete estimate can be determined from

$$\widehat{\widehat{\sigma}}_o^2 = \frac{\Omega + \dfrac{h^2}{\sigma_h^2}}{R+1}$$

In case of the redundancy is large enough this may be used to determine the estimated covariance matrix

$$\widehat{\boldsymbol{\Sigma}}_{\widehat{\widehat{\beta}}\widehat{\widehat{\beta}}} = \widehat{\widehat{\sigma}}_0^2 \boldsymbol{\Sigma}_{\widehat{\widehat{\beta}}\widehat{\widehat{\beta}}}$$

Figure 2: *shows 3D point $\mathbf{X}_7$ observed from 4 cameras. In two of the images the point is observed, namely $\mathbf{x}'_{71}$ in image 1 and $\mathbf{x}'_{73}$ in image 3. In the other two images 2 and 4 two lines $\mathbf{L}_1$ and $\mathbf{L}_2$ are observed leading to $\mathbf{l}'_{12}$ and $\mathbf{l}'_{24}$. Estimation of $\mathbf{X}_7$ uses the incidence of the projecting lines $\mathbf{L}'_{71}$ and $\mathbf{L}'_{73}$ and of the projecting planes $\mathbf{A}'_{12}$ and $\mathbf{A}'_{24}$ with the 3D point $\mathbf{X}_7$.*



4 SUBOPTIMAL ESTIMATES FOR 3D POINTS AND LINES

4.1 Minimizing the Algebraic Distance for Estimating 3D Points

We want to determine the coordinates of a 3D point (cf. Fig. 2). We assume I image points $\mathbf{x}'_{ik}$ and J image lines $\mathbf{l}'_{jk}$ to be observed, the second index indicating the image in which the feature has been observed. The observed points are images of the unknown 3D point, the observed lines are images of 3D lines passing through the 3D point.

We first want to give the solution for a 3D point $\mathbf{X}$ when minimizing the algebraic distance.

We have the *point-line constraint* for the i-th point $\mathbf{x}'_{ik}$ observed in the k-th image using (18)

$$\mathbf{0} = \mathbf{w}_{ik} = \mathbf{X} \wedge \mathbf{L}'_{ik} = \Gamma(\mathbf{L}'_{ik})\mathbf{X}$$

with the projecting ray $\mathbf{L}'_i$ from (24)

$$\mathbf{L}'_{ik} = \tilde{\mathsf{P}}_k^{\mathsf{T}} \mathbf{x}'_{ik}$$

We also have the *point-plane constraint* for the j-th image segment $\mathbf{l}'_{jk}$ in the k-th image using (17)

$$0 = w_{jk} =< \mathbf{A}'_{jk}, \mathbf{X} >= \mathbf{A}'^{\mathsf{T}}_{jk}\mathbf{X}$$

with the projecting plane $\mathbf{A}'_j$ from (25)

$$\mathbf{A}'_{jk} = \mathsf{P}_k^{\mathsf{T}} \mathbf{l}'_{jk}$$

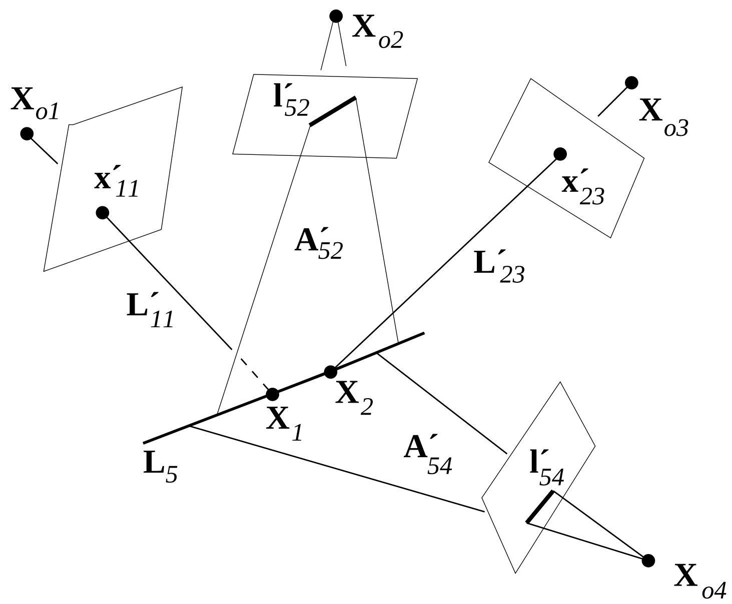
Minimizing

$$\Omega = \sum_{ik} \mathbf{w}_{ik}^{\mathsf{T}} \mathbf{w}_{ik} + \sum_{jk} w_{jk}^2$$

is identical to minimizing

$$\Omega = \mathbf{X}^{\mathsf{T}} \left(\sum_{ik} \Gamma^{\mathsf{T}}(\mathbf{L}'_{ik})\Gamma(\mathbf{L}'_{ik}) + \sum_{jk} \mathbf{A}'_{jk} \mathbf{A}_{jk}'^{\mathsf{T}} \right) \mathbf{X}$$

