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Reliability Analysis of Parameter Estimation in Linear Models with Applications to Mensuration Problems in Computer Vision

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The analysis of a mensuration problem aims at an evaluation of the suitability of the design of the measuring process for a specific task and at an assessment of the actually obtained measurements and of their influence onto the result. The concept of quality control, as it has been developed by the Netherlands geodesist W. Baarda is outlined. This theory provides objective quality measures, which take the geometry of the design and the used estimation and testing procedure into account: The evaluation of the design is based on measures for the precision, the controllability, and the robustness, which themselves can be used for planning purposes. The evaluation of the data is based on a statistical test, the estimated size of possible blunders and on the influence of the observed values onto the result. Three examples, namely template matching and absolute and relative orientation of cameras, demonstrate that the measures make intuitive evaluation precise and that they seem to besuitable for automatic quality control of mensuration problems encountered in computer vision.

0. INTRODUCTION

0.1. General

Many mensuration problems in computer vision lead to or can be reduced to linear regression. Examples are template matching, the determination of the parameters of the camera orientation based on a number of given points in object space which have been identified in an image, the estimation of motion parameters of a solid object observed in an image sequence, the approximation of a surface based on irregularly distributed points using a polynomial or a finite element description of a surface or just determining a best fitting straight line through a set of given points. Least squares techniques are standard for solving such tasks. They are simple to apply as they lead to a system of linear equations, use all information contained in the data simultaneously and under quite general assumptions lead to optimal estimates. Moreover, least squares techniques in conjunction with the statistical testing theory provide means for a thorough evaluation of the data and the design.

During the last two decades, however, this technique had to suffer from continuous attacks. In all cases the reason can be brought to one single point: least squares estimates are not robust enough to be practical, i.e., the basic assumptions of least squares techniques are said to be not realistic enough to yield acceptable results. There are in principle two ways out of this situation, namely the use of robust estimation procedures or the use of diagnostic methods based on a geometric and statistical analysis of data and design.

Robust estimators (cf. [14], [17]) have been developed in order to eliminate the effect of blunders or gross errors, systematic errors, neglected correlations, simplified

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models, weak and uncertain assumptions, etc. These effects actually can prevent least squares estimates from being acceptable and—what is even more important—can hide behind small residuals or excellent fits of data and model. Such deficiencies also need not necessarily produce large variances of the estimates. So the analyst is lulled into security. Robust estimators really can help and be efficient if the number of unknown parameters is small compared with the number of observed values and if a second condition holds which is not so easy to prove: none of the observations has to be decisive for a single parameter as it would be not controllable within the system. Such situations may easily occur and more likely do hide in large systems.

Data analysts early became aware of this situation and required robustness of design ([1–5, 7]). Robustness of design, a notion which precisely characterizes the problem, has to do with "leverage points," "spurious observations," "nondetectable gross errors," "nonseparable effects," etc. The message of this school is: Exploit the full information least squares techniques offer together with statistical analysis and try to plan the experiments properly to avoid nonrobust design. The aim here is to document the quality of the least squares estimates by demonstrating that

—statistical tests have picked out blunders, model errors, etc. and

—nondetectable gross errors or systematic errors only have limited effect on the result.

The approach of robust estimation obviously has to be supported by such an analysis of the design of the experiment in order to check the assumptions of the estimation process. Though robust estimators are effective for data cleaning they prevent a proper analysis of the results because the modelling of expected errors, especially outliers, are integrated into the basic assumptions. On the other hand, the approach of the statistical testing theory provides a clear framework for quality control in design, estimation, evaluation, and improvement of experiments.

Now quality control is an inherent problem in all applications of computer vision for mensuration. The geometrical relations between the images and the objects, especially in stereopsy and image sequence analysis are too heterogeneous to allow a simplified analysis, especially if additional information, such as geometrical or dynamic constraints have to be taken into account. Automation needs self-diagnosis (cf. [12, 19]) which in this context can be based on well-established techniques for describing the uncertainty of measurements, their effect onto the result, and for analyzing the suitability of the design for specified tasks.

0.2 Concept of Statistical Models for Mensuration

The aim of measurements is to get information about reality. Based on a mathematical model one develops a certain measurement procedure including the specification of instruments, the design of the network, etc. This measurement process can be interpreted to pose questions to reality. Answers from reality are the observed values, say x. They only have a meaning with respect to the questions posed in the context of the assumed model.

The mathematical model usually contains several unknown parameters, denoted by a vector y, which generally are not directly observable. It is the task of the design

to achieve a measuring process which allows the derivation of these unknown parameters from the observed quantities with a required quality.

The actual measurement process is too complicated to be described in detail. Calibration of instruments tries to guarantee that the observed values are related to the values in the model up to a certain precision. Usually also other parameters, which characterize the conditions during the measuring process are recorded (e.g., temperature) even if they are not directly included in the basic mathematical model. But they are expected to be used in the evaluation phase.

The mathematical model comprises the assumed geometrical and physical relations between the observations and the unknown parameters. Thus the observations, appearing in the model are abstractions of the observed values. It is common to treat the observed values x as realizations of stochastic variables $\mathbf{x}$, which appear in the model. The model is not set up for the observed values x but for (functions of) the observations $\mathbf{x}$, e.g., the expectations $E(\mathbf{x})$ or the variances $V(\mathbf{x}) = \Sigma_{xx}$. For practical reasons the assumed relations between the unknown parameters y and the observations $\mathbf{x}$ are kept as simple as possible and as complex as necessary for the specific task.

In order to be able to check the model we have to describe alternative models, making the expected errors in the model explicit. The checks and tests based on these alternatives thus can also be interpreted as questions posed to reality where the answers of reality, e.g., in the form of test statistics, again have a meaning only within the context of the model and its alternatives.

Altogether the amount of information we obtain from the observed values and their assessment essentially depends on the experience and creativity of the designer and on the ability to translate his ideas into mathematically formulated models for the measuring and evaluation process.

0.3. Outline of Paper

In the following sections we want to show how one can get answers to questions concerning the actual observed values and concerning the quality of the design of the measurement process with respect to these questions. The outlined concept of quality control has been developed by the Netherlands geodist W. Baarda in the 60'th and early 70'th ([1, 2, 4]). We will concentrate on the error detection problem. The concept, however, can also be used for systematic errors, thus for more general deficiencies in the mathematical model.

In detail we will treat the following questions:

1. (a) How can one decide whether a mathematical model has to be rejected in favor of an alternative model? This leads to a statistical test (with a test statistic w_i), which may be used, e.g., for blunder detection.

(b) How large must the difference between the mathematical model and an alternative model be in order to reject the mathematical model? It is shown that this minimum difference (the noncentrality parameter δ_0 of the distribution of w_i under the alternative hypothesis), in general, is much larger than the critical value of the test statistic, if the decision has to be correct with a high probability.

2. (a) How large is the suspected model error? How large is a gross error in a suspicious observation? In general only a certain percentage of a gross error is shown in the residuals of a least squares fit. With this percentage, namely the

redundancy number r_i, which can be derived from the design of the measuring process the size of a gross error can be estimated from the residuals.

(b) What is the minimum size of a gross error to be detectable? This lower bound $(\nabla_0 x_i)$ for gross errors, which are detectable with a certain minimum probability, in general, is much larger than three times the standard deviation of the observations and describes the controllability of the measuring design or the internal reliability according to Baarda.

3. (a) What influence does an observed value have on the result? The effect $(\nabla_i f)$ of an observation (x_i) on the result (f) may be decisive for the elimination of an observation and for the acceptance of the result.

(b) What maximum influence do nondetectable gross or systematic errors have on the result? This maximum influence $(\nabla_{0i} f)$ of undetectable errors on the result (f) describes the sensitivity or the robustness of the design of the measuring process with respect to gross errors or the external reliability according to Baarda. It may be decisive for the optimization or acceptance of the design.

