Algebraic Projective Geometry and Direct Optimal Estimation of Geometric Entities

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Abstract:
The paper presents a new technique for optimal estimation for statistically uncertain geometric entites. It is an extension of the classical eigenvector solution technique but takes the full covariance information into account to arrive at a ML-estimate. The proposed solution is significantly more transparent than the solution for estimation under heteroscedasticity proposed by Leedan, Matei and Meer (LEEDAN 1997, MATEI & MEER 1997). We give a new representation of algebraic projective geometry easing statistical reasoning. We show how the setup can be used in object reconstruction, especially when estimating points and edges of polyhedra. We explicitely give an example for estimating 3D-points and 3D-lines from image points and image lines. The direct solutions do practically require no approximate values.

1 MOTIVATION

The task of estimating 3D points from image points is a classical one in Computer Vision. Automatic image processing allows to easily extract image line segments enabling to reconstruct 3D lines, besides points. In case of polyhedra, which may be used for a large class of man made objects, image line segments may also be used to determine 3D points and image points may be used to determine 3D lines. Moreover, grouping of 3D-points, 3D-lines and planes for reconstructing polyhedra requires the estimation of 3D points, lines and planes from given 3D points, lines and planes.

These reconstruction tasks have to cope with two problems: (1) optimal estimation taking all available information about the uncertainty of the geometric entities and the spatial relations into account and (2) means for direct optimal or suboptimal solutions, leading to approximate estimates, as the spatial relations are nonlinear. Both problems have been in the focus of research.

Nonlinear estimation techniques have been analysed and adapted to geometric problems, bundle adjustment (TRIGGS ET AL. 2000) becoming a standard reference for object reconstruction and camera calibration (cf. e. g. (FITZGIBBON & ZISSERMAN 2000)). However, nonlinear estimation techniques require good approximate values.

Direct solutions, often based on an eigenvector analysis, are commonly used, either in their original form minimizing the algebraic error or in modified versions leading to least squares estimates. Recently Leedan, Matei and Meer (LEEDAN 1997, MATEI & MEER 1997) extended the eigenvector method to heteroscedastic observations, i. e. to observations which groupwise may be statistically fully correlated. Their approach (1) yields ML-estimates, (2) is based on the solution of a generalized eigenvalue problem which is practically a direct solution, and (3) has shown to be superior to renormalization techniques proposed by Kanatani (KANATANI 1996). However, their approach leads to quite complex expressions.

Algebraic projective geometry, promoted in the last decade (cf. (FAUGERAS & PAPADOPOULO 1998, HARTLEY & ZISSERMAN 2000)), allows to represent 2D points and lines, and 3D points, lines and planes in a consistent way which significantly simplifies geometric reasoning. It also was the basis for direct estimation procedures for camera calibration. The main reason why direct eigenvector methods are feasible is that algebraic projective geometry leads to geometric contraints where the unknown parameters appear linearily, e. g. when estimating the projection matrix or the fundamental matrix. However, statistical spatial reasoning based on algebraic projective geometry has not been presented up to now.

The work closest to ours is the monography by Kanatani (cf. (KANATANI 1996), who presents techniques for statistical geometric reasoning using homogeneous representations of geometric entities, however, does not make use of the neat formulations of algebraic geometry, which leads to cumbersome expressions.

The paper has two goals: (1) it presents a new technique for direct optimal estimation, significantly simplifying the method of Leedan, Matei and Meer, (2) it presents algebraic projective geometry in a way which makes the linearity of the relations explicit, providing the Jacobians which allows rigorous error propagation, and shows that it also significantly simplifies the expressions for statistical estimation. The paper's focus therefore is to simplify computation for statistically uncertain geometric entities and make all relations as transparent as possible.

As a field of application we use the estimation of 3D points and lines, as it is required in all 3D reconstruction and grouping processes. We specifically discuss the optimal estimation of 3D points and lines from image points and lines taking the full covariance information from the feature extraction into account and demonstrate the feasibility with a real example from object reconstruction.

The paper is organized as follows: Section 2 collects the necessary relations from algebraic projective geometry including analytical geometry in 2D and 3D as well as modelling the imaging process. It puts them into a consistent form, largely inspired by the paper (Faugeras & Papadopoulo 1998) and makes all relations directly accessible to statistical reasoning. Section 3 presents the new technique for optimal estimation, relating it to the classical, suboptimal eigenvector solution minimizing the algebraic error. Section 4 links the result of sections 2 and 3 and presents the estimation of 3D points and 3D lines from image points and lines in detail. Section 5 discusses an example with real data in detail.

Notation: We denote coordinate vectors of planar geometric objects with small bold face letters, e. g. $\boldsymbol{x}$, in 3D space with capital bold face letters, e. g. $\boldsymbol{X}$. Vectors and matrices are denoted with slanted letters, thus $\boldsymbol{x}$ or R. Homogeneous vectors and matrices are denoted with upright letters, e. g. $\mathbf{x}$, $\mathbf{A}$ or Π. Proportionality is denoted with $\cong$, e. g. $\mathbf{x} \cong \lambda\mathbf{x}$. We use the skew matrix $\mathsf{S}(\mathbf{x}) = [\mathbf{x}]_\times$ of a 3-vector $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}$ in order to represent the cross product by $\mathbf{a} \times \mathbf{b} = \mathsf{S}(\mathbf{a})\mathbf{b} = -\mathbf{b} \times \mathbf{a} = -\mathsf{S}(\mathbf{b})\mathbf{a}$. We will use the vec-operator collecting the columns of a matrix into a vector, thus $\text{vec}A = \text{vec}(\boldsymbol{a}_1, \boldsymbol{a}_2, ...) = (\boldsymbol{a}_1^\mathsf{T}, \boldsymbol{a}_2^\mathsf{T}, ...)^\mathsf{T}$, and use the relation $\text{vec}(ABC) = (C^\mathsf{T} \otimes A)\text{vec}B^\mathsf{T}$ with the Kronecker product $A \otimes B = \{a_{ij}B\}$.

2 Representations

The goal of this section is to provide the necessary tools from algebraic projective geometry. It gives a representation which eases computation of statistically uncertain geometric entities.

We represent all geometric entities with homogeneous coordinates. We use two basic operations to derive one geometric entity γ from two others α and β: The *join* $\gamma = \alpha \wedge \beta$ of two entities α and β yields the linear space containing both entities. E. g. a line in 2D or 3D can be interpreted as the join of two points. The *intersection* $\delta = \alpha \cap \beta$ of two entities yields the common linear subspace of both entities. E. g. a line in 3D can be seen as the intersection of two planes.

2.1 Points, Lines and Planes

Points $\mathbf{x}^\mathsf{T} = (\boldsymbol{x}_0^\mathsf{T}, x_h)$ and lines $\mathbf{l} = (\boldsymbol{l}_h^\mathsf{T}, l_0)$ in the plane are represented with 3-vectors, splitted into their homogeneous part, indexed with h, and non-homogeneous part indexed with 0. Points $\mathbf{X} = (\boldsymbol{X}_0^\mathsf{T}, X_h)$ and planes $\mathbf{A} = (\boldsymbol{A}_h^\mathsf{T}, A_0)$ in 3D space are represented similarily. Lines $\mathbf{L}$ in 3D are represented with their *Plücker coordinates* $\mathbf{L}^\mathsf{T} = (\boldsymbol{L}_h^\mathsf{T}, \boldsymbol{L}_0^\mathsf{T})$. It can be derived from two points $\mathbf{X}$ and $\mathbf{Y}$ by the direction $\boldsymbol{L}_h = Y_h \boldsymbol{X}_0 - X_h \boldsymbol{Y}_0$ of the line and the normal $\boldsymbol{L}_0 = \boldsymbol{X}_0 \times \boldsymbol{Y}_0$ of the plane through the line and the origin. We will need the dual 3D-line $\overline{\mathbf{L}}^\mathsf{T} = (\boldsymbol{L}_0^\mathsf{T}, \boldsymbol{L}_h^\mathsf{T})$ which has homogeneous and non-homogeneous part interchanged. This can also be written as

$$\overline{\mathbf{L}} = C\mathbf{L} \qquad \text{with} \qquad C = \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}$$