Figure 3: *shows 3D line* $\mathbf{L}_5$ *observed from 4 cameras. In two of the images the line is observed, namely* $\mathbf{l}'_{52}$ *in image 2 and* $\mathbf{l}'_{54}$ *in image 4. In the other two images 1 and 3 two points* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *on* $\mathbf{L}_5$ *are observed leading to* $\mathbf{x}'_{11}$ *and* $\mathbf{x}'_{23}$*. Estimation of* $\mathbf{L}_5$ *uses the incidence of the projecting lines* $\mathbf{L}'_{11}$ *and* $\mathbf{L}'_{23}$ *and of the projecting planes* $\mathbf{A}'_{52}$ *and* $\mathbf{A}'_{54}$ *with the 3D line* $\mathbf{L}_5$*.*



under the constraint $\mathbf{X}^\mathsf{T}\mathbf{X} = 1$. The direct solution is given by the smallest normalized eigenvector of

$$\mathsf{M} = \sum_{ik} \Gamma^\mathsf{T}(\mathbf{L}'_{ik})\Gamma(\mathbf{L}'_{ik}) + \sum_{jk} \mathbf{A}'_{jk}\mathbf{A}_{jk}{}'^\mathsf{T}$$

4.2 Minimizing the Algebraic Distance for Estimating 3D Lines

Analogically we can setup constraints for an unknown 3D line $\mathbf{L}$ (cf. fig. 3).

In case we observe the j-th line $\mathbf{l}'_{jk}$ in image k we have the *line-plane constraint*, namely the projecting plane $\mathbf{A}'_j$ to pass through the 3D line

$$\mathbf{0} = \mathbf{w}_{jk} = \mathbf{A}'_{jk} \cap \mathbf{L} = \Pi^\mathsf{T}(\mathbf{A}'_{jk})\mathbf{L}$$

where the projecting plane again can be determined from $\mathbf{A}'_{jk} = \mathsf{P}_k^\mathsf{T}\mathbf{l}'_{jk}$

In case we have the image point $\mathbf{x}'_{ik}$ of a 3D point lying on the unknown 3D line we have the *line-line constraint*, namely the projection ray $\mathbf{L}'_{ik}$ to pass throught he unknown line

$$0 = w_{ik} = < \mathbf{L}'_{ik}, \mathbf{L} > = \overline{\mathbf{L}}{}'^\mathsf{T}_{ik}\mathbf{L}$$

where again the projecting line can be determined from $\mathbf{L}'_{ik} = \tilde{\mathsf{P}}_k^\mathsf{T}\mathbf{x}'_{ik}$. Minimizing

$$\Omega = \sum_{ik} w_{ik}^2 + \sum_{jk} \mathbf{w}_{jk}^\mathsf{T}\mathbf{w}_{jk}$$

is identical to minimizing

$$\Omega = \mathbf{L}^\mathsf{T}\left(\sum_{ik} \overline{\mathbf{L}}'_{ik}\overline{\mathbf{L}}'^\mathsf{T}_{ik} + \sum_{jk} \Pi(\mathbf{A}'_{jk})\Pi^\mathsf{T}(\mathbf{A}'_{jk})\right)\mathbf{L}$$

under the constraints $\mathbf{L}^\mathsf{T}\mathbf{L} = 1$ and $\mathbf{L}^\mathsf{T}\overline{\mathbf{L}} = 0$ The direct $\widehat{\mathbf{L}}$ solution for $\mathbf{L}$ without the second constraint is given by the normalized smallest eigenvector of

$$\mathsf{N} = \sum_{ik} \overline{\mathbf{L}}'_{ik}\overline{\mathbf{L}}'^\mathsf{T}_{ik} + \sum_{jk} \Pi(\mathbf{A}'_{jk})\Pi^\mathsf{T}(\mathbf{A}'_{jk}) \quad (44)$$

The final solution, which takes the Plücker constraint into account can use $C\widehat{\beta} = C\widehat{\mathbf{L}} = \widehat{\overline{\mathbf{L}}}$ in (42) and is given by

$$\widehat{\widehat{\mathbf{L}}} = \widehat{\mathbf{L}} - \mathbf{\Sigma}_{\widehat{L}\widehat{L}}\widehat{\overline{\mathbf{L}}}\frac{h}{\sigma_h^2}$$

with

$$h = \frac{1}{2} < \widehat{\mathbf{L}}, \widehat{\mathbf{L}} >$$

and

$$\sigma_h^2 = \widehat{\overline{\mathbf{L}}}^{\mathsf{T}} \mathbf{\Sigma}_{\widehat{L}\widehat{L}}\widehat{\overline{\mathbf{L}}}$$

4.3 Minimal Solutions

The above mentioned direct solution require a minimum number of observations to be available.

The estimation of the 3D point is possible if we have

1. at least two non parallel projecting rays $\mathbf{L}'$ or
2. at least three pairwise non parallel projecting planes $\mathbf{A}'$ or
3. at least one projecting ray $\mathbf{L}'$ and one non parallel projecting plane $\mathbf{A}'$

Then the rank of the matrix M is three.

The estimation of the 3D line is possible if we have

1. at least two non parallel projecting planes $\mathbf{A}'$ or
2. at least one projecting plane $\mathbf{A}'$ and two projecting lines $\mathbf{L}'$ which are not parallel to $\mathbf{A}'$ and do not meet the plane in the same 3D point.
3. at least 5 (!) linearily independent projecting lines $\mathbf{L}'$ or

Then the rank of the matrix N is five.

Actually there is also a minimal *solution* for the 3D line, having four degrees of freedom, in case of only *four lines* meeting it. The matix N then has rank 4, thus there is no unique eigenvector corresponding to the smallest eigenvalue. If the nullspace of N is spanned by the two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ the unknown 3D line is

$$\mathbf{L} = \lambda\mathbf{e}_1 + (1-\lambda)\mathbf{e}_2$$

for some λ. The Plücker condition

$$\mathbf{L}^{\mathsf{T}} C\mathbf{L} = 0$$

leads to a quadratic condition for λ, which then yields two solutions for the unknown line. This special solution, however, is not contained in the general setup.