Section 1 discusses the geometry of a design, as answers to questions 2(a) and 3(a) (estimated size of errors, effect of observations) can be given without referring to statistical tests. The statistical test and its power are discussed in Section 2 leading to answers of questions 1 (a, b) (test and noncentrality parameter). The link of geometric and statistical notions in Section 3 then forms the basis for evaluation of the reliability of the design giving answers to questions 2(b) and 3(b) (controllability and sensitivity). Three examples from computer vision, namely template matching, absolute and relative orientation of cameras, provide a demonstration of the usefullness of these concepts.

1. GEOMETRY OF DESIGN

1.1. Precision of Parameter Estimation

Let the linear model

$$E(\mathbf{x}) = Ay \qquad V(\mathbf{x}) = \Sigma = \sigma_0^2 Q \tag{1.1}$$

be given. It relates the n expected values $E(\mathbf{x})$ of the observations $\mathbf{x}$ and the u unknown parameters y in a linear manner. The coefficients (a_{ij}) of the design matrix A are assumed to be known.They could be derived from a linearization of a nonlinear model $E(\mathbf{x}) = f(y)$, A then being the matrix of partial derivatives df_i/dy_j. The matrix A is assumed to have full rank, with $n > u$, thus $\mathrm{rk}(A) = u$. The variance covariance matrix Σ of the observations is assumed to be known except for an unknown variance factor σ_0^2. The weight-coefficient matrix Q is assumed to be regular. For a detailed treatment of this topic, cf. [23, 24].

Starting from observed values $x' = (x_1, x_2, \ldots, x_n)'$ the overconstrained equation system (1.1) can be solved leading to estimates $\hat{y}$ for the unknown parameters (the hat $\hat{}$ stands for estimate):

$$\hat{y} = (A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1}x \tag{1.2a}$$

$$\hat{y} = (A'Q^{-1}A)^{-1}A'Q^{-1}x. \tag{1.2b}$$

If the x are assumed to be realizations of normally distributed stochastic variables $\mathbf{x}$, Eqs. (1.2a) and (1.2b) are the *maximum likelihood estimates* for y. The estimate is independent of the variance factor σ_0^2. $\hat{y}$ also is the *best linear unbiased estimate* (BLUE) for y, where, except for the expectation and the variances, no assumption on the distribution of $\mathbf{x}$ has to be made. Equation (1.1) can also be solved by *least squares* techniques using a weight matrix W of the observations:

$$\hat{y} = (A'WA)^{-1}A'Wx. \tag{1.2c}$$

Obviously the estimates coincide if $W = Q^{-1}$ is chosen.

With $\hat{y}$ we can derive the *fitted values*

$$\hat{x} = A\hat{y} \tag{1.3}$$

which "fit" the model and are estimates for the expectation $E(\mathbf{x})$. The differences

$$\hat{e} = A\hat{y} - x = \hat{x} - x \tag{1.4}$$

are corrections to the observed values or the *residuals*. With them we can obtain an estimate

$$\hat{\sigma}_0^2 = \frac{\hat{e}'Q^{-1}\hat{e}}{r} \tag{1.5}$$

for the *variance factor*, where

$$r = n - u \tag{1.6}$$

is the *redundancy* of the system. If $r = 0$ the observations uniquely determine the parameters y independently of the assumed stochastic properties of the observed values and the residuals are zero.

Now, the *precision* of the estimates $\hat{\mathbf{y}}$, $\hat{\mathbf{x}}$, and $\hat{\mathbf{e}}$ can be derived by error propagation

$$\begin{aligned} V(\hat{\mathbf{y}}) &= \Sigma_{\hat{y}\hat{y}} = (A'\Sigma^{-1}A)^{-1} = \sigma_0^2 Q_{\hat{y}\hat{y}} \\ V(\hat{\mathbf{x}}) &= \Sigma_{\hat{x}\hat{x}} = A\Sigma_{\hat{y}\hat{y}}A' = \sigma_0^2 Q_{\hat{x}\hat{x}} \\ V(\hat{\mathbf{e}}) &= \Sigma_{\hat{e}\hat{e}} = \Sigma - \Sigma_{\hat{x}\hat{x}} = \sigma_0^2 Q_{\hat{e}\hat{e}}. \end{aligned} \tag{1.7a}$$

The standard deviations of the estimates are

$$\begin{aligned} \sigma_{\hat{y}_i} &= \sigma_0\sqrt{q_{\hat{y}_i\hat{y}_i}} = \sqrt{V(\hat{\mathbf{y}}_i)} \\ \sigma_{\hat{x}_i} &= \sigma_0\sqrt{q_{\hat{x}_i\hat{x}_i}} = \sqrt{V(\hat{\mathbf{x}}_i)} \\ \sigma_{\hat{e}_i} &= \sigma_0\sqrt{q_{\hat{e}_i\hat{e}_i}} = \sqrt{V(\hat{\mathbf{e}}_i)}\,. \end{aligned} \tag{1.7b}$$

These measures can be used to evaluate the influence of the inevitable random errors in the observations onto the result. One must keep in mind that the covariance matrices and the standard deviations are properties of the stochastical

variables $\hat{\mathbf{y}}$, $\hat{\mathbf{x}}$, and $\hat{\mathbf{e}}$ in the mathematical model, *not* of the actual estimates $\hat{y}$, $\hat{x}$, and $\hat{e}$. This distinction is made clear in the formulas, not always in the corresponding text.

REMARKS. The linearization of a nonlinear model leads to small errors in the estimates, which in general can be neglected (cf. [25]), especially if the relative accuracy is better than 10%.

EXAMPLES. E1. *Template Matching.* Template matching can be formulated as a least squares estimation problem, also termed the differential approach (cf. [15]). If the one-dimensional and continuous template $g(t)$ and the n observed pixel gray levels ($g_1(t_i)$, $i = 1, \ldots, n$) are given, the nonlinear model reads as $E(\mathbf{g}_1(t_i)) = g(t_i - t)$ with the single unknown shift parameter $y = t$. Linearization at $t_0 = 0$ with the first derivative $\dot{g}'(t) = dg/dt$ leads to the linear model $E(\mathbf{g}_1(t_i)) = g(t_i) - \dot{g}'(t_i) \cdot t$ or $g_1(t_i) + \hat{e}(t_i) = \dot{g}(t_i) - g'(t_i) \cdot \hat{t}$. The design matrix A, here a vector, contains the first derivatives: $A' = (\dot{g}'(t_1), \ldots, \dot{g}'(t_n))'$. Assuming white noise, i.e., uncorrelated, equally distributed random errors in the measured grey values g_1, thus $\Sigma = \sigma_0^2 I$, one obtains the well-known estimate $\hat{t} = [\Sigma \dot{g}'^2(t_i)]^{-1} \cdot [\Sigma(g_1(t_i) - g(t_i)) \cdot \dot{g}'(t_i)]$, which in practical application can be used as an updated approximate value in an iterative solution of the nonlinear problem.

With Eq. (1.7b) and the variance $\sigma_{\dot{g}'}^2 = \Sigma \dot{g}'^2(t_i)/n$ and the signal to noise ratio $\mathrm{SNR} = \sigma_g/\sigma_0$ we can determine the theoretical precision of the estimated shift $\hat{t}$:

$$\sigma_{\hat{t}}^2 = \frac{\sigma_0^2}{\sum_{i=1}^{n} \dot{g}^2(t_i)} = \frac{\sigma_0^2}{n} \cdot \frac{1}{\sigma_{\dot{g}'}^2} = \frac{1}{n} \cdot \frac{1}{\mathrm{SNR}^2} \cdot \frac{\sigma_g^2}{\sigma_{\dot{g}'}^2}.$$

The precision of matching depends on

—the noise level σ_0 of the grey values

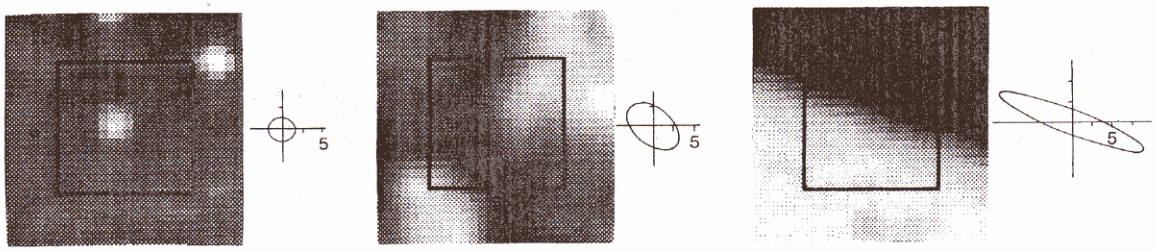
—the number n of pixels or the window size

—the roughness of the texture of the template

which makes an ad hoc evaluation objective. In two-dimensional template matching one obtains a 2×2 covariance matrix for the two unknown shifts or parallaxes which can be visualized by a confidence ellipse (cf. Fig. 1).