As $C^2 = \boldsymbol{I}$ we have $\overline{\overline{\mathbf{L}}} = \mathbf{L}$. The line parameters have to fulfill the *Plücker constraint*

$$L_1 L_4 + L_2 L_5 + L_3 L_6 = \boldsymbol{L}_h^\mathsf{T} \boldsymbol{L}_0 = \frac{1}{2} \mathbf{L}^\mathsf{T} \overline{\mathbf{L}} = \frac{1}{2} \mathbf{L}^\mathsf{T} C \mathbf{L} = 0 \tag{1}$$

which is clear, as $\boldsymbol{L} = Y_h \boldsymbol{X}_0 - X_h \boldsymbol{Y}_0$ is orthogonal to $\boldsymbol{L}_0 = \boldsymbol{X}_0 \times \boldsymbol{Y}_0$. All 6-vectors $\mathbf{L} \neq 0$ fullfilling the Plücker constraint represent a 3D line.

2.2 Relations between Geometric Entities

All links between two geometric elements are bilinear in their homogeneous coordinates, an example being the line joining two points in 3D . Thus the coordinates of new entities can be written in the form

$$\boldsymbol{\gamma} = \mathsf{A}(\boldsymbol{\alpha})\boldsymbol{\beta} = \mathsf{B}(\boldsymbol{\beta})\boldsymbol{\alpha} \qquad \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\beta}} = \mathsf{A}(\boldsymbol{\alpha}) \qquad \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\alpha}} = \mathsf{B}(\boldsymbol{\beta})$$

Thus the matrices $\mathsf{A}(\boldsymbol{\alpha})$ and $\mathsf{B}(\boldsymbol{\beta})$ have entries being linear in the coordinates $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. At the same time they are the Jacobians of $\boldsymbol{\gamma}$.

We then may use error propagation or the propagation of covariances of linear $\boldsymbol{y} = A\boldsymbol{x}$ functions of $\boldsymbol{x}$ with covariance matrix $\boldsymbol{\Sigma}_{xx}$ leading to $\boldsymbol{\Sigma}_{yy} = A\boldsymbol{\Sigma}_{xx}A^\mathsf{T}$ to obtain rigorous expressions for the covariance matrices of constructed entities $\boldsymbol{\gamma}$:

$$\boldsymbol{\Sigma}_{\gamma\gamma} = \mathsf{A}(\boldsymbol{\alpha})\boldsymbol{\Sigma}_{\beta\beta}\mathsf{A}^\mathsf{T}(\boldsymbol{\alpha}) + \mathsf{B}(\boldsymbol{\beta})\boldsymbol{\Sigma}_{\alpha\alpha}\mathsf{B}^\mathsf{T}(\boldsymbol{\beta})$$

in case of stochastic independence.

The construction of new entities are collected in table 1. There we need the two matrices[1)]:

$$\mathsf{\Pi}(\mathbf{X}) = \frac{\partial \mathbf{X} \wedge \mathbf{Y}}{\partial \mathbf{Y}} = \begin{pmatrix} X_h I & -\boldsymbol{X}_0 \\ S_{X_0} & \mathbf{0} \end{pmatrix} \tag{2}$$

useful for 3D-*p*oints and *p*lanes (hence the name $\mathsf{\Pi}$) and the Plücker matrix[2)]

$$\mathsf{\Gamma}(\mathbf{L}) = \frac{\partial \mathbf{L} \wedge \mathbf{X}}{\partial \mathbf{X}} = \begin{pmatrix} S_{L_h} & \boldsymbol{L}_0 \\ -\boldsymbol{L}_0^{\mathsf{T}} & 0 \end{pmatrix} = -\mathsf{\Gamma}^{\mathsf{T}}(\mathbf{L}) \tag{3}$$

useful for 3D lines $\mathbf{L}$ (hence the name $\mathsf{\Gamma}$) and their duals

$$\overline{\mathsf{\Pi}}(\mathbf{X}) = C\mathsf{\Pi}(\mathbf{X}) \qquad \text{and} \qquad \overline{\mathsf{\Gamma}}(\mathbf{L}) = \mathsf{\Gamma}(C\mathbf{L}) = \mathsf{\Gamma}(\overline{\mathbf{L}})$$

The possible constraints between pairs of entities are collected in table 2. There we need the inner products which are definded by

$$< \mathbf{x}, \mathbf{l} > \doteq \mathbf{x}^{\mathsf{T}}\mathbf{l} \qquad < \mathbf{X}, \mathbf{A} > \doteq \mathbf{X}^{\mathsf{T}}\mathbf{A} \qquad < \mathbf{L}, \mathbf{M} > \doteq \mathbf{L}^{\mathsf{T}} C \mathbf{M} = \mathbf{L}^{\mathsf{T}}\overline{\mathbf{M}}$$

for 2D points and lines, 3D points and planes, and for 3D lines. Observe, the Plücker constraint can be written as $< \mathbf{L}, \mathbf{L} >= 0$

Table 1: ***Construction of new geometric entities. The matrices*** $\mathsf{\Pi}$, $\overline{\mathsf{\Pi}}$, $\mathsf{\Gamma}$ ***and*** $\overline{\mathsf{\Gamma}}$ ***are given in eqs. (2) and (3) resp. All forms are linear in the coordinates of the given entities allowing rigorous error propagation.***

link	expression
$\mathbf{l} = \mathbf{x} \wedge \mathbf{y}$	$\mathbf{l} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$
$\mathbf{x} = \mathbf{l} \cap \mathbf{m}$	$\mathbf{x} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$
$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$	$\mathbf{L} = \mathsf{\Pi}(\mathbf{X})\mathbf{Y} = -\mathsf{\Pi}(\mathbf{Y})\mathbf{X}$
$\mathbf{L} = \mathbf{A} \cap \mathbf{B}$	$\mathbf{L} = \overline{\mathsf{\Pi}}(\mathbf{A})\mathbf{B} = -\overline{\mathsf{\Pi}}(\mathbf{B})\mathbf{A}$
$\mathbf{A} = \mathbf{L} \wedge \mathbf{X}$	$\mathbf{A} = \mathsf{\Gamma}(\mathbf{L})\mathbf{X} = \overline{\mathsf{\Pi}}^{\mathsf{T}}(\mathbf{X})\mathbf{L}$
$\mathbf{X} = \mathbf{L} \cap \mathbf{A}$	$\mathbf{X} = \overline{\mathsf{\Gamma}}(\mathbf{L})\mathbf{A} = \mathsf{\Pi}^{\mathsf{T}}(\mathbf{A})\mathbf{L}$

[1)] For an interpretation of these two matrices see (Heuel 2000)

[2)] This is one of several possible conventions, in (Hartley & Zisserman 2000) the dual matrix is taken as the Plücker matrix. This has no effect on the relations discussed below.

