Uncertain Neighborhood Relations of Point Sets and Fuzzy Delaunay Triangulation*

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Abstract

Voronoi diagrams are a classical tool for analyzing spatial neighborhood relations. For point fields the spatial proximity can be easily visualized by the dual graph, the Delaunay triangulation. In image analysis VDs and DTs are commonly used to derive neighborhoods for grouping or for relational matching. Neighborhood relations derived from the VD, however, are uncertain in case the common side of two Voronoi cells is comparably short or, equivalently, in case four points of two neighboring triangles in a DT are close to a circle. We propose a measure for characterizing the uncertainty of neighborhoods in a plane point field. As a side result we show the measure to be invariant to the numbering of the four points, though being dependent on the cross ratio of four points. Defining a fuzzy Delaunay triangulation is taken as an example.

1 Motivation

Voronoi Diagrams (VDs) are a classical tool for analyzing spatial neighborhood relations. For two dimensional point sets the spatial proximity easily can be visualized by the dual graph, the Delaunay Triangulation (DT), being extensible to higher dimensions [Preparata and Shamos 1985] or to more general patterns [Mehlhorn *et al.* 1991].

In image analysis VDs and DTs are commonly used to derive neighborhoods for grouping (e. g. [Ahuja and Tuceryan 1989], [Heuel and Förstner 1998]) or for relational matching (e. g. [Ogniewicz 1993]). No thresholds are required for establishing neighborhoods using VD which allows to postpone decisions on the adequateness of derived neighborhoods to a later stage. One of the primary criteria for grouping image features or other data is proximity, which can be established by a DT. Many procedures involving relational matching use neighborhood relations as a first choice.

Now, neighborhood relations derived from the VD are uncertain in case the common side of two Voronoi cells is comparably short or, equivalently, in case

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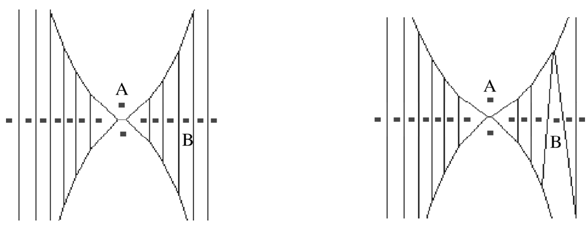
four points of two neighboring triangles in a DT are close to a circle. In grouping, this easily occurs e. g. in case the distance between two sets of collinear points is large compared to two disturbing points (cf. fig. 1).

Figure 1: *shows left two sets of collinear points and two additional points being nearly cocircular with the two inner points. In case one uses the Voronoi Diagram for establishing neighborhood relations, the two additional points prevent the two groups as being identified as neighbors. Right the same set of points is shown, but the top middle (A) and the third from the right (B) slightly moved: The vertical connection between the middle points still is present, while the connection of A and B is broken: The length of the common side of the Voronoi diagram obviously is no good measure for this type of uncertainty (generated by* VoronoiGlide, *[Icking* et al. *1996]).*



When matching two spatial structures based on the properties of the carrying features and their proximity, e. g. using the region adjacency graph, the same situation may occur. Neighborhood relations then may be evaluated with respect to their uncertainty or their sensitivity with respect to small changes or noise.

A typical example is given in Fig. 2, where a small change of the position of the point D leads to a structural change of the VD, thus also of the neighborhood relations derived from it.

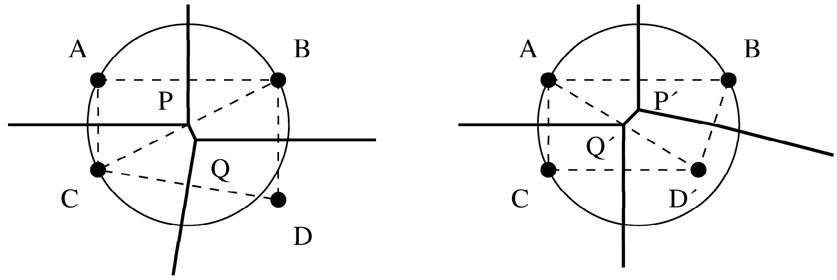
We propose a quantitative measure for the uncertainty of a neighborhood relation derived from the VD of a planar point set. It allows to define a soft or *fuzzy Delaunay triangulation*: In the example of Fig. 2 both diagonals of a quadrangle, composed of two triangles with a common edge, are then part of the fuzzy DT, but with a certainty less than 1. The certainty measure can be used in grouping or matching.