One can show that the texture parameter $\sigma_{\dot{g}'}^2/\sigma_g^2$ is proportional to both the effective bandwidth of the signal and the curvature of the autocorrelation function (cf. [20, 11]). The theoretical studies (e.g., [13, 9]), the computer simulations (cf. [16, 21]), and the empirical results (e.g., [6, 22]) consistently prove that subpixel accuracy in object location and parallax determination is reachable.

E2. *Absolute orientation of a single camera.* The orientation of a single camera can be determined from an image of an object if the spatial coordinates of at least 3 object points and the calibration of the camera (focal length, etc.) are given. Figure 2 shows a square shaped block with side length 30 cm and thickness 18 cm which is tilted by 30° with respect to the coordinate system. Assume, an image is taken by a digital camera with a focal length of 15 mm and a pixel size of $\Delta x = 20$ μm from a



(a) (b) (c)

FIG. 1. Three templates and expected precision of matching precision of gray value measurements. σ_x = gray levels: (1) only the gray values within the 16 × 16 window are assumed to be used, pixel size 20 μm; (2) confidence ellipse. The true value of the estimated shifts will lie within the ellipse with a probability of 99%, unit: μm. a. target: small and round error ellipse; b. unsharp natural feature: large ellipse; c. edge: elongated ellipse.

distance of approximately 90 cm. Let us assume that the image coordinates x' and y' of the 7 visible corner points are measured with an accuracy of $\sigma_{x'} = \sigma_{y'} = 3\ \mu\text{m} \approx \frac{1}{7}\ \Delta x$ and the object coordinates (X, Y, Z) of these points are precisely known. The least squares fit of the 2×14 observations for the 6 orientation parameters (shifts X_0, Y_0, Z_0 and rotations ω, φ, κ) then leads to a covariance matrix

$$\Sigma_{yy} = D\begin{bmatrix} \hat{\mathbf{X}}_0 \\ \hat{\mathbf{Y}}_0 \\ \hat{\mathbf{Z}}_0 \\ \hat{\omega}_0 \\ \hat{\varphi}_0 \\ \hat{\kappa}_0 \end{bmatrix} = \sigma_0^2 Q =$$

$$(3\mu\text{m})^2 \times \begin{bmatrix} 43204. & & & & & \\ 0 & 37301. & & & \text{symmetric} & \\ -3957.4 & 0 & 8718.0 & & & \\ 0 & -624.24 & 0 & & & \\ -728.68 & 0 & 61.882 & & & \\ 0 & -72.548 & 0 & & & \end{bmatrix} \quad (r/m) \quad \begin{bmatrix} 10.582 & & \\ 0 & 12.429 & \\ 1.1497 & 0 & 2.6433 \end{bmatrix}$$

$((r \Rightarrow \text{rad}))$

from which one can derive the standard deviations

$$\sigma_{\hat{X}_0} = 0.62 \quad \text{mm}, \sigma_{\hat{Y}_0} = 0.58 \quad \text{mm}, \sigma_{\hat{Z}_0} = 0.28 \text{ mm}$$

$$\sigma_{\hat{\omega}} = 0.037°, \qquad \sigma_{\hat{\varphi}} = 0.040°, \qquad \sigma_{\hat{\kappa}} = 0.018°$$

for the estimated orientation parameters. There are high correlations $r_{ij} = \sigma_{ij}/(\sigma_i \cdot \sigma_j)$ between $\hat{\omega}$ and $\hat{\mathbf{Y}}_0$, namely -0.993, and between φ and $\mathbf{X}_0$, namely -0.994, which indicate that the effects of errors onto the correlated estimates are not separable. This causes high standard deviations for the estimates except for κ and

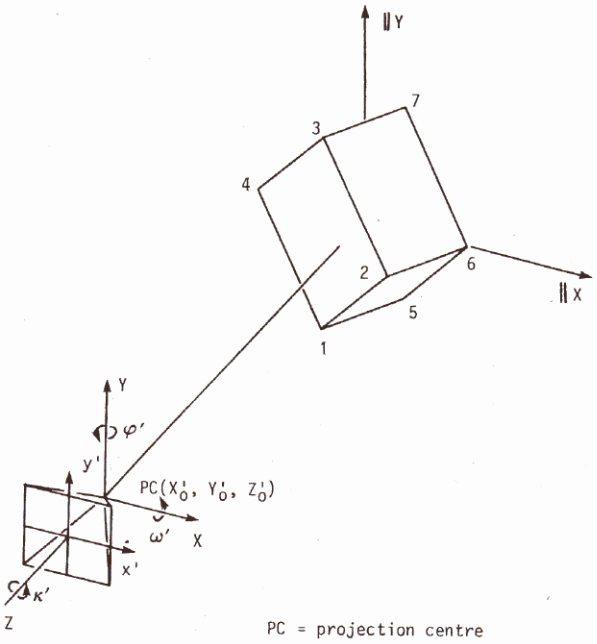


FIG. 2. Absolute orientation of camera in space. Given: object coordinates (X, Y, Z) of 7 points; measured: image coordinates (x', y'); unknown: orientation parameters: (1) shifts X_0', Y_0', Z_0'; (2) rotation angles $\omega', \varphi', \kappa'$.

Z_0. The high correlations also indicate that the numerical solution may be unstable if not performed with a sufficient word length. However, if the determined standard deviations are sufficient for a specified task and the calculations do not introduce disturbing rounding errors, the high correlations do not contaminate the result.

E3. *Relative orientation of two cameras.* If the object is unknown we need at least two images taken from two different positions to be able to reconstruct the form of the object, except for an unknown scale factor. At the same time we are able to determine the relative orientation of the two cameras in space, again except for an unknown scale. Let us assume the position and orientation of the left camera to be known and to be 30 cm left of the origin (cf. Fig. 3). The right is assumed to be 30 cm right to the origin approximately on the x-axis while the orientation may slightly change. The distance between the two camera positions is assumed to be known in order to fix the scale; then we can derive the position Y_0'' and Z_0'' and the orientation angles ω'', φ'', and κ'' of the second camera and the form of the object simultaneously, if the image coordinates (x', y') and (x'', y'') of at least 5 points are measured by a matching procedure. Let us now assume the image coordinates of the 7 front points are measurable with a standard deviation of 3 μm again, then the least squares fit of the $2 \times 2 \times 7 = 28$ image coordinates for the $5 + 7 \times 3 = 36$ unknown orientation parameters and object coordinates lead to a precision of the orientation parameters of

$$\sigma_{\hat{Y}_0''} = 2.9 \text{ mm}, \qquad \sigma_{\hat{Z}_0''} = 12. \text{ mm},$$
$$\sigma_{\hat{\omega}''} = 0.058°, \qquad \sigma_{\hat{\varphi}''} = 0.68°, \qquad \sigma_{\hat{\kappa}''} = 0.042°.$$

The precision of the positions $\hat{Y}_0''$ and $\hat{Z}_0''$ and the rotation angle φ'' around the