Table 2: ***shows incidence constraints between pairs of entities and their algebraic representation. The inner product for lines requires uses C as weight matrix. The relations for the last two contraints use the algebraic expressions in table 1.***

constraint	representation
$\mathbf{x} \in \mathbf{l}$	$< \mathbf{x}, \mathbf{l} >= 0$
$\mathbf{X} \in \mathbf{A}$	$< \mathbf{X}, \mathbf{A} >= 0$
$\mathbf{L} \cap \mathbf{M} \neq \emptyset$	$< \mathbf{L}, \mathbf{M} >= 0$
$\mathbf{X} \in \mathbf{L}$	$\mathbf{X} \wedge \mathbf{L} = \mathbf{0}$
$\mathbf{L} \in \mathbf{A}$	$\mathbf{L} \cap \mathbf{A} = \mathbf{0}$

2.3 Projection and Inverse Projection

2.3.1 Points

The projection of a 3D point $P(\mathbf{X})$ onto the image plane yields the image point $p'(\mathbf{x}')$ via a direct linear transformation (cf. Fig. 1).

$$\mathbf{x}' = \mathsf{P}\mathbf{X} = (I_3 \otimes \mathbf{X}^\mathsf{T})\mathbf{p} \qquad \text{with} \qquad \mathbf{p} = \text{vec}\mathsf{P}^\mathsf{T}$$

or

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} \mathbf{1}^\mathsf{T} \\ \mathbf{2}^\mathsf{T} \\ \mathbf{3}^\mathsf{T} \end{pmatrix} \mathbf{X} = \begin{pmatrix} < \mathbf{1}, \mathbf{X} > \\ < \mathbf{2}, \mathbf{X} > \\ < \mathbf{3}, \mathbf{X} > \end{pmatrix} = \begin{pmatrix} \mathbf{X}^\mathsf{T} & \mathbf{0}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ \mathbf{0}^\mathsf{T} & \mathbf{X}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ \mathbf{0}^\mathsf{T} & \mathbf{0}^\mathsf{T} & \mathbf{X}^\mathsf{T} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

with the projection matrix

$$\mathsf{P} = \frac{\partial \mathbf{x}'}{\partial \mathbf{X}} = \mathsf{K}R(I| - \boldsymbol{X}_o)$$

It is directly related to the parameters of the interior and exterior orientation, the calibration matrix K, the rotation matrix R and the projection centre $\mathbf{X}_o$, a relation which we, however, do not need in our context. The matrix P at the same time is one of the two Jacobians, the other one is

$$\frac{\partial \mathbf{x}'}{\partial \mathbf{p}} = I_3 \otimes \mathbf{X}^\mathsf{T}$$

The projection matrix in general has rank 3 and the null space of its transpose is the homogeneous vector $\mathbf{X}_o$ of the projection centre as $\mathsf{P}\mathbf{X}_o = \mathbf{0}$. Therefore the three row vectors $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ of the projection matrix P can be interpreted as the parameters of the planes of the camera coordiante system (cf. the discussion in (Hartley & Zisserman 2000)).

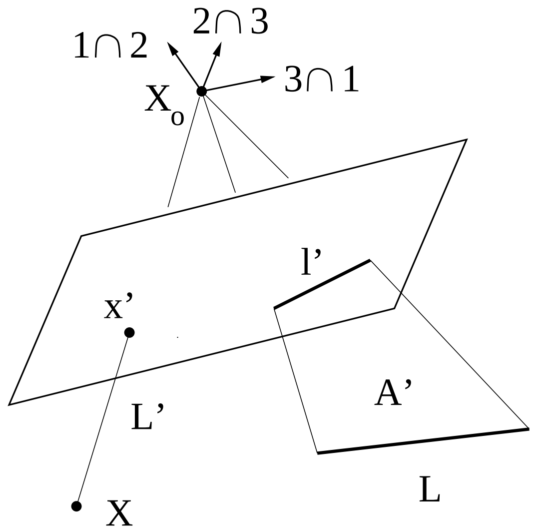


Figure 1: ***shows the geometric situation for the projection of a 3D point*** $\mathbf{X}$ ***and a 3D line*** $\mathbf{L}$ ***into one image, yielding the image point*** $\mathbf{x}' = \mathsf{P}\mathbf{X}$ ***and the image line*** $\mathbf{l} = \tilde{\mathsf{P}}\overline{\mathbf{L}}$***. The projection ray*** $\mathbf{L}' = \tilde{\mathsf{P}}^{\mathsf{T}}\mathbf{x}'$ ***and the projection plane*** $\mathbf{A}' = \mathsf{P}^{\mathsf{T}}\mathbf{l}'$ ***can easily be determined using the projection matrices for points and lines.***

2.3.2 Lines

A similar projection relation holds for 3D lines. We obtain the direct linear transformation of 3D lines (Faugeras & Papadopoulo 1998, Förstner 2000)

$$\mathbf{l}' = \mathsf{Q}\overline{\mathbf{L}} = (I_3 \otimes \overline{\mathbf{L}}^{\mathsf{T}})\mathbf{q} \qquad \text{with} \qquad \mathbf{q} = \text{vec}(\mathsf{Q}^{\mathsf{T}}) \tag{4}$$

or

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{1}}^{\mathsf{T}} \\ \tilde{\mathbf{2}}^{\mathsf{T}} \\ \tilde{\mathbf{3}}^{\mathsf{T}} \end{pmatrix} \overline{\mathbf{L}} = \begin{pmatrix} < \tilde{\mathbf{1}}, \mathbf{L} > \\ < \tilde{\mathbf{2}}, \mathbf{L} > \\ < \tilde{\mathbf{3}}, \mathbf{L} > \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{L}}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0}^{\mathsf{T}} & \overline{\mathbf{L}}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} & \overline{\mathbf{L}}^{\mathsf{T}} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{1}} \\ \tilde{\mathbf{2}} \\ \tilde{\mathbf{3}} \end{pmatrix}$$

with the 3×6 projection matrix[3)] Q

$$\mathsf{Q} = \frac{\partial \mathbf{l}'}{\partial \overline{\mathbf{L}}} = \begin{pmatrix} (\mathbf{2} \cap \mathbf{3})^{\mathsf{T}} \\ (\mathbf{3} \cap \mathbf{1})^{\mathsf{T}} \\ (\mathbf{1} \cap \mathbf{2})^{\mathsf{T}} \end{pmatrix} \tag{5}$$

at the same time being one of the two Jacobians, the other being

$$\frac{\partial \mathbf{l}'}{\partial \mathbf{q}} = I_3 \otimes \overline{\mathbf{L}}^{\mathsf{T}}$$

[3)] called $\tilde{\mathsf{P}}$ in (Faugeras & Papadopoulo 1998) where the projection is given as product of $\tilde{\mathsf{P}}$ with the line $\mathbf{L}$ instead of its dual $\overline{\mathbf{L}}$ as in (4).

The three rows of Q are 6-vectors representing 3D lines, namely the intersections of the principle planes, thus the three coordinate axes of the camera system.

2.3.3 Inversion

Inversion of the projection leads to projection rays $\mathbf{L}'$ for image points $\mathbf{x}'$

$$\mathbf{L}' = \mathsf{Q}^\mathsf{T}\mathbf{x}' = u'\tilde{\mathbf{1}} + v'\tilde{\mathbf{2}} + w'\tilde{\mathbf{3}} = (\mathbf{x}'^\mathsf{T} \otimes I_6)\mathbf{q} = u'\mathbf{2} \cap \mathbf{3} + v'\mathbf{3} \cap \mathbf{1} + w'\mathbf{1} \cap \mathbf{2} \qquad (6)$$

The expression for $\mathbf{L}'$ results from the incidence relation $\mathbf{x}'^\mathsf{T}\mathbf{l}' = 0$ for all lines $\mathbf{l}' = \mathsf{Q}\overline{\mathbf{L}}$ passing through $\mathbf{x}'$, leading to $(\mathbf{x}'^\mathsf{T}\mathsf{Q})\ \overline{\mathbf{L}} = <\mathbf{L}', \mathbf{L}> = 0$. Observe the two Jacobians to be

$$\frac{\partial \mathbf{L}'}{\partial \mathbf{x}'} = \mathsf{Q}^\mathsf{T} \qquad \frac{\partial \mathbf{L}'}{\partial \mathbf{q}} = \mathbf{x}'^\mathsf{T} \otimes I_6$$