2 The Problem

Checking the stability of the neighborhoods of geometric features, especially points, derived from a Voronoi diagram or a Delaunay triangulation (cf. Fig.

2) can use the geometric configuration of the four points causing the endpoints of each edge PQ: the two points B and C of the two neighboring Voronoi cells and the two points A and D neighbored to these two points.

Figure 2: *shows four points A, B, C and D (left) or D' (right) together with their Voronoi Diagram and their Delaunay Triangulation. Obviously the neighborhood relations are uncertain in case D is uncertain, i. e. slightly shifts to D'. Both diagonals should be considered as valid neighborhoods, however, with a certainty less than 1.*



In case Point D is slightly shifted to D' the edge PQ will disappear and change into the edge $P'Q'$, indicating A and D' to be neighbored.

The transition appears when D passes the circle through (ABC). This is due to the fact that the circle, defined by the three neighboring vertices, does not contain another point (cf. theorem 5.8 in [Preparata and Shamos 1985]). Therefore it is reasonable to check the closeness of the four points to a circle. The closeness may then be transferred into a certainty measure for the diagonal of the DT to actually represent the true neighborhood. This certainty measure should be dependent on the uncertainty of the position of the given points and invariant to the numbering of the points.

A similar reasoning holds for edges of the convex hull. Here the certainty will depend on the collinearity of the three points or the area of the boundary triangle.

The idea is to determine the distance of one point to the circle through the other three points and the area of the boundary triangles, transfer it into a test statistic and use the significance of the test statistic as certainty measure.

3 The Test Statistics

In both cases we assume the uncertainty of the points, to be small, e. g. the standard deviation of their coordinates to be smaller than the smallest distances in concern. Especially we assumed that it is small enough not to influence more than the neighboring edges of the DT. In case of large uncertainties this might lead to complex changes of the neighborhood relations. This case we do not consider here.

3.1 Interior Edges

We start with the test statistic for interior edges of the Delaunay triangulation, thus edges not belonging to the convex hull.

Given the planar coordinates (x_i, y_i) of the four points $P_i, i = 1, 2, 3, 4$ collected in complex numbers

$$z_i = x_i + jy_i \tag{1}$$

the four points lie on a circle in case the cross ratio

$$c(z_1, z_2, z_3, z_4) = \frac{z_1 - z_4}{z_2 - z_4} : \frac{z_1 - z_3}{z_2 - z_3} \tag{2}$$

is real, or if

$$t = \Im\left(c(z_1, z_2, z_3, z_4)\right) \tag{3}$$

equals 0, where $\Im(\cdot)$ denotes the imaginary part of a complex number.

The proof of this result uses the fact that a homography $z' = (a+bz)/(c+dz)$ of the complex plane is identical to the Möbius-transform ([Bronstein *et al.* 1996], pp.584 ff.), which is circle preserving. Therefore one can always find a unique homography, which maps a circle to a straight line. E. g. using the correspondence of the points z_1, z_2 and z_3 of the circle with the points having coordinates $z'_1 = 1$, $z'_2 = 2$, and $z'_3 = 3$ maps the circle to the real axis. If the 4 points lie on a circle, the fourth point needs to map to a point on the real axis. As the cross ratio of any four points on the real axis it real, and the cross ratio is an invariant of a homography, the cross ratio of the 4 points on the circle needs to be real.

Assuming the given points are uncertain with covariance matrix $\boldsymbol{C}$ which in the most simple case could be $\boldsymbol{C} = \sigma^2\boldsymbol{I}$ one can derive the standard deviation σ_t of t:

$$\sigma_t^2 = \boldsymbol{a}^{\mathsf{T}}\boldsymbol{C}\boldsymbol{a} \tag{4}$$

with the Jacobian

$$\boldsymbol{a} = (a_i) \doteq \left(\frac{\partial t}{\partial u_i}\right) \tag{5}$$

where the coordinates are collected in the vector $\boldsymbol{u} = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)^{\mathsf{T}}$.

Thus we obtain the optimal test statistic [Kreyszig 1968]

$$T = \frac{t}{\sigma_t} \sim N(0, 1) \tag{6}$$

which is normally distributed in case one can assume Gaussian distribution of the given points and the uncertainty is small enough that a first order approximation is sufficient. This condition is fulfilled if the distance of the points is at least 3 times larger than the standard deviation of their coordinates. It is of no importance at this place, whether the uncertainty can be estimated from the data or is given by the user.