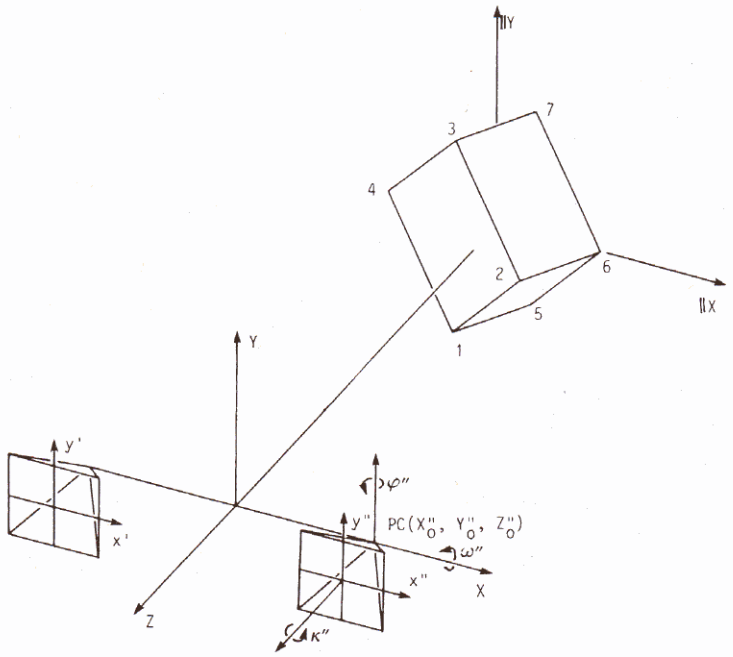


FIG. 3. Relative orientation of two cameras and reconstruction of object. Given: (1) position and orientation of left camera and (2) distance between cameras; measured: image coordinates $(x'\ y')$ and $(x'',\ y'')$ of 7 corresponding points; unknown: (1) orientation of right camera (shifts Y_0'', Z_0'', rotation angles ω'', φ'', κ''); (2) object coordinates (X, Y, Z).

y-axis obviously is much poorer than in the previous example due to the lack of object information. The weakness of this design already is revealed in the low redundancy of $2 = 28 - 26$ for the 26 unknowns, compared to $8 = 14 - 6$ in the previous example. On the other hand, the angles ω'' and κ'' can very well be determined. The reason is, that only the vertical or y-parallaxes determine the orientation parameters, whereas the x-coordinate directly determines the object coordinates, especially the depth. We will make this statement precise later. Again we can observe the same high correlations between the orientation parameters ω'' and Y_0'' and between φ'' and Z_0''.

The analysis of the precision only yields realistic statements if no model errors, especially no gross errors are present. We therefore investigate the influence of blunders on the estimation and on the evaluation procedure.

1.2. Gross Errors and Their Effects

Let us now assume that observational value x_i is erroneous by ∇x_i. (With the sign ∇ all errors in the mathematical model are designated, in contrast to differences Δ. The notation goes back to Baarda and is not to be confused with the operator in physics). In order to use the matrix formulas we introduce the vector $\nabla_i x' = (0, 0, \ldots, \nabla x_i, 0, \ldots, 0)$, containing zeros except for ∇x_i at the ith position. Thus the observational vector x is changed by $\nabla_i x$.

We are now ready to state our basic model and the assumed alternatives precisely. According to statistical terminology we call Eq. (1.1) the *null hypothesis* H_0:

$$H_0\colon E(\mathbf{x}|H_0) = Ay \tag{1.1b}$$

with, in general, several *alternative hypotheses* H_{ai}:

$$H_{ai}: E(\mathbf{x}|H_{ai}) = E(\mathbf{x}|H_0) + \nabla_i x, \qquad \nabla_i x \neq 0 \tag{1.8a}$$

or

$$H_{ai}: E(\mathbf{x}|H_{ai}) = Ay + h_i \nabla x_i, \qquad \nabla x_i \neq 0 \tag{1.8b}$$

with the vector h_i describing the effect of the error source onto the observation. Thus h_i is a unit vector $h_i' = (\ldots, 0, 1, 0, \ldots)'$.

This model for describing gross errors is the most simple case because only the expectation or mean value of the observations is assumed to be influenced by an error. But important practical situations may be described by this type of alternative model, e.g.:

—a wrong correspondence in a matching algorithm for automatic stereo

—a wrongly measured distance between two points

—a single systematic effect (in this case h_i is full).

In all cases one assumes that the measuring accuracy is not influenced by the error and only one parameter is sufficient for describing the effect of the error source onto the observations.

There are, however, important counterexamples which do not fit into this type of alternative:

—Errors during identification of targets or of control points influence 2 or 3 observed values, i.e., coordinates, in the image or in object space

—A movement of the camera during the exposure may lead to systematic errors where both the position and—due to image blur—the precision of the measurement of the points is changed.

—Finally only the variance of an observation may be influenced by an error while leaving the mean unchanged, e.g., if two types of measuring or matching procedures are used where the difference in precision is not known and one has to assume equal standard deviation of both procedures. We will concentrate here on the alternative of the type of Eqs. (1.8) in order to keep the derivations simple. (For generalizations cf. Section 5.)

Let us now investigate the effect of errors in the observations, as described by the alternative hypothesis, onto the different estimates within our procedure. The influence of the error $\nabla_i x$ onto the estimates is

$$\nabla_i \hat{y} = Q_{\hat{y}\hat{y}} A' Q^{-1} \nabla_i x \tag{1.9a}$$

$$\nabla_i \hat{x} = Q_{\hat{x}\hat{x}} Q^{-1} \nabla_i x \tag{1.9b}$$

$$\nabla_i \hat{e} = -Q_{\hat{e}\hat{e}} Q^{-1} \nabla_i x. \tag{1.9c}$$

Figure 4 shows the effect of a $2\frac{1}{2}$ pixel or 50 μm error in the y-coordinate of point 5 of the example in Fig. 3 onto the form of the reconstructed object. All points are

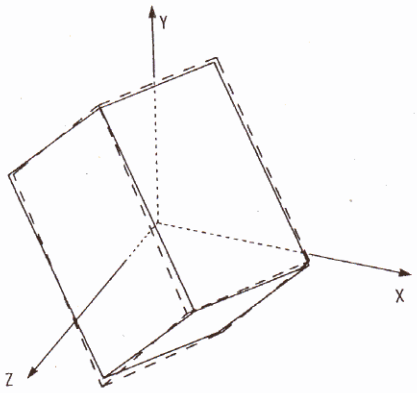


FIG. 4. Effect of a $2\frac{1}{2}$ pixel- or 50 μm-error in the y-coordinate of point 5 onto the form of the reconstructed object. Solid lines: form of object if no gross error is present; dashed lines: form of object if gross error is present. Assuming a precision of the measurements of 3 μm or $\frac{1}{7}$ pixel the 50 μm-error is not detectable.

influenced. As ∇x_i in practical cases is unknown, its influence generally can only be realized if the true coordinates of the new points are known. The same holds for the influence of the gross error onto the fitted values because they are just a linear function of the unknown parameters.

At least, however, the effect $\nabla_i \hat{e}$ of the error ∇x_i onto the residuals $\hat{e}$ is visible, as —presuming the validity of the mathematical model—the expectation

$$E(\hat{\mathbf{e}}) = 0 \tag{1.10}$$

of $\hat{\mathbf{e}}$ is known, in contrast to $E(\hat{\mathbf{y}})$ or $E(\hat{\mathbf{x}})$. We therefore study the effect of an error onto the residuals. It will be shown that we are able to obtain an estimate for the size of a gross error which then can be used to estimate its influence onto the fitted values and the unknown parameters.

1.3. Effect of Observational Errors onto the Residuals

In order to simplify the notation we introduce the matrix

$$R = \Sigma_{\hat{e}\hat{e}}\Sigma^{-1} = Q_{\hat{e}\hat{e}}Q^{-1} = I - A(A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1} \tag{1.11}$$

Then Eq. (1.9c) reads as

$$\nabla_i \hat{e} = -R\,\nabla_i x \quad \text{or} \quad \nabla_i e_j = -r_{ji} \cdot \nabla x_i. \tag{1.12}$$

The index i could be omitted here, as Eq. (1.12) holds for general ∇x also for random errors. R, in general, is a full matrix. Therefore, in general,

(a) an error in one observation influences *all* residuals (columns)

(b) one residual is influenced by *all* observational errors (rows).

(c) the effect $\nabla_i \hat{e}_i$ of an error ∇x_i in observation x_i onto the corresponding residual $\hat{e}_i$ is determined by the ith diagonal element of R.