A similar expression can be given for the projection planes $\mathbf{A}'$ for image lines $\mathbf{l}'$

$$\mathbf{A}' = \mathsf{P}^\mathsf{T}\mathbf{l}' = a'\mathbf{1} + b'\mathbf{2} + c'\mathbf{3} = (\mathbf{l}'^\mathsf{T} \otimes I_4)\mathbf{p} \qquad (7)$$

The expression results from the incidence relation $\mathbf{l}'^\mathsf{T}\mathbf{x}' = 0$ for all points $\mathbf{x}' = \mathsf{P}\mathbf{X}$ on the line $\mathbf{l}'$, leading to $(\mathbf{l}'^\mathsf{T}\mathsf{P})\ \mathbf{X} = <\mathbf{A}', \mathbf{X}> = 0$. The two Jacobians therefore are

$$\frac{\partial \mathbf{A}'}{\partial \mathbf{l}'} = \mathsf{P}^\mathsf{T} \qquad \frac{\partial \mathbf{A}'}{\partial \mathbf{p}} = \mathbf{l}'^\mathsf{T} \otimes I_4$$

2.3.4 Relation between the Projection Matrices

We also will need the Jacobian of Q with respect to P. It is given by the explicit expression of the vector $\mathbf{q} = \mathrm{vec}(\mathsf{Q}^\mathsf{T})$ as a function of $\mathbf{p} = \mathrm{vec}(\mathsf{Q}^\mathsf{T})$ using $\mathbf{A} \cap \mathbf{B} = \overline{\Pi}(A)\mathbf{B}$ in table 1:

$$\mathbf{q} = \begin{pmatrix} \mathbf{2} \cap \mathbf{3} \\ \mathbf{3} \cap \mathbf{1} \\ \mathbf{1} \cap \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \overline{\Pi}(\mathbf{2}) \\ \overline{\Pi}(\mathbf{3}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\Pi}(\mathbf{1}) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{0} & -\overline{\Pi}(\mathbf{3}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\overline{\Pi}(\mathbf{1}) \\ -\overline{\Pi}(\mathbf{2}) & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

therefore

$$\frac{\partial \mathbf{q}}{\partial \mathbf{p}} = \begin{pmatrix} \mathbf{0} & -\overline{\Pi}(\mathbf{3}) & \overline{\Pi}(\mathbf{2}) \\ \overline{\Pi}(\mathbf{3}) & \mathbf{0} & -\overline{\Pi}(\mathbf{1}) \\ -\overline{\Pi}(\mathbf{2}) & \overline{\Pi}(\mathbf{1}) & \mathbf{0} \end{pmatrix} \qquad (8)$$

Summarizing this section, all mentioned relations appeared to be bilinear, resulting from consequently using algebraic projective geometry. This results in *rigorous expressions for all Jacobians* and allows *rigorous error propagation.*

3 ESTIMATION

3.1 Best Estimates

The task is to estimate best fits of unknown parameters from observed values having known stochastic properties. We assume our estimation problem to have a special structure:

1. We want to determine U unknown parameters $\beta_u, u = 1, ..., U$, collected in the vector $\boldsymbol{\beta}$. We have I groups of observations $\boldsymbol{y}_i, i = 1, ..., I$ having n_i observations each. They may be collected in the vector $\boldsymbol{y}$ with length $N = \sum_i n_i$.

2. In our application we assume each of the I groups of observations to be linked with the unknown parameters $\boldsymbol{\beta}$ by a set of m_i constraints $w_{ij}, j = 1, ..., m_i$ collected in the vector
$$\mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta}) = \mathsf{A}_i^\mathsf{T}(\boldsymbol{y}_i)\ \boldsymbol{\beta} = \mathbf{0} \tag{9}$$
leading to the $M = \sum_i m_i$ constraints
$$\mathbf{w}(\boldsymbol{y}, \boldsymbol{\beta}) = \mathsf{A}^\mathsf{T}(\boldsymbol{y})\ \boldsymbol{\beta} = \mathbf{0} \tag{10}$$
The essential part is the *linearity* of these constraints in the *unknown parameters* and their *homogeneity*[4)]. We later assume the constraints also to depend linearly on the observations, thus being of the structure
$$\mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta}) = \mathsf{A}_i(\boldsymbol{y}_i)\boldsymbol{\beta} = \mathsf{B}_i(\boldsymbol{\beta})\boldsymbol{y}_i = \mathbf{0} \tag{11}$$

   The constraints are supposed to be valid for the true values of the unknown parameters and the observations. They should also hold for the fitted values $\widehat{\boldsymbol{y}}$ and $\widehat{\boldsymbol{\beta}}$. With the matrices
$$\mathsf{A}^\mathsf{T}(\boldsymbol{y}) = (\mathsf{A}_1^\mathsf{T}(\boldsymbol{y}_1), ..., \mathsf{A}_I^\mathsf{T}(\boldsymbol{y}_I)) \qquad \mathsf{B}(\boldsymbol{\beta}) = \mathrm{Diag}(\mathsf{B}_i(\boldsymbol{\beta}))$$
the bilinear constraints can be written as
$$\mathbf{w}(\boldsymbol{y}, \boldsymbol{\beta}) = \mathsf{A}(\boldsymbol{y})\boldsymbol{\beta} = \mathsf{B}(\boldsymbol{\beta})\boldsymbol{y} = \mathbf{0}$$

4) In case they are not homogeneous, the following expressions become more involved (MATEI & MEER 1997)

3. Due to the homogeneity of the constraints (9) we need the additional constraint between the unknown parameters only

$$\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{\beta} = 1 \tag{12}$$

4. In addition to the homogeneity constraint, we might need more constraints for the unkown parameters. Here we only discuss one of them. In case of estimated 3D lines we in addition have the Plücker condition $\mathbf{L}^{\mathsf{T}}\overline{\mathbf{L}} = 0$ being a constraint of the form

$$\frac{1}{2}\boldsymbol{\beta}^{\mathsf{T}} C \boldsymbol{\beta} = 0 \tag{13}$$

5. The observed values are uncertain, their uncertainty is given by

$$\underline{\boldsymbol{y}} \sim N(\tilde{\boldsymbol{y}}, \boldsymbol{\Sigma}_{yy}) = N(\{\tilde{\boldsymbol{y}}_i\}, \mathrm{Diag}(\boldsymbol{\Sigma}_{y_i y_i}))$$

   stating the groups to be mutually independent, however allow for full covariance matrices among the observations within the groups $\boldsymbol{y}_i$, the tilde $\tilde{}$ indicating the true value.

The optimal estimate $\widehat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$ is given by finding the minimum

$$\Omega = (\widehat{\boldsymbol{y}} - \boldsymbol{y})^{\mathsf{T}} \boldsymbol{\Sigma}_{yy}^{+} (\widehat{\boldsymbol{y}} - \boldsymbol{y}) = \sum_{i=1}^{I} (\widehat{\boldsymbol{y}}_i - \boldsymbol{y}_i)^{\mathsf{T}} \boldsymbol{\Sigma}_{y_i y_i}^{+} (\widehat{\boldsymbol{y}}_i - \boldsymbol{y}_i) \tag{14}$$

under the given constraints. It is known to be the ML-estimate in case the random perturbations of the observed values are normally distributed. If we do not impose this assumption, one knows it is the best linear unbiased estimate. Observe, we allow the observations to be correlated with a covariance matrix not having full rank. The effective number of observations then will be lower than N namely $\sum_i \mathrm{rk}\,(\boldsymbol{\Sigma}_{y_i y_i})$.

We will split the estimation problem into two steps: The first one takes only the basic constraints (10) and the normalization constraint (12) into account. Instead of iteratively solving a set of normal equations we iteratively solve an eigenvalue problem, which practically needs no approximate values. The solution is a direct generalization of the classical procedure for directly solving a problem of type (9) with constraint groups of size 1. The second step then updates this estimate based on the additional constraints (13).