Observe T^2 is the Mahalanobis-distance of a point to the circle through the other three points, and χ_1^2-distributed [Kreyszig 1968].

3.2 Edges of the Convex Hull

An edge belonging to the convex hull is uncertain if the corresponding boundary triangle has a small area. Therefore we test the area

$$A = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \tag{7}$$

to be zero leading to the test statistic

$$T_b = \frac{A}{\sigma_A} \sim N(0, 1) \tag{8}$$

where the standard deviation again can be derived by error propagation. Instead of using the area of the three point we, analogously to the cocircularity test, we also could have tested the imaginary part $\Im(r)$ of the ratio $r = (z_1 - z_3)/(z_2 - z_3)$ to be zero.

We now show the test statistics to be invariant to the numbering of the points.

3.3 The Invariance of the Test Statistic T of Interior Edges

There are $\binom{6}{4} = 24$ cross ratios, which form 6 groups with 4 having the same value, as $c(z_1, z_2, z_3, z_4) = c(z_2, z_1, z_4, z_3) = c(z_4, z_3, z_2, z_1) = c(z_3, z_4, z_1, z_2)$. If c is one value for the cross ratio depending on the first numbering, the other 5 possible values are $c_2 = 1 - c$, $c_3 = 1/c$, $c_4 = 1 - 1/c$, $c_5 = 1/(1 - c)$ and $c_6 = 1 - 1/(1 - c)$ ([Fischer 1985], p. 153 ff.) Thus all possible cross ratios are functionally dependent.

Now, assume we have two stochastic variables x and y, related by $x = f(y)$, where the first two moments are $E(x) = \mu_x$ and $D(x) = \sigma_x^2$. Then by error propagation the first two moments of y are $E(y) = \mu_y = f(\mu_x)$ and $D(y) = \sigma_y^2 = (df/dx)^2 \sigma_x^2$. Now, testing a value of x to be equal to μ_x leads to the optimal test statistic

$$t_x = \frac{x - \mu_x}{\sigma_x} \tag{9}$$

whereas testing value y to be equal to μ_y leads to

$$t_y = \frac{y - \mu_y}{\sigma_y} \tag{10}$$

Obviously $|t_x| = |t_y|$ up to second order terms as

$$|t_y| = \left| \frac{f(x) - f(\mu_x)}{|\frac{df}{dx}|\sigma_x} \right| = \left| \frac{f(\mu_x) + \frac{df}{dx}(x - \mu_x) - f(\mu_x)}{|\frac{df}{dx}|\sigma_x} \right| = \left| \frac{x - \mu_x}{\sigma_x} \right| \tag{11}$$

This holds for arbitrary f.

In our case, the test statistics $t_i^2/\sigma_{t_i}^2$, with $t_i = \Im(c_i)$, for all six cases are identical, thus not only identical up to second order terms, as can be verified e. g. using MAPLE.

This confirms intuition: The test on cocircularity is independent on the numbering.

Example: We demonstrate the invariance of the test statistic T by an example: Given the four equally distant points (0,0), (1,0), (2,0) and (3,0), the cross ratio is 3/4, being a real number, indicating them to be cocircular.

If the fourth point has coordinates $(3, s)$, the cross ratio is

$$c(z_1, z_2, z_3, z_4) = \frac{1}{2} \frac{3 + j\,s}{2 + j\,s} \tag{12}$$

its imaginary part is

$$t = \Im(z_1, z_2, z_3, z_4) = -\frac{1}{2} \frac{s}{4 + s^2} \tag{13}$$

Assuming the points to be uncertain by σ in all coordinates, the variance of t is given by:

$$\sigma_t^2 = \frac{1}{8} \frac{3\,s^4 + 29\,s^2 + 40}{(4 + s^2)^2} \sigma^2 \tag{14}$$

as the Jacobian of t is (using MAPLE)

$$\boldsymbol{a} = \left(\frac{\partial c}{\partial u_i}\right) = \frac{1}{4(4 + s^2)^2} \begin{pmatrix} s(4 + s^2) \\ (2 + s^2)(4 + s^2) \\ -8\,s \\ -2\,(12 + 9\,s^2 + s^4) \\ -s(4 + s^2) \\ (s^2 + 6)(4 + s^2) \\ 8\,s \\ 2\,(-4 + s^2) \end{pmatrix} \tag{15}$$