5 ML-ESTIMATION OF 3D POINTS AND LINES

5.1 Optimal 3D Points

Now take covariances into account.

The covariance of the i-th residual $\mathbf{w}_{ik} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L}'_{ik}$ of the point-point constraint is given by

$$\mathbf{\Sigma}^{(\nu)}_{w_{ik}w_{ik}} = \overline{\Pi}^{\mathsf{T}}(\widehat{\mathbf{X}}^{(\nu)})\mathbf{\Sigma}_{L'_{ik}L'_{ik}}\overline{\Pi}(\widehat{\mathbf{X}}^{(\nu)})$$

where the Jacobian $\partial\mathbf{w}_{ik}/\partial \boldsymbol{y} = \overline{\Pi}^{\mathsf{T}}(\widehat{\mathbf{X}}^{(\nu)})$ needs to be evaluated at the estimated 3D point. The covariance matrix of the projecting line $\mathbf{L}'_{ik}$ can be determined from

$$\mathbf{\Sigma}_{L'_{ik}L'_{ik}} = \tilde{\mathsf{P}}_k^{\mathsf{T}}\mathbf{\Sigma}_{x'_{ik}x'_{ik}}\tilde{\mathsf{P}}_k$$

assuming the orientation parameters to be error free. The variance $\sigma^2_{w_{jk}}$ of the j-th residual $w_{jk} = \mathbf{X}^{\mathsf{T}}\mathbf{A}'_{jk}$ of the point-plane constraint is given by

$$\sigma^2_{w_{jk}} = \left(\widehat{\mathbf{X}}^{(\nu)}\right)^{\mathsf{T}} \boldsymbol{\Sigma}_{A'_{jk}A'_{jk}} \left(\widehat{\mathbf{X}}^{(\nu)}\right)$$

The covaraince matrix of the projection plane $\mathbf{A}'_j$ can be determined from

$$\boldsymbol{\Sigma}_{A'_{jk}A'_{jk}} = \mathsf{P}_k^{\mathsf{T}} \boldsymbol{\Sigma}_{l'_{jk}l'_{jk}} \mathsf{P}_k$$

Omitting the iteration index we need to minimize

$$\Omega = \sum_{ik} \mathbf{w}_{ik}^{\mathsf{T}} \boldsymbol{\Sigma}^{+}_{w_{ik}w_{ik}} \mathbf{w}_{ik} + \sum_{j} \frac{w^2_{jk}}{\sigma^2_{w_j k}}$$

or

$$\Omega = \mathbf{X}^{\mathsf{T}} \left(\sum_{ik} \Gamma^{\mathsf{T}}(\mathbf{L}'_{ik}) \boldsymbol{\Sigma}^{+}_{w_{ik}w_{ik}} \Gamma(\mathbf{L}'_{ik}) + \sum_{jk} \frac{\mathbf{A}'_{jk}\mathbf{A}'^{\mathsf{T}}_{jk}}{\sigma^2_{w_{jk}}} \right) \mathbf{X}$$

under the constraint $\widehat{\boldsymbol{\beta}}^{\mathsf{T}} \widehat{\boldsymbol{\beta}} = 1$.

The solution therefore is the smallest normalized eigenvalue of

$$\mathsf{M} = \sum_{ik} \Gamma^{\mathsf{T}}(\widehat{\mathbf{L}}'_{ik}) \left(\overline{\Pi}^{\mathsf{T}}(\widehat{\mathbf{X}}) \boldsymbol{\Sigma}_{L'_{ik}L'_{ik}} \overline{\Pi}(\widehat{\mathbf{X}}) \right)^{+} \Gamma(\mathbf{L}'_{ik}) + \sum_{jk} \frac{\widehat{\mathbf{A}}'_{jk}\mathbf{A}'^{\mathsf{T}}_{jk}}{\widehat{\mathbf{X}}^{\mathsf{T}} \boldsymbol{\Sigma}_{A'_{jk}A'_{jk}} \widehat{\mathbf{X}}} \tag{45}$$

The determination of M needs to use the fitted values $\widehat{\mathbf{X}}$, $\widehat{\mathbf{L}}'_{ik}$ and $\widehat{\mathbf{A}}'_{jk}$, initiating an iteration scheme.

For determining the pseudoinverse of $\boldsymbol{\Sigma}_{ww} = \overline{\Pi}^{\mathsf{T}}(\widehat{\mathbf{X}}) \boldsymbol{\Sigma}_{L'L'} \overline{\Pi}(\widehat{\mathbf{X}})$ we need its rank. The point-point constraint $\mathbf{w} = \mathbf{0}$ has two degrees of freedom, which easily can be seen in case the line is approximately parallel to the Z-axis through the origin: then a point in the XY-plane close to the origin lies on the line if its two coordinates are identical with those of the intersection point of the line with the XY-plane. Therefore the rank of $\boldsymbol{\Sigma}_{ww}$ is 2, and the null space has dimension 2.

The covariance matrix of the optimal point is

$$\boldsymbol{\Sigma}_{\widehat{X}\widehat{X}} = \mathsf{M}^{+}$$

using (45) where the nullspace $\widehat{\mathbf{X}}$ is taken to determine the pseudo inverse with (19).

5.2 Optimal 3D Lines

The procedure is similar for the estimation of 3D lines.