This two step procedure is also given in (Matei & Meer 1997, Leedan 1997).

3.2 Minimizing the Algebraic Distance

We first give the classical solution which does not take the uncertainty of the observed values and the additional constraint into account. In order to show the simplicity of the extension we write this well known solution more in detail.

We give a solution to the problem

$$\mathsf{A}_i^{\mathsf{T}}(\widehat{\boldsymbol{y}}_i)\,\widehat{\boldsymbol{\beta}} = 0,\ I = 1, ..., I \qquad \widehat{\boldsymbol{\beta}}^{\mathsf{T}}\widehat{\boldsymbol{\beta}} = 1$$

The first constraint will not be fulfilled by the given observations $\boldsymbol{y}_i$ leading to the residuals $\mathbf{w}_i$ of the constraints

$$\mathbf{w}_i = \mathbf{w}(\boldsymbol{y}_i, \widehat{\boldsymbol{\beta}}) = \mathsf{A}_i^{\mathsf{T}}(\boldsymbol{y}_i)\,\widehat{\boldsymbol{\beta}},\ i = 1, ..., I$$

Therefore we minimize the algebraic distance

$$\Omega_1 = \sum_{i=1}^{I} \mathbf{w}_i^{\mathsf{T}}\,\mathbf{w}_i = \boldsymbol{\beta}^{\mathsf{T}} \left(\sum_{i=1}^{I} \mathsf{A}_i(\boldsymbol{y}_i)\,\mathsf{A}_i^{\mathsf{T}}(\boldsymbol{y}_i) \right) \boldsymbol{\beta} \qquad \text{under} \qquad \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{\beta} = 1$$

This minimization problem leads to the solution: $\widehat{\boldsymbol{\beta}} = \mathbf{e}_i(\lambda_i)|\lambda_i(\mathsf{M}) = \min$ stating the optimal estimate $\widehat{\boldsymbol{\beta}}$ to be the smallest normalized eigenvector of the matrix

$$\mathsf{M} = \sum_{i=1}^{I} \mathsf{A}_i(\boldsymbol{y}_i)\,\mathsf{A}_i^{\mathsf{T}}(\boldsymbol{y}_i) = \mathsf{A}(\boldsymbol{y})\,\mathsf{A}^{\mathsf{T}}(\boldsymbol{y})$$

This is a *direct* solution, as no approximate values are necessary. This solution can also be used in case the constraints are linear only in the unknown parameters but possibly nonlinear in the observations (cf. (DUDA & HART 1973), pp. 332, pp. 377 and (TAUBIN 1993)).

The solution obviously is suboptimal as it depends on the mutual scaling of the observations and does not take into account their uncertainty. Taubin gives a solution which is invariant to the scaling of the variables (TAUBIN 1993), which is identical to the optimal solution for uncorrelated observations with the same variance. Matei and Meer (MATEI & MEER 1997) give a solution to the case of observations of different weights.

3.3 Minimizing the Weighted Algebraic Distance

The following solution is equivalent to the one of Matei and Meer [5], but it is much simpler.

We just need to take the uncertainty of the residuals $\mathbf{w}_i(\boldsymbol{y}_i, \widehat{\boldsymbol{\beta}})$ of the constraints into account. The covariance matrix follows from (9) and is given by

$$\boldsymbol{\Sigma}_{w_i w_i} = \left(\left. \frac{\partial \mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta})}{\partial \boldsymbol{y}_i} \right|_{y=\widehat{y}, \beta=\widehat{\beta}} \right) \boldsymbol{\Sigma}_{y_i y_i} \left(\left. \frac{\partial \mathbf{w}_i(\boldsymbol{y}_i, \boldsymbol{\beta})}{\partial \boldsymbol{y}_i} \right|_{y=\widehat{y}, \beta=\widehat{\beta}} \right)^{\mathsf{T}} \tag{15}$$

This leads to the following optimization problem which in case of normally distributed observations yields the ML-estimate: Minimize the form

$$\Omega = \mathbf{w}^{\mathsf{T}} \boldsymbol{\Sigma}_{ww}^{+} \mathbf{w} = \sum_{i=1}^{I} \mathbf{w}_i^{\mathsf{T}} \boldsymbol{\Sigma}_{w_i w_i}^{+} \mathbf{w}_i \to \min \tag{16}$$

under the given constraints. The pseudo inverse is to be taken in case the constraints are linearily dependent. It is known from estimation theory, that the minimum of

$$\Omega = \widehat{\boldsymbol{e}}^{\mathsf{T}} \boldsymbol{\Sigma}_{yy}^{+} \widehat{\boldsymbol{e}} = \mathbf{w}^{\mathsf{T}} \boldsymbol{\Sigma}_{ww}^{+} \mathbf{w}$$

from (14) is identical to the minimum of Ω in (16) in case the same constraints are used.

3.4 Specialization to bilinear constraints

We now specialize to the structure of our constraints. Together with the factorization of $\mathbf{w}$ we therefore need to find the optimal value for $\widehat{\boldsymbol{\beta}}$ minimizing

$$\Omega = \boldsymbol{\beta}^{\mathsf{T}} \mathsf{A}^{\mathsf{T}}(\boldsymbol{y}) \left(\mathsf{B}(\boldsymbol{\beta}) \boldsymbol{\Sigma}_{yy} \mathsf{B}^{\mathsf{T}}(\boldsymbol{\beta}) \right)^{+} \mathsf{A}(\boldsymbol{y})\, \boldsymbol{\beta}$$

under the constraint $\widehat{\boldsymbol{\beta}}^{\mathsf{T}} \widehat{\boldsymbol{\beta}} = 1$. This leads to the optimal estimate $\widehat{\boldsymbol{\beta}}$ being the normalized eigenvector corresponding to the smallest eigenvalue of the matrix[6] (Förstner 2001)

$$\mathsf{M} = \mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}}) \left(\mathsf{B}(\widehat{\boldsymbol{\beta}}) \boldsymbol{\Sigma}_{yy} \mathsf{B}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}) \right)^{+} \mathsf{A}(\boldsymbol{y})$$

---

[5] except for the bias, which is taken into account in theirs, which, however, according to a personal communication with P. Meer, has only minor influence on the result.

[6] Matai and Meer first take the Jacobian of Ω with respect to the unknowns $\boldsymbol{\beta}$ and from that derive a generalized eigenvalue problem to solve for $\boldsymbol{\beta}$. Here we obtain an ordinary eigenvalue problem.

Observe, we need the estimated values $\widehat{\boldsymbol{\beta}}$ for the error propagation of $\mathbf{w}(\boldsymbol{y}, \widehat{\boldsymbol{\beta}})$ in $\mathsf{B}(\widehat{\boldsymbol{\beta}})\boldsymbol{\Sigma}_{yy}\mathsf{B}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}})$ and the fitted observations $\widehat{\boldsymbol{y}}$ in the left factor $\mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}})$. Therefore we need to iterate. This is done by using the unknown parameters $\widehat{\boldsymbol{\beta}}^{(\nu-1)}$ and the fitted observations $\widehat{\boldsymbol{y}}^{(\nu-1)}$ from the previous iteration and determine the minimum eigenvector $\widehat{\boldsymbol{\beta}}^{(\nu)}$ of

$$\boxed{\mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}}^{(\nu-1)}) \left(\mathsf{B}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\boldsymbol{\Sigma}_{yy}\mathsf{B}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\right)^{+} \mathsf{A}(\boldsymbol{y})\, \widehat{\boldsymbol{\beta}}^{(\nu)} = \lambda\, \widehat{\boldsymbol{\beta}}^{(\nu)}} \tag{17}$$