The test statistic is

$$T(s) = \frac{t}{\sigma_t} = \frac{t}{\sigma\sqrt{\boldsymbol{a}^\mathsf{T}\boldsymbol{a}}} = \frac{s}{\sigma} \frac{-\sqrt{2}}{\sqrt{3\,s^4 + 29\,s^2 + 40}} \tag{16}$$

If we now exchange the second and the third point, thus $\{(0,0), (2,0), (1,0), (3,s)\}$, we obtain the cross ratio

$$c'(z_1, z_2, z_3, z_4') = -\frac{3 + j\,s}{1 + j\,s} \tag{17}$$

which is -3 for $s = 0$,compared to 3/4 before The imaginary part is

$$t' = 2\,\frac{s}{1 + s^2} \tag{18}$$

We obtain the variance of t'

$$\sigma_t'^2 = 2\,\frac{3\,s^4 + 29\,s^2 + 40}{(1 + s^2)^2} \sigma^2 \tag{19}$$

which is different than before exchanging points 2 and 3. We now have – except for the sign – the same test statistic

$$T'(s) = \frac{t'}{\sigma_t'} = \frac{s}{\sigma} \frac{\sqrt{2}}{\sqrt{3\,s^4 + 29\,s^2 + 40}} \tag{20}$$

3.4 The Invariance of the Test Statistic T_b of Edges of the Convex Hull

The test statistic T_b of edges of the convex hull is invariant to the numbering, except for the sign, as the absolute value of the determinant is invariant to the sequence of columns.

4 Fuzzy Delaunay Triangulation

We now want to apply this concept to a complete Delaunay triangulation. We want to achieve a smooth transition between triangulations if a point is moved smoothly. This is not the case in the classical Delaunay triangulation, as a small shift, e. g. of point D in Fig. 1 leads to a large change in the triangulation, namely a change of the diagonal. The key idea is to already include the upcoming edge before the point D actually crosses the circle (ABC) and give both diagonals a significance value between 0 and 1.

We start with the classical Delaunay triangulation. For each edge not being part of the convex hull, we have two neighboring triangles. If the resulting quadrangle is significantly concave, we accept the edge with significance 1. If the quadrangle is not significantly concave we determine the test statistic T of the diagonal and derive a significance number by using a sigmoid function, here the error function $\mathrm{erf}(T)$, yielding a transition between 0 and 1 for test statistics between $-\infty$ and $+\infty$:

$$S(T) = \mathrm{erf}(T) = \int_{x=-\infty}^{T} \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}} dx \tag{21}$$

The significance of the other diagonal is $1 - S$. In case $S < S_0$ is smaller than some significance level, e. g. 0.9, the edge is assumed to be uncertain, and the other diagonal is taken as *also* being part of the *fuzzy triangulation.*

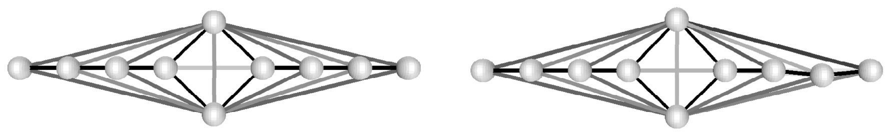
The significance $S(T_b)$ for the boundary edges is calculated similarly.

A few examples demonstrate the concept.

The fuzzy Delaunay triangulation of cocircular points: The different uncertainty of the neighborhoods in fig. 1 is made objective by the corresponding fuzzy Delaunay triangulations in fig. 3. Observe the move of the second left point significantly changes the certainty of the link to the center points.

Changing the Uncertainty: Fig. 4 shows a sequence of fuzzy Delaunay triangulations with increasing uncertainty. Obviously the left triangulation is equal to the classical one except for one quadrilateral with points nearly lying on a circle. Increasing the uncertainty reveals only a few of the original edges to be stable.

Figure 3: *shows the fuzzy Delaunay triangulation of a point set similar to that in fig. 1. Compared to a. the top middle point and the second right point are slightly shipted upwards and downwards resp. in b. Observe the edges between the two middle points and between the second right point and the top middle one to be uncertain in b.. Also observe the uncertainty of the lower right edge of the convex hull in b.*



Moving a Point. Fig. 5 shows a sequence of point sets with the Delaunay triangulation and the Voronoi diagram generated by the sofware package VoronoiGlide ([Icking *et al.* 1996]). The different point sets are generated by moving one point. Observe in Fig. 5b and d the VD contains a vertex with degree 4 indicating the triangulation be uncertain.