Thus the matrix R contains the full information on the geometry of the design with

respect to the influence of observational errors onto the residuals. It only depends on the design matrix A and the covariance matrix Σ of the observations, not on the individual observational values. R therefore can be used for planning the design of the measuring process.

The matrix R has a number of useful properties which we can use to advantage for the interpretation in our context. They all result from the fact that R is *idempotent*, i.e., $R^2 = R$:

(a) The matrix

$$U = I - R = Q_{\hat{x}\hat{x}}Q^{-1} = \Sigma_{\hat{x}\hat{x}}\Sigma^{-1} = A(A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1} \tag{1.13}$$

which is used in Eq. (1.11) for the influence the errors ∇x onto the fitted values $\hat{x}$ is also idempotent. Because of $\hat{x} = Ux$ in statistical literature U is often called the hat matrix.

(b) The eigenvalues of R and U are 0 or 1, which can be shown using the eigenvalue decomposition.

(c) Therefore the trace and the rank of R and U are identical

$$\operatorname{tr} R = \operatorname{rk} R \qquad \operatorname{tr} U = \operatorname{rk} U.$$

(d) From

$$\operatorname{tr} U = \operatorname{rk} U = \operatorname{rk}\left(A(A'\Sigma^{-1}A)^{-1}\right)A'\Sigma^{-1}\Big) = \operatorname{rk} A = u \tag{1.14}$$

follows

$$\operatorname{tr} R = \sum_{i}^{n} r_{ii} = \operatorname{tr}(I - (I - R)) = \operatorname{tr} I - \operatorname{tr} U = n - u = r.$$

Thus the diagonal elements r_{ii} of R sum up to the redundancy r of the system showing how r is distributed over the n observations. Analogously the diagonal elements u_{ii} of U sum up to the total number u of the unknown parameters showing in how far each observation contributes to the determination of the unknown parameters. In Section 1.5 we will investigate the effect onto a subvector of $\hat{y}$.

(e) The diagonal elements r_{ii} and u_{ii} lie between 0 and 1:

$$0 \le r_{ii} \le 1 \qquad 0 \le u_{ii} \le 1. \tag{1.16}$$

If the diagonal elements are equal to 0 or 1 then all off-diagonal elements in the corresponding row and column of R *and* U are 0. R and U are symmetric only if $Q \equiv I$.

The diagonal elements of R obviously are decisive for our analysis. As they sum up to the redundancy we call them "*redundancy numbers*" and define

DEFINITION (r_i).

The redundancy number r_i of an observation x_i is defined by

$$r_i \doteq \left(I - A(A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1}\right)_{ii} = \left(\Sigma_{\hat{e}\hat{e}}\Sigma^{-1}\right)_{ii} = \left(Q_{\hat{e}\hat{e}}Q^{-1}\right)_{ii} \tag{1.17}$$

and is the contribution of the observation to the redundancy r of the system with $\Sigma r_i = r$.

Analogously we define

DEFINITION (u_i)

The contribution u_i of the observation x_i to the number u of unknown parameters of the system is defined by

$$u_i \doteq \left(A(A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1}\right)_{ii} = \left(\Sigma_{\hat{x}\hat{x}}\Sigma^{-1}\right)_{ii} = \left(Q_{\hat{x}\hat{x}}Q^{-1}\right)_{ii} \quad (1.18)$$

with $\Sigma u_i = u$.

Thus we obtain the symmetric relations

—for the *number* of all observations, unknowns, and redundant observations

$$n = u + r \quad (1.6b)$$

—for the individual *contributions*

$$1 = u_i + r_i \quad (1.19)$$

—for the idempotent *matrices*

$$I = U + R. \quad (1.20)$$

Thus the average value r for the redundancy numbers r_i is equal to the relative redundancy r/u.

Coming back to the relation between observational errors and the residuals we now obtain the influence $\nabla_i \hat{e}_i$ of an error ∇x_i in x_i onto the corresponding residual $\hat{e}_i$

$$\nabla_i \hat{e}_i = -r_i \cdot \nabla x_i. \quad (1.21a)$$

As r_i always is between 0 and 1 only a—possibly small—part of an error shows in the residual.

EXAMPLES. E1. *Template matching* (continued). Let us assume a one-dimensional edge with a known characteristic is located. Let the $n = 13$ first derivatives be $A' = (0, 0, 0, 0, 10, 30, 60, 30, 10, 0, 0, 0, 0)$. The redundancy numbers can be derived from $r_i = 1 - \dot{g}'^2(t_i)/[\Sigma \dot{g}'^2(t_i)]$, which in our case yields $\{r_i\} = \{1, 1, 1, 1, 0.98, 0.84, 0.36, 0.84, 0.98, 1, 1, 1, 1\}$. The average redundancy $r/n = 12/13 = 0.92$ is misleading, pretending a high controllability. The smallest redundancy number at the steepest point of the edge indicates the weakness of this design. Only a third of an error in the grey level of this pixel is revealed in the residual; the other part, namely 66%, directly influences the determination of the shift parameter. The example also shows that only pixels with a first derivative greater than 0 contribute to the shift determination which of course is to be expected.

E2. *Absolute orientation of a single camera* (continued). The redundancy numbers of the 14 image coordinates for the absolute orientation of a single camera (cf. Fig. 2) are given in the following table:

Pt. #	1	2	3	4	5	6	7
r_{xi}	0.569	0.345	0.569	0.590	0.628	0.652	0.628
r_{yi}	0.617	0.246	0.617	0.680	0.566	0.726	0.566

$$\Sigma r_i = 8.000.$$

The average redundancy $r/n = \frac{8}{14} = 0.57$ is high, the smallest redundancy number in y-coordinate of point 2 being approximately $\frac{1}{4}$ is acceptable. A quarter of an error in this coordinate shows in the corresponding residual.

E3. *Relative orientation of two cameras* (continued). The redundancy numbers for the 14 coordinates of the points in the left one of the two images for relative orientation are given in the following table (the values for the coordinates of the right image are identical):

Pt. #	1	2	3	4	5	6	7
r_{xi}	0	0	0	0	0	0	0
r_{yi}	0.150	0.160	0.150	0.185	0.048	0.259	0.048

As already pointed out the relative redundancy of $r/n = \frac{2}{28} = 0.07$ is not acceptable, which is confirmed by the individual redundancy numbers. The residual $\hat{e}_5$ caused by the 50 μm-error in x_5 would be approximately 3 μm; without knowing the geometry one would not suspect an error. The situation is even worse, as errors in the direction of the base line (x-axis) do not show in the residuals at all, independently on their size. They fully determine the object coordinates, namely the depth (z-coordinate). But also the errors in the direction of the y-axis, thus errors in the vertical parallaxes, hardly show in the residuals. One can expect that gross errors in the image coordinates must be quite large to be detectable, independently on the used test or search strategy.

We will now look at some applications of the geometrical relations established above.

If we have an indication that observation x_i is erroneous then from the residual $\hat{e}_i$ we can obtain an estimate $\hat{\nabla}x_i$ for the size of the error:

$$\hat{\nabla}x_i = -\hat{e}_i/r_i. \tag{1.22}$$

The same estimate would be obtained in the extended model, Eq. (1.8b) of the alternative hypothesis, where we have one additional unknown parameter ∇x_i for the size of the error source, i.e., the gross error. Moreover, this estimate is identical to the difference between the observed value x_i and the predicted value $\hat{x}_i^{(i)}$ obtained from an estimation leaving out x_i or obtained from an estimation where the weight of x_i is set to zero. This is of utmost practical importance as Eq. (1.22)

can be used for all observations simultaneously based on one single solution which is not possible in the other cases. Thus a repetition of estimations leaving individual observations out to estimate their size is not necessary any more.

In order to evaluate the residuals $\hat{e}_i$ and possibly the size of the estimated errors $\hat{\nabla}x_i$ we determine their standard deviations

$$\sigma_{\hat{e}_i} = \sigma_{x_i}\sqrt{r_i} \tag{1.23}$$

$$\sigma_{\hat{\nabla}x_i} = \frac{\sigma_{x_i}}{\sqrt{r_i}} \tag{1.24}$$

Thus the standard deviations of the residuals are always—possibly much—smaller than the standard deviation of the observations, whereas the precision of the estimated size of the errors always is—possibly much—lower than the precision of the observations.