The fitted values of the observations can be determined individually from

$$\boxed{\widehat{\boldsymbol{y}}_i^{(\nu-1)} = \left(I - \boldsymbol{\Sigma}_{y_iy_i}\mathsf{B}_i^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)}) \left(\mathsf{B}_i(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\boldsymbol{\Sigma}_{y_iy_i}\mathsf{B}_i^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\right)^{+} \mathsf{B}_i^{\mathsf{T}}(\widehat{\boldsymbol{\beta}}^{(\nu-1)})\right) \boldsymbol{y}_i} \tag{18}$$

Taking the constraint $\widehat{\boldsymbol{\beta}}^{\mathsf{T}}\widehat{\boldsymbol{\beta}} = 1$ into account we also can determine the covariance matrix $\boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}}$ of the estimated value from $\boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}} = [\mathsf{A}^{\mathsf{T}}(\widehat{\boldsymbol{y}}) \left(\mathsf{B}(\widehat{\boldsymbol{\beta}})\boldsymbol{\Sigma}_{yy}\mathsf{B}^{\mathsf{T}}(\widehat{\boldsymbol{\beta}})\right)^{+} \mathsf{A}(\widehat{\boldsymbol{y}})]^{+}$ using its null space $\widehat{\boldsymbol{\beta}}$. The estimated variance factor is given by $\widehat{\sigma}_0^2 = \Omega/R$. The optimum value of Ω is taken from (16) and the redundancy R is the number of effective constraints G_{eff} reduced by the number of effective unknowns $U-1$, i. e. $R = \sum_i \text{rk}\,(\boldsymbol{\Sigma}_{w_iw_i}) - (U-1)$ the number of unknown parameters β_i being U. In case the redundancy is large enough, say > 30, this can be used to determine the estimated covariance matrix of the unknown parameters

$$\widehat{\boldsymbol{\Sigma}}_{\widehat{\beta}\widehat{\beta}} = \widehat{\sigma}_0^2 \boldsymbol{\Sigma}_{\widehat{\beta}\widehat{\beta}}$$

3.5 Further Constraints

In case of further constraints are to be fullfilled we need to update the estimate. This is performed by taking the estimates $\widehat{\boldsymbol{\beta}}$ as observations and impose the desired constraints (cf. (Matei & Meer 1997) leading to new estimates $\widehat{\widehat{\boldsymbol{\beta}}}$.

4 Estimates for 3D Lines

We now present estimates for 3D lines, the procedures for points are similar (Förstner 2001).

4.1 Minimizing the Algebraic Distance for Estimating 3D Lines

We want to determine the coordinates of a 3D line (cf. Fig. 4.1). We assume I image points $\mathbf{x}'_{ik}$ and J image lines $\mathbf{l}'_{jk}$ to be observed in up to K images, the second index indicating the image in which the feature has been observed. The observed lines are images of parts of the 3D line, the observed points are images of 3D points sitting on the 3D line.

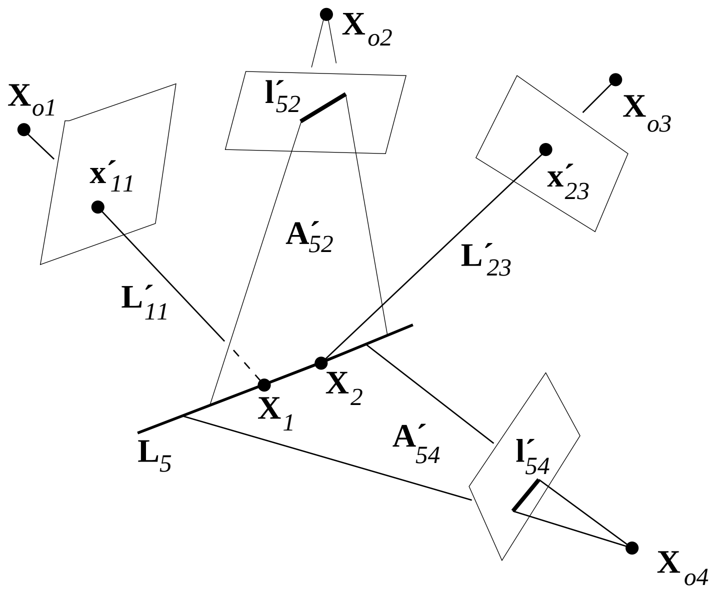


Figure 2: ***3D line*** $\mathbf{L}_5$ ***observed from 4 cameras. In two of the images the line is observed, namely*** $\mathbf{l}'_{52}$ ***in image 2 and*** $\mathbf{l}'_{54}$ ***in image 4. In the other two images 1 and 3 two points*** $\mathbf{X}_1$ ***and*** $\mathbf{X}_2$ ***on*** $\mathbf{L}_5$ ***are observed leading to*** $\mathbf{x}'_{11}$ ***and*** $\mathbf{x}'_{23}$***. Estimation of*** $\mathbf{L}_5$ ***uses the incidence of the projecting lines*** $\mathbf{L}'_{11}$ ***and*** $\mathbf{L}'_{23}$ ***and of the projecting planes*** $\mathbf{A}'_{52}$ ***and*** $\mathbf{A}'_{54}$ ***with the 3D line*** $\mathbf{L}_5$***.***

We first want to give the solution for a 3D line $\mathbf{L}$ when minimizing the algebraic distance.

In case we observe the j-th line $\mathbf{l}'_{jk}$ in image k we have the *line-plane constraint*, namely the projecting plane $\mathbf{A}'_j$ to pass through the 3D line

$$\mathbf{0} = \mathbf{w}_{jk} = \mathbf{A}'_{jk} \cap \mathbf{L} = \mathsf{\Pi}^{\mathsf{T}}(\mathbf{A}'_{jk})\mathbf{L}$$

where the projecting plane again can be determined from $\mathbf{A}'_{jk} = \mathsf{P}_k^{\mathsf{T}}\mathbf{l}'_{jk}$

In case we have the image point $\mathbf{x}'_{ik}$ of a 3D point lying on the unknown 3D line we have the *line-line constraint*, namely the projection ray $\mathbf{L}'_{ik}$ to pass through he unknown line

$$0 = w_{ik} =< \mathbf{L}'_{ik}, \mathbf{L} >= \overline{\mathbf{L}}'^{\mathsf{T}}_{ik}\mathbf{L}$$

where again the projecting line can be determined from $\mathbf{L}'_{ik} = \tilde{\mathsf{P}}_k^{\mathsf{T}}\mathbf{x}'_{ik}$. Minimizing

$$\Omega = \sum_{ik} w_{ik}^2 + \sum_{jk} \mathbf{w}_{jk}^{\mathsf{T}}\mathbf{w}_{jk}$$

is identical to minimizing

$$\Omega = \mathbf{L}^{\mathsf{T}} \left(\sum_{ik} \overline{\mathbf{L}}'_{ik} \overline{\mathbf{L}}'^{\mathsf{T}}_{ik} + \sum_{jk} \Pi(\mathbf{A}'_{jk}) \Pi^{\mathsf{T}}(\mathbf{A}'_{jk}) \right) \mathbf{L}$$

under the constraints $\mathbf{L}^{\mathsf{T}}\mathbf{L} = 1$ and $\mathbf{L}^{\mathsf{T}}\overline{\mathbf{L}} = 0$. The direct $\widehat{\mathbf{L}}$ solution for $\mathbf{L}$ without the second constraint is given by the normalized smallest eigenvector of

$$\mathsf{N} = \sum_{ik} \overline{\mathbf{L}}'_{ik} \overline{\mathbf{L}}'^{\mathsf{T}}_{ik} + \sum_{jk} \Pi(\mathbf{A}'_{jk}) \Pi^{\mathsf{T}}(\mathbf{A}'_{jk}) \tag{19}$$

The final solution, which takes the Plücker constraint into account is given by

$$\widehat{\widehat{\mathbf{L}}} = \widehat{\mathbf{L}} - \frac{1}{2} \Sigma_{\widehat{L}\widehat{L}} \widehat{\overline{\mathbf{L}}} (\widehat{\overline{\mathbf{L}}}^{\mathsf{T}} \Sigma_{\widehat{L}\widehat{L}} \widehat{\overline{\mathbf{L}}})^{-1} < \widehat{\mathbf{L}}, \widehat{\mathbf{L}} >$$

Iteration might be neccessary as the correction of the line due to the Plücker constraint changes the matrix N. A second iteration has been shown to be sufficient.