The covariance of the j-th residual $\mathbf{w}_{jk} = \Pi^{\mathsf{T}}(\mathbf{A}'_{jk})\mathbf{L} = \overline{\Gamma}(\mathbf{L})\mathbf{A}'_{jk}$ of the line-plane constraint is given by

$$\boldsymbol{\Sigma}^{(\nu)}_{w_{jk}w_{jk}} = \overline{\Gamma}(\widehat{\mathbf{L}}^{(\nu)}) \boldsymbol{\Sigma}_{A'_{jk}A'_{jk}} \overline{\Gamma}^{\mathsf{T}}(\widehat{\mathbf{L}}^{(\nu)})$$

where the Jacobian $\partial \mathbf{w}_{jk} / \partial \boldsymbol{y} = \overline{\Gamma}(\widehat{\mathbf{L}}^{(\nu)})$ needs to be evaluated at the estimated 3D line. The covariance matrix of the projecting line $\mathbf{A}'_j$ can be determined from

$$\boldsymbol{\Sigma}_{A'_{jk}A'_{jk}} = \mathsf{P}_k^{\mathsf{T}} \boldsymbol{\Sigma}_{l'_{jk}l'_{jk}} \mathsf{P}_k$$

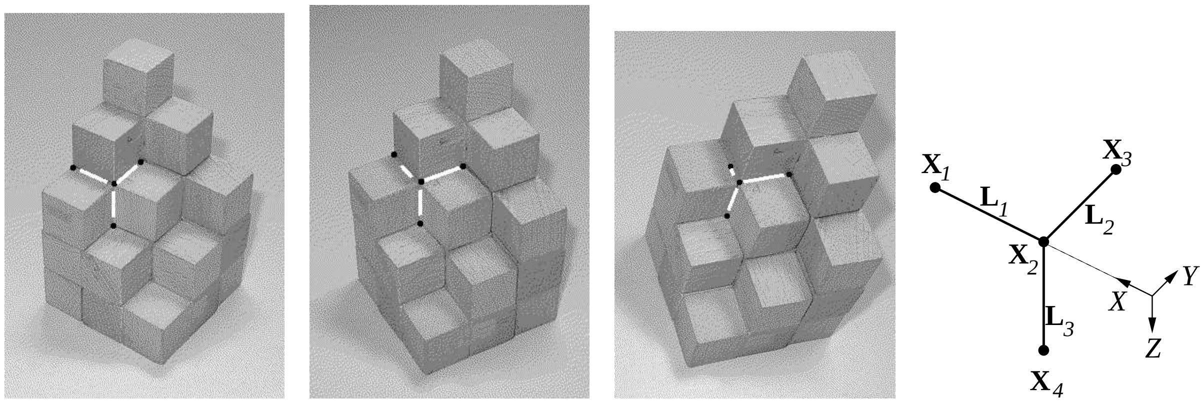
The variance $\sigma^2_{w_{ik}}$ of the i-th residual $w_{ik} = \overline{\mathbf{L}}'^{\mathsf{T}}_{ik}\mathbf{L}$ of the line-line constraint is given by

$$\sigma^2_{w_{ik}} = \left(\widehat{\overline{\mathbf{L}}}^{(\nu)}\right)^{\mathsf{T}} \boldsymbol{\Sigma}_{L'_{ik}L'_{ik}} \left(\widehat{\overline{\mathbf{L}}}^{(\nu)}\right)$$

The covariance matrix of the projection line $\mathbf{L}'_{ik}$ can be determined from

$$\boldsymbol{\Sigma}_{L'_{ik}L'_{ik}} = \tilde{\mathsf{P}}_k^{\mathsf{T}} \boldsymbol{\Sigma}_{x'_{ik}x'_{ik}} \tilde{\mathsf{P}}_k$$

Figure 4: *shows three images of a polyhedron. In each image the four points* $\mathbf{X}_i, i = 1,2,3,4$ *and the three lines* $\mathbf{L}_j, j = 1,2,3$ *have been observed. Based on hypotheses about the object, image points may be used to determine the geometry of the 3D line and image line segments can be used to determine 3D points. Observe, point 1 in the right image 3 does not correspond to the others. The coordinate system is in the centre of the polyhedron. On the right hand the numbering of the points and lines is given together with the coordinate system.*



Omitting the iteration index we need to minimize

$$\Omega = \sum_{ik} \frac{w_{ik}^2}{\sigma_{w_{ik}^2}} + \sum_{j} \mathbf{w}_{jk}^{\mathsf{T}} \boldsymbol{\Sigma}^{+}_{w_{jk}w_{jk}} \mathbf{w}_{jk}$$

or

$$\Omega = \mathbf{L}^{\mathsf{T}} \left(\sum_{ik} \frac{\overline{\mathbf{L}}'_{ik} \overline{\mathbf{L}}'^{\mathsf{T}}_{ik}}{\sigma^2_{w_{ik}}} + \sum_{jk} \Pi(\mathbf{A}'_{jk}) \boldsymbol{\Sigma}^{+}_{w_{jk}w_{jk}} \Pi^{\mathsf{T}}(\mathbf{A}'_{jk}) \right) \mathbf{L}$$

under the constraints $\mathbf{L}^{\mathsf{T}}\mathbf{L} = 1$ and $\mathbf{L}^{\mathsf{T}} C \mathbf{L} = 0$

When only taking the first constraint into account we therefore obtain the estimate $\widehat{\mathbf{L}}$ as the smallest normalized eigenvalue of

$$\mathsf{N} = \sum_{ik} \frac{\widehat{\overline{\mathbf{L}}}'_{ik} \overline{\mathbf{L}}'^{\mathsf{T}}_{ik}}{\widehat{\mathbf{L}}^{\mathsf{T}} \boldsymbol{\Sigma}_{L'_{ik}L'_{ik}} \widehat{\mathbf{L}}} + \sum_{jk} \Pi(\widehat{\mathbf{A}}'_{jk}) \left(\overline{\Gamma}(\widehat{\mathbf{L}}) \boldsymbol{\Sigma}_{A'_{jk}A'_{jk}} \overline{\Gamma}^{\mathsf{T}}(\widehat{\mathbf{L}}) \right)^{+} \Pi^{\mathsf{T}}(\mathbf{A}'_{jk}) \tag{46}$$

Observe, the determination of the matrices $\Gamma(\widehat{\mathbf{L}})$ in N needs to be based on the fitted values $\widehat{\mathbf{L}}$, $\widehat{\overline{\mathbf{L}}}'_{ik}$ and $\widehat{\mathbf{A}}'_{jk}$, initiating an iteration scheme.