We also can standardize the estimated size $\hat{\nabla}x_i$ of the error relating it to the precision of the observations. The ratio

$$\delta_i' = \hat{\nabla}x_i/\sigma_{x_i}. \tag{1.25a}$$

is a dimensionless quantity and measures the estimated size of an error in x_i in units of the standard deviation of the observation. Its precision is

$$\sigma_{\delta_i'} = 1/\sqrt{r_i}\,. \tag{1.26}$$

Thus a gross error must be at least 25 to 100 times ($\delta_i' > 25$ or 100) larger than the standard deviation of the observation to be estimable with a reasonable precision, e.g., a relative error < 25% to 10%.

The estimated size of the error can now also be written in the form

$$\hat{\nabla}x_i = \delta_i' \cdot \sigma_{x_i}. \tag{1.25b}$$

Equation (1.21a) also can be used to determine the redundancy numbers r_i by computer simulation with a standard software package. By only changing the observation x_i by ∇x_i the redundancy number r_i can be computed from

$$r_i = -\nabla e_i/\nabla x_i, \tag{1.21b}$$

where ∇e_i is the resulting change in the corresponding residual.

EXAMPLES E4. $\hat{e}_i = -1.5$ cm, $r_i = 0.25$, $\sigma_{x_i} = 0.8$ cm
$\rightarrow \sigma_{\hat{e}_i} = 0.8$ cm $\sqrt{0.25} = 0.4$ cm
$\hat{\nabla}x_i = -1.5$ cm$/0.25 = -6$ cm, $\sigma_{\hat{\nabla}x_i} = 1.6$ cm, $\delta'_i = -7.5$, $\sigma_{\delta_i'} = 2$ or:
$\hat{\nabla}x_i = -6 \pm 1.6$ cm, $\delta_i' = -7.5 \pm 2$.

The small residual—compared to the precision of the observation—has a standard deviation only 50% of σ_{x_i} and may result from an error of 6 cm. The precision of this estimate, however, is not very high, the relative precision is $1.6/6 = 2/7.5 = 27\%$.

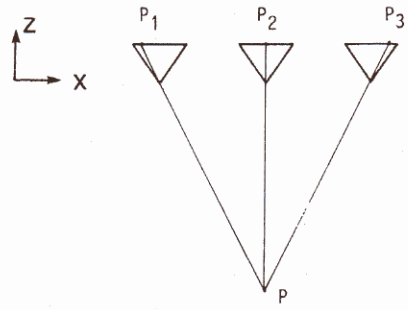


FIG. 5. Intersection of three rays. Given: position and orientation of three cameras projection centres aligned; measured: coordinates of image points P_1, P_2, and P_3 (here only x-coordinates are of concern); unknown: coordinates of object point P. Errors in the x-coordinates are not locatable.

E5. $\hat{e}_i = -16.3$ cm, $r_i = 1/6$, $\sigma_{x_i} = 0.8$ cm
$\rightarrow \sigma_{\hat{e}_i} = 0.33$ cm
$\hat{\nabla} x_i = 97.8$ cm, $\sigma_{\hat{\nabla} x_i} = 0.8$ cm $\sqrt{6} \approx 2$ cm, thus $\hat{\nabla} x_i = 97.8 \pm 2$ cm
$\delta_i' = 97.8$ cm$/0.8$ cm ≈ 122, $\sigma_{\delta_i'} \approx 2.4$, thus $\delta_i' = 122 \pm 2.4$.
The large residual may be caused by an error of approximately 1 m. As the error is quite large, namely $\delta_i' \approx 122$ times larger than σ_{x_i} this estimate is comparably precise and may be a reason to suspect a copying error.

E6. Figure 5 shows the intersection point P of three rays from P_1, P_2, and P_3 as they may occur if a point in space is observed from 3 points lying in a plane. The R-Matrix in this case is

$$R = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

The redundancy numbers are $\frac{1}{6}$, $\frac{2}{3}$, and $\frac{1}{6}$. They sum up to the total redundancy $r = 3 - 2 = 1$ of this system with $n = 3$ observations and $u = 2$ unknown coordinates. R is singular with rk $R = r = 1$. All rows and columns are proportional to each other. Therefore *any* combination of errors of any size leads to a vector $\nabla \hat{e} = R \nabla x$ of residuals with the same ratio $1 : -2 : 1$ between the residuals. An error localization is not possible. The largest residual always belongs to the center ray. The maximum residual can *not* be used for error localization.

Let the residual vector be $\hat{e}' = (-12\ \mu\text{m}, 24\ \mu\text{m}, -12\ \mu\text{m})'$ and the precision of the image coordinates be 10 μm, then the individual estimates for the causing error and their precision would be

$$\hat{\nabla} x_1 = -12\ \mu\text{m}/\left(\tfrac{1}{6}\right) = 72\ \mu\text{m} \pm 24\ \mu\text{m}$$

or

$$\hat{\nabla} x_2 = 24\ \mu\text{m}/\left(\tfrac{2}{3}\right) = -36\ \mu\text{m} \pm 12\ \mu\text{m}$$

or

$$\hat{\nabla} x_3 = -12\ \mu\text{m}/\left(\tfrac{1}{6}\right) = 72\ \mu\text{m} \pm 24\ \mu\text{m}$$

Thus this residual pattern may be caused by either a single error in the border rays of 72 μm, by an error of 36 μm in the center ray or other more complicated error sources. The estimates ∇x_i assume the random errors to cancel out. The accuracy of the estimate with a smaller redundancy number is less than that of the estimate with a larger redundancy number.

We will come back to this example in Section 2.

1.4. Effect of Observational Errors onto the Unknown Parameters

The effect of observational errors comprized in the vector $\nabla_i x$ can be calculated from Eq. (1.9a). There we were not able to use this equation as we had no information on the actual size of the error in the observation. In the previous section we have developed an estimate for just this size and can now use it to derive the influence of an observation onto the estimate $\hat{y}$ of the unknown parameters

$$\hat{\nabla}_i \hat{y} = Q_{\hat{y}\hat{y}} A' Q^{-1} \hat{\nabla}_i x. \tag{1.27}$$

Figure 4 would show such an influence if the 50 μm-error in the coordinate had been estimated from the residual via Eq. (1.22). Thus $\hat{y}$ would change by $\hat{\nabla}_i \hat{y}$ if the observation were eliminated from the estimation process or if its weight were set to 0. Therefore, Eq. (1.27) not only is the influence of the estimated size of an error in x_i onto the estimate $\hat{y}$, but $\hat{\nabla}_i \hat{y}$ can also be termed the influence of the observation onto $\hat{y}$.

In large systems the calculation of the $n \times u$ influence numbers $(Q_{\hat{y}\hat{y}} A' Q^{-1})_{ji}$ is prohibitive. Therefore we want to study this influence in three ways regarding

—the fitted values $\hat{x}$,

—all unknown parameters $\hat{y}$, and

—an arbitrary function $f(\hat{y})$ of the unknown parameters.

(a) Let us first ask for the influence of observation x_i onto its fitted value $\hat{x}_i$. From Eq. (1.11a) one obtains

$$\hat{\nabla}_i \hat{x}_i = u_i \hat{\nabla} x_i = \hat{\nabla} x_i + \hat{e}_i \tag{1.28a}$$

in full analogy to Eq. (1.21). Equation (1.28) clearly shows that from the estimated error one part ($\hat{e}_i$) shows in the residual while the other part ($\hat{\nabla}_i \hat{x}_i$) goes into the fitted value. The standard deviations of the fitted value x_i and of the influence $\hat{\nabla}_i \hat{x}_i$ are

$$\sigma_{\hat{x}_i} = \sigma_{x_i} \sqrt{u_i} \tag{1.29}$$

$$\sigma_{\hat{\nabla}_i \hat{x}_i} = \sigma_{x_i} \sqrt{\frac{u_i}{r_i}} \tag{1.30}$$

respectively again in analogy to Eq. (1.24). We also want to standardize the influence $\hat{\nabla}_i \hat{x}_i$ referring it to the precision of the fitted value. The standardized influence onto the fitted value is

$$\bar{\delta}_i = \frac{\hat{\nabla}_i \hat{x}_i}{\sigma_{\hat{x}_i}} = \frac{u_i \hat{\nabla} x_i}{\sigma_{x_i} \sqrt{u_i}} \tag{1.31a}$$

$$\bar{\delta}_i = \delta_i' \cdot \sqrt{u_i}\,. \tag{1.31b}$$

Thus the influence of an observation x_i onto its fitted value is equal to $\bar{\delta}_i$ times the standard deviation of the fitted value. As $u_i \leq 1$ the influence factor $\bar{\delta}_i$ is always smaller than the standardized size δ_i' of the estimated error.