4.2 Minimal Solutions

The above mentioned direct solution require a minimum number of observations to be available.

The estimation of the 3D line is possible if we have

1. at least two non parallel projecting planes $\mathbf{A}'$ or
2. at least one projecting plane $\mathbf{A}'$ and two projecting lines $\mathbf{L}'$ which are not parallel to $\mathbf{A}'$ and do not meet the plane in the same 3D point or
3. at least 5 (!) linearily independent projecting lines $\mathbf{L}'$.

Then the rank of the matrix N is five.

Actually there is also a minimal *solution* for the 3D line, having four degrees of freedom, in case of only *four lines* meeting it. The matrix N in general then has rank 4, thus there is no unique eigenvector corresponding to the smallest eigenvalue. If the nullspace of N is spanned by the two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ the unknown 3D line is

$$\mathbf{L} = \lambda \mathbf{e}_1 + (1 - \lambda) \mathbf{e}_2$$

for some λ. The Plücker condition

$$\mathbf{L}^{\mathsf{T}} C \mathbf{L} = 0$$

leads to a quadratic condition for λ, which then yields two solutions for the unknown line. This special solution, however, is not contained in the general setup.

4.3 Optimal Estimation of 3D Lines

Now we take covariances of the observation into account.

The covariance of the j-th residual $\mathbf{w}_{jk} = \mathsf{\Pi}^{\mathsf{T}}(\mathbf{A}'_{jk})\mathbf{L} = -\overline{\mathsf{\Gamma}}(\mathbf{L})\mathbf{A}'_{jk}$ of the line-plane constraint is given by

$$\mathbf{\Sigma}^{(\nu)}_{w_{jk}w_{jk}} = \overline{\mathsf{\Gamma}}(\widehat{\mathbf{L}}^{(\nu)})\mathbf{\Sigma}_{A'_{jk}A'_{jk}}\overline{\mathsf{\Gamma}}^{\mathsf{T}}(\widehat{\mathbf{L}}^{(\nu)})$$

where the Jacobian $\partial \mathbf{w}_{jk}/\partial \boldsymbol{y} = \overline{\mathsf{\Gamma}}(\widehat{\mathbf{L}}^{(\nu)})$ needs to be evaluated at the estimated 3D line. The covariance matrix of the projecting plane $\mathbf{A}'_j$ can be determined from

$$\mathbf{\Sigma}_{A'_{jk}A'_{jk}} = \mathsf{P}_k^{\mathsf{T}}\mathbf{\Sigma}_{l'_{jk}l'_{jk}}\mathsf{P}_k + ({\mathbf{l}_{jk}}'^{\mathsf{T}} \otimes I_4)\mathbf{\Sigma}_{\mathrm{pp}}(\mathbf{l}'_{jk} \otimes I_4) \tag{20}$$

assuming independence of $\mathbf{l}_{jk}$ and P_k and the index $\mathbf{p} = \mathrm{vec}(\mathsf{P}^{\mathsf{T}})$ indicating the vector containing the elements of the projection matrix P row-wise. The variance $\sigma^2_{w_{ik}}$ of the i-th residual $w_{ik} = \overline{\mathbf{L}}'^{\mathsf{T}}_{ik}\mathbf{L}$ of the line-line constraint is given by

$$\sigma^2_{w_{ik}} = \left(\widehat{\overline{\mathbf{L}}}^{(\nu)}\right)^{\mathsf{T}} \mathbf{\Sigma}_{L'_{ik}L'_{ik}} \left(\widehat{\overline{\mathbf{L}}}^{(\nu)}\right)$$

The covariance matrix of the projection line $\mathbf{L}'_{ik}$ can be determined from

$$\mathbf{\Sigma}_{L'_{ik}L'_{ik}} = \mathsf{Q}_k^{\mathsf{T}}\mathbf{\Sigma}_{x'_{ik}x'_{ik}}\mathsf{Q}_k + ({\mathbf{x}_{ik}}'^{\mathsf{T}} \otimes I_6)\mathbf{\Sigma}_{\mathrm{qq}}(\mathbf{x}'_{ik} \otimes I_6) \tag{21}$$

with the index $\mathbf{q} = \mathrm{vec}(\mathsf{Q}^{\mathsf{T}})$ indicating the elements of the projection matrix Q collected row-wise. The covariance matrix $\mathbf{\Sigma}_{\mathrm{qq}}$ can be determined from the covariance matrix $\mathbf{\Sigma}_{\mathrm{pp}}$ using the Jacobian (8). Omitting the iteration index we need to minimize $\Omega = \sum_{ik} w^2_{ik}/\sigma_{w^2_{ik}} + \sum_j \mathbf{w}^{\mathsf{T}}_{jk}\mathbf{\Sigma}^{+}_{w_{jk}w_{jk}}\mathbf{w}_{jk}$ or

$$\Omega = \mathbf{L}^{\mathsf{T}}\left(\sum_{ik}\frac{\overline{\mathbf{L}}'_{ik}\overline{\mathbf{L}}'^{\mathsf{T}}_{ik}}{\sigma^2_{w_{ik}}} + \sum_{jk}\mathsf{\Pi}(\mathbf{A}'_{jk})\mathbf{\Sigma}^{+}_{w_{jk}w_{jk}}\mathsf{\Pi}^{\mathsf{T}}(\mathbf{A}'_{jk})\right)\mathbf{L}$$

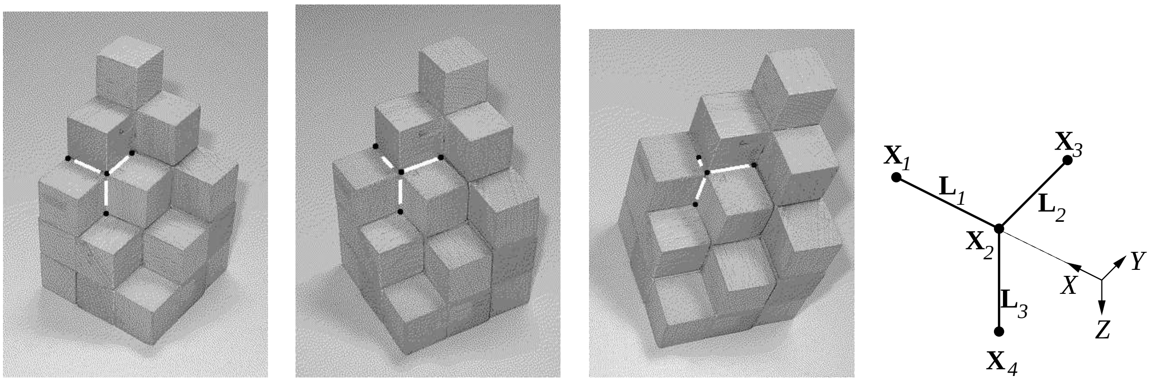
under the constraints $\mathbf{L}^{\mathsf{T}}\mathbf{L} = 1$ and $\mathbf{L}^{\mathsf{T}}C\mathbf{L} = 0$

When only taking the first constraint into account we therefore obtain the estimate $\widehat{\mathbf{L}}$ as the smallest normalized eigenvalue of

$$\mathsf{N} = \sum_{ik}\frac{\widehat{\overline{\mathbf{L}}}'_{ik}\overline{\mathbf{L}}'^{\mathsf{T}}_{ik}}{\widehat{\overline{\mathbf{L}}}^{\mathsf{T}}\mathbf{\Sigma}_{L'_{ik}L'_{ik}}\widehat{\overline{\mathbf{L}}}} + \sum_{jk}\mathsf{\Pi}(\widehat{\mathbf{A}}'_{jk})\left(\overline{\mathsf{\Gamma}}(\widehat{\mathbf{L}})\mathbf{\Sigma}_{A'_{jk}A'_{jk}}\overline{\mathsf{\Gamma}}^{\mathsf{T}}(\widehat{\mathbf{L}})\right)^{+}\mathsf{\Pi}^{\mathsf{T}}(\mathbf{A}'_{jk}) \tag{22}$$

Observe, the determination of the matrices $\mathsf{\Gamma}(\widehat{\mathbf{L}})$ in N needs to be based on the fitted values $\widehat{\mathbf{L}}$, $\widehat{\overline{\mathbf{L}}}'_{ik}$ and $\widehat{\mathbf{A}}'_{jk}$, initiating an iteration scheme.