The rank of $\boldsymbol{\Sigma}_{w_{jk}w_{jk}}$ again is 2 allowing to determine the pseudo inverse.

Taking the Plücker constraint into account we obtain the final estimate

$$\widehat{\widehat{\mathbf{L}}} = \widehat{\mathbf{L}} - \frac{1}{2} \frac{\widehat{\mathbf{L}}^{\mathsf{T}} C \widehat{\mathbf{L}}}{\widehat{\mathbf{L}} C \mathsf{N}^{+} C \widehat{\mathbf{L}}} \mathsf{N}^{+} C \widehat{\mathbf{L}} = \widehat{\mathbf{L}} - \frac{1}{2} \frac{\widehat{\overline{\mathbf{L}}}^{\mathsf{T}} \widehat{\mathbf{L}}}{\widehat{\overline{\mathbf{L}}}^{\mathsf{T}} \mathsf{N}^{+} \widehat{\overline{\mathbf{L}}}} \mathsf{N}^{+} \widehat{\overline{\mathbf{L}}}$$

The covariance matrix of the final estimate now is

$$\boldsymbol{\Sigma}_{\hat{\hat{L}}\hat{\hat{L}}} = \boldsymbol{\Sigma}_{\hat{L}\hat{L}} - \frac{1}{\sigma_h^2} \boldsymbol{\Sigma}_{\hat{L}\hat{L}} \widehat{\overline{\mathbf{L}}}\, \widehat{\overline{\mathbf{L}}}^{\mathsf{T}} \boldsymbol{\Sigma}_{\hat{L}\hat{L}}$$

using $\boldsymbol{\Sigma}_{\hat{L}\hat{L}} = \mathsf{N}^{+}$ in (46) with its nullspace $\widehat{\mathbf{L}}$.

6 EXAMPLES

Fig. 4 shows three images of a polyhedron. The three projection matrices P_i for points have been estimated using a DLT based on 13 observed points exploiting the special structure of the polyhedron. The object coordinate system is in the centre of the polyhedron.

We give the projection matrix P_3 here

$$\mathsf{P}_3 = \begin{pmatrix} 0.062307122 & 0.0070222277 & -0.064360979 & -0.61856139 \\ 0.021465382 & -0.086378179 & 0.0037555823 & -0.77547722 \\ -0.026349715 & -0.023352877 & -0.061182468 & -2.6455633 \end{pmatrix}$$

The three projection matrices $\tilde{\mathsf{P}}_i$ have been derived from P_i based on (23). The transposed projection matrix $\tilde{\mathsf{P}}_3^{\mathsf{T}}$ is:

$$\tilde{\mathsf{P}}_3^{\mathsf{T}} = \begin{pmatrix} 0.0053725339 & 0.0019326512 & -0.0055330116 \\ 0.0012143465 & -0.0055079970 & -0.0016155325 \\ -0.0027773188 & 0.0012700169 & -0.0055327105 \\ 0.077221631 & -0.18113636 & 0.035040097 \\ -0.21040932 & -0.033022936 & 0.058875784 \\ 0.057381241 & 0.13242593 & -0.052233531 \end{pmatrix}$$

We analyse the aggregate of four points with the given object coordinates:

$$E(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4) = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The three neigbouring lines $\mathbf{L}_i$ are

$$E(\mathbf{L}_1) = E(\mathbf{X}_2) \wedge E(\mathbf{X}_1) \cong (1,0,0,0,0,0)^{\mathsf{T}}$$
$$E(\mathbf{L}_2) = E(\mathbf{X}_2) \wedge E(\mathbf{X}_3) \cong (0,1,0,0,0,2)^{\mathsf{T}}$$
$$E(\mathbf{L}_3) = E(\mathbf{X}_2) \wedge E(\mathbf{X}_4) \cong (0,0,1,0,-2,0)^{\mathsf{T}}$$

where $\cong$ denotes *proportional to.*

We automatically measured and numbered the following points in the three images using the software package FEX (FUCHS 1998) (cf. fig. 4:

$$(\boldsymbol{x}_1', \boldsymbol{x}_2', \boldsymbol{x}_3', \boldsymbol{x}_4')_1 = \frac{1}{1000}\begin{pmatrix} 178.653 & 195.386 & 172.133 & 240.403 \\ 211.444 & 255.175 & 283.147 & 254.897 \end{pmatrix}$$

$$(\boldsymbol{x}_1', \boldsymbol{x}_2', \boldsymbol{x}_3', \boldsymbol{x}_4')_2 = \frac{1}{1000}\begin{pmatrix} 168.716 & 196.003 & 180.674 & 239.210 \\ 230.514 & 258.052 & 301.273 & 257.671 \end{pmatrix}$$

$$(\boldsymbol{x}_1', \boldsymbol{x}_2', \boldsymbol{x}_3', \boldsymbol{x}_4')_3 = \frac{1}{1000}\begin{pmatrix} 164.776 & 182.537 & 173.359 & 220.103 \\ 261.775 & 271.594 & 327.718 & 257.569 \end{pmatrix}$$