(b) We now want to investigate the total influence of the error $\hat{\nabla} x_i$ onto the estimated parameters $\hat{y}$. In order to arrive at a scalar measure we determine the length of the vector $\hat{\nabla}_i \hat{y}$, weighted with the normal equation matrix $(\Sigma_{\hat{y}\hat{y}})^{-1}$. We obtain

$$\|\hat{\nabla}_i \hat{y}\| = \sqrt{\hat{\nabla}_i \hat{y}' \left(\Sigma_{\hat{y}\hat{y}}\right)^{-1} \hat{\nabla}_i \hat{y}} \tag{1.32}$$

If we now assume

$$\Sigma = \mathrm{Diag}\left(\sigma_{x_i}^2\right) \tag{1.33}$$

the observations to be uncorrelated, which is common practice, then one can show that

$$\bar{\delta}_i = \|\hat{\nabla}_i \hat{y}\|. \tag{1.31c}$$

Thus the normalized length of the influence vector $\hat{\nabla}_i \hat{y}$ can be derived from Eq. (1.31b) using (1.25) and (1.22) if the observations are uncorrelated. This is a numerical advantage, as only the residual $\hat{e}_i$, the redundancy number r_i and the standard deviation σ_{x_i} are necessary:

$$\bar{\delta}_i = -\frac{\hat{e}_i}{\sigma_{x_i} \cdot r_i} \sqrt{1 - r_i}\,. \tag{1.31d}$$

(c) As $\bar{\delta}_i$ still seems to be an abstract measure we will now analyse the influence of the observation x_i onto an arbitrary function $f = g' \hat{y}$ of the unknown parameters, e.g., onto a distance, a direction, or an area derived from estimated coordinates. Thus we want to determine

$$\hat{\nabla}_i f = g' \hat{\nabla}_i \hat{y} = g' \Sigma_{\hat{y}\hat{y}} A' \Sigma^{-1} \hat{\nabla}_i x. \tag{1.34}$$

As we assume uncorrelated observations and only one error $\hat{\nabla} x_i$ in x_i we can simplify Eq. (1.34) to

$$\hat{\nabla}_i f = g' \Sigma_{\hat{y}\hat{y}} a_i \cdot \hat{\nabla} x_i / \sigma_{x_i}^2, \tag{1.35}$$

where a_i' is the ith row in the design matrix A.

Using Schwarz's inequality

$$u'Vw \leq \sqrt{u'Vu}\sqrt{w'Vw}$$

we obtain

$$\hat{\nabla}_i f \leq \sqrt{g'\Sigma_{\hat{y}\hat{y}}g}\sqrt{a_i'\Sigma_{\hat{y}\hat{y}}a_i}\,\hat{\nabla}x_i/\sigma_{x_i}^2$$

which, using $\hat{\nabla}x_i/\sigma_{x_i} = \delta_i'$, $\sqrt{a_i'\Sigma_{\hat{y}\hat{y}}a_i} = \sigma_{\hat{x}_i} = \sigma_{x_i}\sqrt{u_i}$, and $\sqrt{g'\Sigma_{\hat{y}\hat{y}}g} = \sigma_f$, reduces to

$$\hat{\nabla}_i f \leq \bar{\delta}_i \cdot \sigma_f. \tag{1.36}$$

This is the main formula for evaluating the effect of errors onto the result of an estimation procedure. It can be termed the *empirical sensitivity* of the result. Equation (1.36) states that the effect of errors $\hat{\nabla}_i x$ in the observations onto an arbitrary function f is smaller than $\bar{\delta}_i$ times the standard deviation of that function. Thus if the values $\bar{\delta}_i$ are small for all observations we need not expect large deteriorations of the result due to observational errors. If one eliminates the observation x_i from the data then a value $f(\hat{y})$ derived from the estimates $\hat{y}$ does change f not more than $\bar{\delta}_i \cdot \sigma_f$. As these statements refer to the standard deviation of f, the design of the measurement process has to guarantee acceptable precision for all functions of the estimates which are of interest.

The inequality sign in Eq. (1.36) reflects the fact that a function may not be influenced at all by an observation. The equal sign holds for the fitted values $\hat{x}_i = a_i'\,\hat{y}$ as has been shown above (cf. Eq. (1.31)), here $\hat{\nabla}_i\hat{x}_i = \bar{\delta}_i\,\sigma_{\hat{x}_i}$ is valid.

The next section discusses the case where one is interested only in the influence onto a subset $\hat{k} = \hat{y}_1$ of the unknown parameters, e.g., the coordinates. The formulas essentially remain the same except that the influence value u_i has to be replaced by $\bar{u}_{k_i}$. A first reading could omit this subsection and directly skip to the example before Section 1.6.

1.5. Effect of Observational Errors onto a Subset of the Parameters

Sometimes one wants to determine the influence of observations only onto a subset of the unknown parameters, e.g., the coordinates. This influence should, in general, be smaller as a part of the errors is absorbed by other—nuisance—parameters, e.g., orientation parameters or parameters compensating for systematic errors. In order to achieve a simple relation for this influence we reduce the original model with all parameters onto a model with only the subset of parameters in concern.

Let the unknown parameters consist of two parts $y' = (t'\ k')$ representing u_t transformation parameters and u_k coordinates with $u = u_t + u_k$, e.g., in a image sequence, where one wants to simultaneously estimate the orientation parameters of the camera positions and the object coordinates of common points. Let us assume that we are only interested in the determination of the coordinates. Thus we treat the orientation parameters as nuisance parameters (Of course, the situation may be vice versa, e.g., in navigation problems. Then the role of t and k are to be interchanged.). The design matrix A correspondingly is split into $A = (B\ C)$. The

model Eq. (1.1) then reads as

$$E(\mathbf{x}) = Ay = Bt + Ck \qquad V(\mathbf{x}) = \Sigma = \sigma_0^2 Q. \tag{1.37}$$

The normal equations are

$$\begin{aligned} N\hat{y} - h &= \begin{bmatrix} N_{tt} & N_{tk} \\ N_{kt} & N_{kk} \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{k} \end{bmatrix} - \begin{bmatrix} h_t \\ h_k \end{bmatrix} \\ &= \begin{bmatrix} B'\Sigma^{-1}B & B'\Sigma^{-1}C \\ C'\Sigma^{-1}B & C'\Sigma^{-1}C \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{k} \end{bmatrix} \begin{bmatrix} B'\Sigma^{-1}x \\ C'\Sigma^{-1}x \end{bmatrix} \\ &= (A'\Sigma^{-1}A)\hat{y} - A'\Sigma^{-1}x = 0. \end{aligned} \tag{1.38}$$

If we reduce this system onto the parameters k by eliminating t we obtain the reduced system

$$N_k\hat{k} - h_k = \left(N_{kk} - N_{kt}(N_{tt})^{-1}N_{tk}\right)\hat{k} - \left(h_k - N_{kt}(N_{tt})^{-1}h_t\right) = 0 \tag{1.39}$$

By matrix multiplication one can show that with the reduced design matrix

$$\bar{C} = C - B(N_{tt})^{-1}N_{tk} = \left(I - B(B'\Sigma^{-1}B)^{-1}B'\Sigma^{-1}\right)C \tag{1.40}$$

this reduced system is equal to

$$(\bar{C}'\Sigma^{-1}\bar{C})\hat{k} - \bar{C}'\Sigma^{-1}x = 0. \tag{1.41}$$

This system is totally analogous to the original system Eq. (1.38) if we substitute the matrix A by $\bar{C}$ and the dimension u by $\bar{u}_k$.