Figures 6 and 7 show the fuzzy triangulation of the sequence of point sets with a bit higher resolution. Observe the gradual change of the triangulation. Especially the uncertainty of the triangulation 3 in fig. 6 is visualized by the two diagonals having the same grey value, indicating the certainty to be around 0.5 for both diagonals. Also observe the *smooth* switch between the diagonals between the triangulations 2 and 3.

5 Conclusions

We have presented a method to determine the uncertainty of neighborhood relations of point fields. The uncertainty measure is based on the Voronoi diagram of point sets in the plane or the planar Delaunay triangulation and takes the real or fictitious locational uncertainty of the points into account. The underlying test statistic for interior edges of the triangulation depends on the complex cross ratio and has been shown to be invariant to the numbering. We developed the concept of a fuzzy Delaunay triangulation and gave examples which follow intuition. The concept can be applied in grouping and matching. It may be extended to abstract Voronoi diagrams for including linear or area type features ([Mehlhorn *et al.* 1991]).

Acknowledgments: I thank Andre Braunmandl for the implementation of the fuzzy Delaunay triangulation.

Figure 4: *shows fuzzy Delaunay traingulations with increasing uncertainty. The average point distance is 1, the standard deviations shown are 0.1, 0.3, 0.5, 1.0, 2.0, 3.0.*

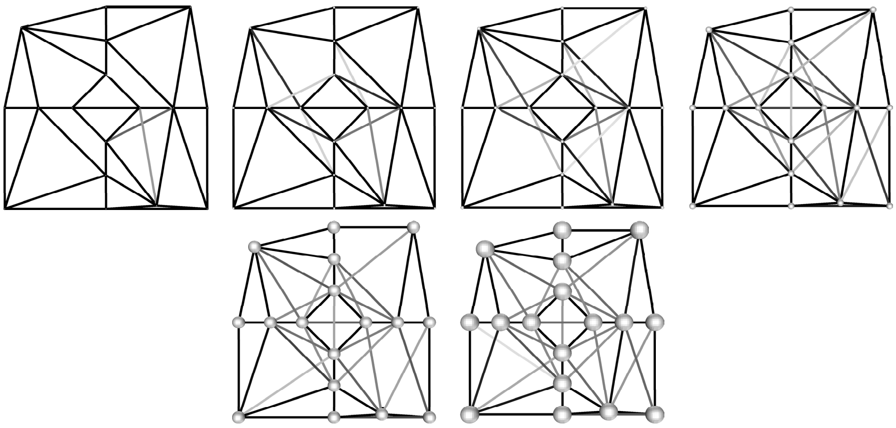


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Figure 5: *shows a sequence of 5 VD together with the DT, caused by moving one point from left to right. In VD b. and d. the VD contains a vertex with 4 edges, indicating the triangulation to be uncertain [Icking* et al. *1996]*

a. *b.* *c.* *d.* *e.*

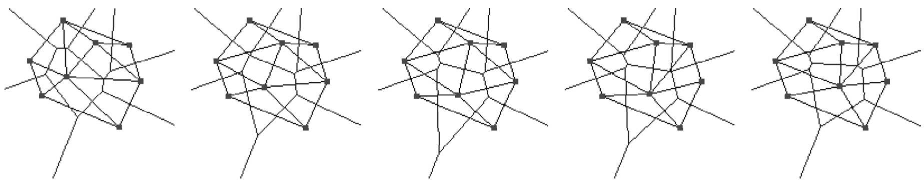

Figure 6: *shows the first part of the sequence of fuzzy Delaunay triangulations (FDT) of fig. 5. In order to increase the resolution, the sequence is calculated for a denser sequence of points. Observe the gradual change of the triangulation.*

a. *b.* *c.* *d.*

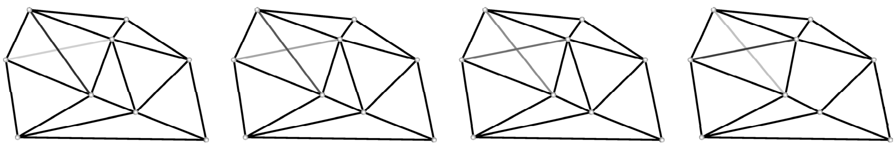

Figure 7: *shows the second part of the sequence of fuzzy Delaunay triangulations (FDT) of fig. 5. Observe the* smooth *switch between the diagonals between the triangulations f and g.*

e. *f.* *g.* *h.*