We might confirm point 2 in image 3 by projecting the ideal point:

$$E(\mathbf{x}_{23}') = \mathsf{P}_3 E(\mathbf{X}_2) = \begin{pmatrix} -.493947 \\ -.732546 \\ -2.69826 \end{pmatrix} \cong \begin{pmatrix} 183.06_{-3} \\ 271.49_{-3} \\ 1 \end{pmatrix}$$

We also measured and numbered the following three line segments in image 1:

$$(\mathbf{l}_1', \mathbf{l}_2', \mathbf{l}_3')_1 = \begin{pmatrix} 0.914567 & 0.768382 & 0.014422 \\ -0.404434 & 0.639991 & 0.999895 \\ -76.67_{-3} & -314.36_{-3} & -258.05_{-3} \end{pmatrix}$$

The image line segments in image two are:

$$(\mathbf{l}'_1, \mathbf{l}'_2, \mathbf{l}'_3)_2 = \begin{pmatrix} -0.685392 & 0.934881 & -0.031488 \\ 0.728173 & 0.354959 & 0.999504 \\ -50.14_{-3} & -275.61_{-3} & -250.13_{-3} \end{pmatrix}$$

The image lines in image three are

$$(\mathbf{l}'_1, \mathbf{l}'_2, \mathbf{l}'_3)_3 = \begin{pmatrix} -0.377482 & 0.985130 & 0.360059 \\ 0.926016 & 0.171807 & 0.932929 \\ -179.60_{-3} & -227.09_{-3} & -318.85_{-3} \end{pmatrix}$$

We might confirm line 3 in image 3 by backprojecting the ideal line

$$E(\mathbf{L}_3) = E(\mathbf{X}_2) \wedge E(\mathbf{X}_4) \cong (0, 0, 1, 0, -2, 0)^{\mathsf{T}}$$

using the projection matrix for lines:

$$E(\mathbf{l}'_{33}) = \tilde{\mathsf{P}}_3 E(\overline{\mathbf{L}}_3) = \begin{pmatrix} -0.985263 \\ -0.171042 \\ 0.336387 \end{pmatrix} = \begin{pmatrix} 0.357745 \\ 0.933819 \\ -319.01_{-3} \end{pmatrix}$$

We now give the results for the estimated points, once when minimizing the algebraic error, once when using the ML-estimate. As point 1 in image 3 is not corresponding to the points in the other images, it is not used. We also give the estimated $\widehat{\sigma}_0$, being the factor by which the standard deviations provided by the feature extraction are too optimistic. The results for 3D points are:

point	type	X [mm]	Y [mm]	Z [mm]	red.	$\widehat{\sigma}_0$[1]	$\sigma_{\widehat{X}}$ [mm]	$\sigma_{\widehat{Y}}$ [mm]	$\sigma_{\widehat{Z}}$ [mm]
1	alg.	3.90	-0.16	-0.14	4				
	opt.	4.20	0.03	0.09	4	1.56	0.35	0.27	0.31
2	alg.	1.95	-0.09	-0.06	12				
	opt.	1.99	-0.01	-0.07	12	2.08	0.88	0.70	0.96
3	alg.	2.04	1.89	-0.07	6				
	opt.	2.13	1.95	0.03	6	3.92	0.57	0.46	0.62
4	alg.	2.25	0.14	2.21	6				
	opt.	2.08	0.04	1.04	6	4.41	0.49	0.41	0.58

The ML-estimates appear to be better than the algebraic estimates. This is confirmed by the average of the 6 normalized distances compared with their mean. The r. m. s. distance reduces from 1.31 mm to 0.96 mm. The estimated values for $\widehat{\sigma}$ are in the range of 1.5 to 4.5 which suggests the estimates of the feature extraction to be a bit optimistic. The standard deviations of the estimated 3D coordinates are in the range betweeen 0.3 and 1.0 mm, point 1 being most precise and point 2 being worst.

The results for 3D lines are:

line	type	L_1[1]	L_2 [1]	L_3 [1]	L_4 [mm]	L_5 [mm]	L_6 [mm]	red.	$\widehat{\sigma}_o$[1]
1	alg.	0.999	0.009	0.008	0.000	-0.157	0.187	8	
	opt.	0.993	0.041	0.105	-0.007	0.443	0.238	8	2.10
2	alg.	-0.024	-1.000	0.019	-0.042	-0.378	-1.981	8	
	opt.	-0.081	-0.995	-0.062	-0.146	0.131	-1.908	8	1.02
3	alg.	0.102	0.067	0.992	-0.011	-2.018	0.138	8	
	opt.	0.072	0.047	0.996	0.054	-2.085	0.095	8	2.00

The average angular between all pairs of 3D lines error increases from 5.5^{gon} to 6.4^{gon}. The reason might be, that the precision of the observed edges is not very high in this case, but due to the non adequate weighting in the algebraic minimization do not influence the result, as the coefficients $\overline{\mathbf{L}}_{ik}$ are two orders of magnitude lower than the corresponding coefficients $\Pi(\mathbf{A}_{jk})$.

The results suggests that the a priori estimates for the variances of the lines might be too optimistic.

7 CONCLUSIONS

The paper showed the advantage of new geometric and statistical techniques for classical problems in photogrammetry. The rigorous use of projective geometry simplifies object reconstruction. The linearity of estimation models w. r. t. the unknown pameters obviously is also of advantage in case the model is nonlinear as such and in case the model also is nonlinear in the observations, as eigenvalue solutions become possible which pose much less restrictions on the quality of approximate values, without loosing tools for quality evaluation. These techniques need to be explored in further applications.