Figure 3: ***three images of a polyhedron. In each image the four points*** $\mathbf{X}_i, i = 1,2,3,4$ ***and the three lines*** $\mathbf{L}_j, j = 1,2,3$ ***have been observed. Based on hypotheses about the object, image points may be used to determine the geometry of the 3D line and image line segments can be used to determine 3D points. Observe, point 1 in the right image 3 does not correspond to the others. The coordinate system is in the centre of the polyhedron. On the right hand the numbering of the points and lines is given together with the coordinate system.***



Taking the Plücker constraint into account we obtain the final estimate

$$\hat{\hat{\mathbf{L}}} = \hat{\mathbf{L}} - \frac{1}{2}\mathsf{N}^+\hat{\mathbf{L}}(\hat{\mathbf{L}}^\mathsf{T}\mathsf{N}^+\hat{\mathbf{L}})^{-1}\hat{\mathbf{L}}^\mathsf{T}\hat{\mathbf{L}}$$

The covariance matrix of the final estimate now is

$$\mathbf{\Sigma}_{\hat{\hat{L}}\hat{\hat{L}}} = \mathbf{\Sigma}_{\hat{L}\hat{L}} - \mathbf{\Sigma}_{\hat{L}\hat{L}}\hat{\mathbf{L}}\hat{\mathbf{L}}^\mathsf{T}\mathbf{\Sigma}_{\hat{L}\hat{L}}/\sigma_h^2$$

using $\mathbf{\Sigma}_{\hat{L}\hat{L}} = \mathsf{N}^+$ (cf. 22) with its nullspace $\hat{\mathbf{L}}$.

For determining the pseudoinverse of $\mathbf{\Sigma}_{ww} = \overline{\Pi}^\mathsf{T}(\hat{\mathbf{X}})\mathbf{\Sigma}_{L'L'}\overline{\Pi}(\hat{\mathbf{X}})$ we need its rank. The point-point constraint $\mathbf{w} = \mathbf{0}$ has two degrees of freedom, which easily can be seen in case the line is approximately parallel to the Z-axis through the origin: then a point in the XY-plane close to the origin lies on the line if its two coordinates are identical with those of the intersection point of the line with the XY-plane. Therefore the rank of $\mathbf{\Sigma}_{ww}$ is 2, and the null space has dimension 2.

5 Example

Fig. 3 shows three images of a polyhedron. The three projection matrices P_i for points have been estimated using a DLT based on 13 observed points exploiting the special structure of the polyhedron. The object coordinate system is in the centre of the polyhedron with the axes as shown in the figure.

The covariance matrices for the image points and the image lines are given by the feature extraction program. The image coordinates of the points are measured with a standard deviation of 0.2 to 0.3 pixels. The lines are measured with a standard deviation in the position across the line between 0.2 and 0.9 pixels and a standard deviation in direction between 0.5^o and 2^o, except for the two short lines 1 in image 2 and 3, which have a standard deviation of appr. 10^o and 20^o in direction.

We give the results for the estimated points, once when minimizing the algebraic error, once when using the ML-estimate. In all cases we assumed the orientation of the cameras to be error free. As point 1 in image 3 is not corresponding to the points in the other images, it is not used. We also give the estimated $\widehat{\sigma}_0$, being the factor by which the standard deviations provided by the feature extraction are too optimistic. The results for 3D points are:

point	type	X [mm]	Y [mm]	Z [mm]	red.	$\widehat{\sigma}_0[1]$	$\sigma_{\widehat{X}}$ [mm]	$\sigma_{\widehat{Y}}$ [mm]	$\sigma_{\widehat{Z}}$ [mm]
1	alg.	3.90	-0.16	-0.14	4				
	opt.	4.20	0.03	0.09	4	1.56	0.35	0.27	0.31
2	alg.	1.95	-0.09	-0.06	12				
	opt.	1.99	-0.01	-0.07	12	2.08	0.88	0.70	0.96
3	alg.	2.04	1.89	-0.07	6				
	opt.	2.13	1.95	0.03	6	3.92	0.57	0.46	0.62
4	alg.	2.25	0.14	2.21	6				
	opt.	2.08	0.04	1.04	6	4.41	0.49	0.41	0.58

The ML-estimates appear to be better than the algebraic estimates. This is confirmed by the average of the 6 normalized distances compared with their mean. The r. m. s. distance reduces from 1.31 mm to 0.96 mm. The estimated values for $\widehat{\sigma}$ are in the range of 1.5 to 4.5 which suggests the estimates of the feature extraction to be a bit optimistic. The standard deviations of the estimated 3D coordinates are in the range betweeen 0.3 and 1.0 mm, point 1 being most precise and point 2 being worst.

The results for 3D lines are:

line	type	$L_1[1]$	L_2 [1]	L_3 [1]	L_4 [mm]	L_5 [mm]	L_6 [mm]	red.	$\widehat{\sigma}_o[1]$
1	alg.	0.999	0.009	0.008	0.000	-0.157	0.187	8	
	opt.	0.993	0.041	0.105	-0.007	0.443	0.238	8	2.10
2	alg.	-0.024	-1.000	0.019	-0.042	-0.378	-1.981	8	
	opt.	-0.081	-0.995	-0.062	-0.146	0.131	-1.908	8	1.02
3	alg.	0.102	0.067	0.992	-0.011	-2.018	0.138	8	
	opt.	0.072	0.047	0.996	0.054	-2.085	0.095	8	2.00

The average angular between all pairs of 3D lines error increases from 5.5^{gon} to 6.4^{gon}. The reason might be, that the precision of the observed edges is not very high in this case, but due to the non adequate weighting in the algebraic minimization do not influence the result, as the coefficients $\overline{\mathbf{L}}_{ik}$ are two orders of magnitude lower than the corresponding coefficients $\Pi(\mathbf{A}_{jk})$.

The results suggests that the a priori estimates for the variances of the lines might be too optimistic.

6 Conclusions

The paper presented a new technique for nearly direct optimal estimation and thus significantly simplified the procedure by Leedan, Matei and Meer. It gave a new representation for geometric relations in the framework of algebraic projective geometry. This allows to exploit the bilinearity of all geometric relations in an explicit manner and to perform rigorous error propagation of variances and covariances. The observations may be arbitrary correlated, even may have singular covariance matrix. An example with real data demonstrates the feasibility of the approach.

The results can be transferred to all estimation problems where the unknown parameters appear linearily in the constraints, such as the estimation of the projection matrix or the fundamental matrix, but also for general estimation of points, lines and planes for given points, lines and planes incident to the unknown entities.

In all cases the solution can be initiated by the algebraic minimization, practically providing direct optimal estimates. Finally, the estimates yield estimated covariances, which may be used to characterize the statistical uncertainty of the result.

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