APPENDIX

Minimizing the quadratic form (31) under the constraints $\boldsymbol{w}(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{\beta}}) = 0$ and $\boldsymbol{h}(\widehat{\boldsymbol{\beta}}) = 0$ we have to minimize the form

$$\Phi(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{\beta}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{1}{2}(\boldsymbol{y} - \widehat{\boldsymbol{y}})^{\mathsf{T}} \boldsymbol{\Sigma}_{yy}^{+} (\boldsymbol{y} - \widehat{\boldsymbol{y}}) + \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{w}(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{\beta}}) + \boldsymbol{\mu}^{\mathsf{T}} \boldsymbol{h}(\widehat{\boldsymbol{\beta}}) \tag{47}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are Lagrangian multipliers.

For solving this nonlinear problem in the classical iterative manner we need approximate values $\widehat{\boldsymbol{\beta}}^{(0)}$ and $\widehat{\boldsymbol{y}}^{(0)}$ for the unknowns $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}^{(0)} + \widehat{\boldsymbol{\Delta\beta}}$ and $\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{y}}^{(0)} + \widehat{\boldsymbol{\Delta y}}$ which contain corrections $\widehat{\boldsymbol{\Delta\beta}}$ and $\widehat{\boldsymbol{\Delta y}}$. With the Jacobians

$$A = \left(\frac{\partial \boldsymbol{w}(\boldsymbol{y}, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}}\right)\bigg|_{\substack{\beta=\widehat{\beta}^{(0)} \\ y=\widehat{y}^{(0)}}} \quad B = \left(\frac{\partial \boldsymbol{w}(\boldsymbol{y}, \boldsymbol{\beta})}{\partial \boldsymbol{y}}\right)\bigg|_{\substack{\beta=\widehat{\beta}^{(0)} \\ y=\widehat{y}^{(0)}}} \quad H = \left(\frac{\partial \boldsymbol{h}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}\right)\bigg|_{\beta=\widehat{\beta}^{(0)}} \tag{48}$$

and the relation $\widehat{\boldsymbol{\Delta y}} = (\boldsymbol{y} - \widehat{\boldsymbol{y}}^{(0)}) - \widehat{\boldsymbol{e}}$ we obtain the linear constraints $\boldsymbol{w}(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{\beta}}) = \boldsymbol{w}(\widehat{\boldsymbol{y}}^{(0)}, \widehat{\boldsymbol{\beta}}^{(0)}) + A\widehat{\boldsymbol{\Delta\beta}} + B\widehat{\boldsymbol{\Delta y}}$ or $\boldsymbol{w}(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{\beta}}) = \boldsymbol{c}_w + A\widehat{\boldsymbol{\Delta\beta}} - B\widehat{\boldsymbol{e}}$ and $\boldsymbol{h}(\widehat{\boldsymbol{\beta}}) = \boldsymbol{c}_h + H\widehat{\boldsymbol{\Delta\beta}}$ with

$$\boldsymbol{c}_w = \boldsymbol{w}(\widehat{\boldsymbol{y}}^{(0)}, \widehat{\boldsymbol{\beta}}^{(0)}) + B(\boldsymbol{y} - \widehat{\boldsymbol{y}}^{(0)}) = \boldsymbol{w}(\boldsymbol{y}, \widehat{\boldsymbol{\beta}}^{(0)}) \qquad \text{and} \qquad \boldsymbol{c}_h = \boldsymbol{h}(\widehat{\boldsymbol{\beta}}^{(0)}) \tag{49}$$

are the contradictions between the approximate values for the unknown parameters and the given observations and among the approximate values for the unknowns. Setting the partials of Φ (47) zero yields

$$\frac{\partial \Phi}{\partial \widehat{\boldsymbol{y}}^{\mathsf{T}}} = -\boldsymbol{\Sigma}_{yy}^{+} \widehat{\boldsymbol{e}} + B^{\mathsf{T}} \boldsymbol{\lambda} = \boldsymbol{0} \qquad \frac{\partial \Phi}{\partial \widehat{\boldsymbol{\beta}}^{\mathsf{T}}} = A^{\mathsf{T}} \boldsymbol{\lambda} + H^{\mathsf{T}} \boldsymbol{\mu} = \boldsymbol{0} \tag{50}$$

$$\frac{\partial \Phi}{\partial \boldsymbol{\lambda}^{\mathsf{T}}} = \boldsymbol{c}_w + A\widehat{\boldsymbol{\Delta\beta}} - \boldsymbol{B}\widehat{\boldsymbol{e}} = \boldsymbol{0} \qquad \frac{\partial \Phi}{\partial \boldsymbol{\mu}^{\mathsf{T}}} = \boldsymbol{c}_h + H\widehat{\boldsymbol{\Delta\beta}} = \boldsymbol{0} \tag{51}$$

From (50a) follows the relation

$$\widehat{\boldsymbol{e}} = \boldsymbol{\Sigma}_{yy} B^{\mathsf{T}} \boldsymbol{\lambda} \tag{52}$$

When substituting (52) into (51a), solving for $\boldsymbol{\lambda}$ yields

$$\boldsymbol{\lambda} = (B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{-1}(\boldsymbol{c}_w + A\widehat{\boldsymbol{\Delta\beta}}) \tag{53}$$

Substitution in (50b) yields the symmetric normal equation system

$$\begin{pmatrix} A^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{-1}A & H^{\mathsf{T}} \\ H & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\Delta\beta}} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} -A^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{-1}\boldsymbol{c}_w \\ -\boldsymbol{c}_h \end{pmatrix} \tag{54}$$

The Lagrangian multipliers can be obtained from (53) which then yields the estimated residuals in (52).

Up to this point all derivations are known. In our context the contraints are $\boldsymbol{w}(\widehat{\boldsymbol{y}}, \widehat{\boldsymbol{\beta}}) = A(\widehat{\boldsymbol{y}})\widehat{\boldsymbol{\beta}} = 0$ and $h = \frac{1}{2}\widehat{\boldsymbol{\beta}}^{\mathsf{T}}\widehat{\boldsymbol{\beta}} - 1$. This leads to $H^{\mathsf{T}} = \widehat{\boldsymbol{\beta}}$. In the case of convergence we have $\widehat{\boldsymbol{\Delta\beta}} = \mathbf{0}$ and $\boldsymbol{c}_w = A(\boldsymbol{y})\widehat{\boldsymbol{\beta}}$ and therefore the first equation of (54) leads to

$$\mu \cdot \widehat{\boldsymbol{\beta}} = A(\widehat{\boldsymbol{y}})^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{-1}A(\boldsymbol{y}) \cdot \widehat{\boldsymbol{\beta}} \tag{55}$$

This shows the unknown parameter vector to be an eigenvector of an unsymmetric matrix. We however also find the following: As in the case of convergence $\boldsymbol{\Delta\beta} = \mathbf{0}$ the weighted sum of the squared residuals can be rewritten

$$\begin{aligned} \Omega &= \widehat{\boldsymbol{e}}^{\mathsf{T}}\boldsymbol{\Sigma}_{yy}^{+}\widehat{\boldsymbol{e}} = \boldsymbol{c}_w^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{+}B\boldsymbol{\Sigma}_{yy} \cdot \boldsymbol{\Sigma}_{yy}^{+} \cdot \boldsymbol{\Sigma}_{yy}B^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})\boldsymbol{c}_w \\ &= \boldsymbol{c}_w^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{+}\boldsymbol{c}_w = \widehat{\boldsymbol{\beta}}^{\mathsf{T}}\mathsf{A}(\boldsymbol{y})^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{+}\mathsf{A}(\boldsymbol{y})\widehat{\boldsymbol{\beta}} = \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}_{ww}^{+}\boldsymbol{w} \end{aligned}$$

This proves minimizing $\Omega = \boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}_{ww}^{+}\boldsymbol{w}$ leads to the ML-estimate.

The estimated variance factor is given by

$$\widehat{\sigma}^2 = \frac{\Omega}{G_{\text{eff}} - U_{\text{eff}}} = \frac{\widehat{\boldsymbol{e}}^{\mathsf{T}}\boldsymbol{\Sigma}_{yy}^{-1}\widehat{\boldsymbol{e}}}{G_{\text{eff}} - U_{\text{eff}}} = \frac{\boldsymbol{w}^{\mathsf{T}}\boldsymbol{\Sigma}_{ww}^{+}\boldsymbol{w}}{G_{\text{eff}} - U_{\text{eff}}} \tag{56}$$

The redundancy R is the number of effective contraints

$$G_{\text{eff}} = \sum_i \text{rk}\,(\boldsymbol{\Sigma}_{w_i w_i})$$

above the effective number U_{eff}

$$U_{\text{eff}} = U - H$$

of unknown parameters, which is neccessary for determinimg the unknown parameters.

We finally obtain the *estimated* covariance matrix

$$\widehat{\boldsymbol{\Sigma}}_{\widehat{\beta}\widehat{\beta}} = \widehat{\sigma}^2\boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} \tag{57}$$

of the estimated parameters, where $\boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}}$ results from $\overline{N}^{+} = [A^{\mathsf{T}}(B\boldsymbol{\Sigma}_{yy}B^{\mathsf{T}})^{-1}A]^{+}$.

References

Duda, R. O. & Hart, P. E. (1973): *Pattern Classification and Scene Analysis*, Wiley.

Faugeras, O. & Papadopoulo, T. (1998): Grassmann-Caley Algebra for Modeling Systems of Cameras and the Algebraic Equations of the Manifold of Trifocal Tensors, *Trans. of the ROYAL SOCIETY A* **356**: 1123–1152.

Förstner, W. (2000): New Orientation Procedures, *Int. Archives for Photogrammetry and Remote Sensing*, vol. XXXII.

Förstner, W., Brunn, A. & Heuel, S. (2000): Statistically Testing Uncertain Geometric Relationsships, *Mustererkennung ´00*, Informatik Aktuell, Springer.

Fuchs, C. (1998): *Extraktion polymorpher Bildstrukturen und ihre topologische und geometrische Gruppierung*, DGK, Bayer. Akademie der Wissenschaften, Reihe C, Heft 502.

Leedan, Y. (1997): *Statistical analysis of quadratic problems in computer vision*, PhD thesis, Dept. of Electrical Engineering and Computer Engineering, Rutgers University, available at web-site http://www.caip.rutgers.edu/˜meer/RIUL/uploads.html.

Matei, B. & Meer, P. (1997): A General Method for Errors-in-Variables Problems in Computer Vision, *CVPR ´97*.

Taubin, G. (1993): An improved algorithm for algebraic curve and surface fitting, *Fourth ICCV, Berlin*, pp. 658–665.

Wrobel, B. (1971): Die ebene Blockausgleichung nach der Anblock-Methode, *Nachrichten aus dem öffentlichen Vermessungswesen Nordrhein-Westfalen* pp. 203–218.