The line of thought for deriving the influence of an observation onto all unknown parameters can thus directly be applied to the reduced system. Then instead of u_i in Eq. (1.18) one has to use $\bar{u}_{k_i}$:

DEFINITION ($\bar{u}_{k_i}$).

The contribution $\bar{u}_{k_i}$ of x_i to the determination of the subset k of the unknown parameters is defined by

$$\bar{u}_{k_i} \doteq \left(\bar{U}_k\right)_{ii} = \left(\bar{C}(\bar{C}'\Sigma^{-1}\bar{C})^{-1}\bar{C}'\Sigma^{-1}\right)_{ii} \tag{1.42}$$

with $\sum_1^n \bar{u}_{k_i} = u_k$.

We finally obtain the relation $\bar{\delta}_i = \delta_i' \sqrt{\bar{u}_{k_i}}$ for the influence of observation x_i onto the subset $\hat{k}$ of the parameters $\hat{y}$, which replaces Eq. (1.31b).

In order to visualize that $\bar{u}_{k_i} \leq u_i$, i.e., that the influence onto a subset of the unknown parameters is smaller than onto all parameters, we determine the difference $u_i - \bar{u}_{k_i}$. Using Eq. (1.40) we obtain the remaining contribution $u_{t_i} = u_k -$

$\bar{u}_{k_i}$ of the observations to the determination of the rest, t, of the parameters with

DEFINITION (u_{t_i}).

The remaining contribution u_{t_i} of x_i to the determination of the subvector t of the parameters is defined by

$$u_{t_i} \doteq (U_t)_{ii} = \left(B\left(B'\Sigma^{-1}B\right)^{-1}B'\Sigma^{-1}\right)_{ii} \tag{1.43}$$

with $\sum_1^n u_{t_i} = u_t$.

If no transformation parameters t are to be determined then $u_{t_i} = 0$. If no coordinates k are to be determined then $u_{t_i} = u_i$. Altogether this leads to the following analogous equations:

—for the number of observations, transformation parameters, coordinates, and redundant observations

$$n = u_t + u_k + r \tag{1.44}$$

—for the individual contributions of the n observations

$$1 = u_{t_i} + \bar{u}_{k_i} + r_i \tag{1.45a}$$

—and for the idempotent matrices

$$I = U_t + \bar{U}_k + U_r \tag{1.46}$$

(cf. the discussion in Section 1.3).

The components u_{t_i}, $\bar{u}_{k_i}$, and r_i completely show how the information of each observation is distributed on the two groups of unknown parameters and the residuals.

By reordering Eq. (1.45) we can arrive at a different interpretation of the contribution number $\bar{u}_{k_i}$. If we would assume the coordinates $\hat{k}$ to be fixed, i.e., to be known, we would obtain different redundancy numbers, namely $r_i' = 1 - u_{t_i}$, which always are larger than the redundancy numbers r_i. This is because the controllability of the observations is increased by fixing the coordinates. The loss in local redundancy

$$r_i' - r_i = \bar{u}_{k_i} \tag{1.45b}$$

in the complete model equation (1.37) is just the contribution of the observation which is needed for determining the coordinates.

The final formula for the empirical sensitivity factor can now be given

$$\begin{aligned}\bar{\delta}_i &= -\frac{\hat{e}_i}{\sigma_{x_i} \cdot r_i}\sqrt{1 - r_i - \bar{u}_{t_i}} \\ &= \delta_i' \cdot \sqrt{1 - r_i - \bar{u}_{t_i}} = \delta_i' \cdot \sqrt{\bar{u}_{k_i}} = \delta_i' \cdot \sqrt{r_i' - r_i}\,.\end{aligned} \tag{1.47}$$

Together with Eq. (1.37) the effect of each observation onto the result can be calculated and be used for the evaluation of the result with respect to observational errors.

EXAMPLES. E4 (continued). $\hat{e}_i = 1.5$ cm, $r_i = 0.25$, $\sigma_{x_i} = 0.8$ cm

$$\left(\hat{\nabla}x_i = -6 \text{ cm}, \delta_i' = 7.5\right)$$

assuming $u_{t_i} = 0$ follows:

$$\rightarrow u_i = 0.75, \hat{\nabla}_i\hat{x}_i = -6 \text{ cm} \cdot 0.75 = -4.5 \text{ cm}$$

$$\sigma_{\hat{x}_i} = 0.8 \cdot \sqrt{0.75} = 0.7 \text{ cm}$$

$$\bar{\delta}_i = \delta_i' \cdot \sqrt{u_i} = \hat{\nabla}_i\hat{x}_i/\sigma_{\hat{x}_i} = 6.5.$$

The contribution u_i of x_i to the determination of the unknown parameters is 0.75, i.e., $\frac{3}{4}$ (4.5 cm) of the error (6 cm) in x_i go into the fitted value. This is $\bar{\delta}_i = 6.5$ times its standard deviation, thus quite a bit larger than random errors only would influence it and what would not be expected from the small residual of 1.5 cm.

E5 (continued). $\hat{e}_i = -16.3$ cm, $r_i = \frac{1}{6}$, $\sigma_{x_i} = 0.8$ cm

$$\left(\hat{\nabla}x_i = 97.8 \text{ cm}, \delta_i' = 122\right)$$

$$\rightarrow u_i = \tfrac{5}{6}, \hat{\nabla}_i\hat{x}_i = 97.8 \text{ cm} - 16.3 \text{ cm} = 81.5 \text{ cm}$$

$$\sigma_{\hat{x}_i} = 0.8 \text{ cm} \cdot \sqrt{5/6} = 0.73 \text{ cm}$$

$$\bar{\delta}_i = 81.5 \text{ cm}/0.73 \text{ cm} = 112.$$

Here $\frac{5}{6}$ of the gross error goes into the fitted value. If x_i would be a distance with a 1 m-error, the corresponding points would be shifted in a way that their distance calculated from the estimated coordinates would be wrong by approximately 83 cm, which is more than 112 times the standard deviation of this distance. Of course there is no difficulty in finding this error and then to eliminate or—even better—remeasuring the observation.

1.6. Summary of Section 1

The result of this section is twofold:

(1) We have developed objective measures to evaluate the design of a measuring procedure based on least squares estimates. The redundancy numbers r_i (Eq. (1.17)) and the contribution numbers $\bar{u}_{k_i}$ (Eq. (1.42)) give conspicuous insight in how far errors in an observation show up in the corresponding residual and in how far it may deteriorate the result. These values only depend on the design matrix A and on the covariance matrix Σ or the weight coefficient matrix Q alone, not on the actual observations. The design therefore can be planned to maximize the visible

part of errors and to minimize their effect or—combining both criteria—to optimize

$$\frac{\bar{u}_{k_i}}{r_i} \rightarrow \min.$$

We will come back to this ratio in Section 3. It is the decisive factor for the external reliability according to Baarda, measuring the *sensitivity* or the *robustness* of the design.

(2) The evaluation of an actually observed value can be based on the estimated size of a possible gross error in this observation and on its effect onto the result. The estimated size $\hat{\nabla} x_i$ (Eq. (1.22)) of a gross error is a valuable help for the diagnosis of the error source. The precision of this estimate is high enough to yield a reasonable relative accuracy for not too small gross errors (> 20 the measuring precision). In case one is not certain that x_i is erroneous its effect onto the result may be decisive. The empirical sensitivity of the result is comprised in the values $\bar{\delta}_i$ (Eq. (1.47)). They give the maximum distortion of any function of a predefined subset $\hat{k}$ of the unknown parameters $\hat{y}$ in units of the standard deviation of that function (Eq. (1.37)). Values $\bar{\delta}_i < 1$ or 2 guarantee the estimated coordinates to be not more distorted than 1 or 2 times their standard deviation.

Though these measures are useful for assessing the observations we up to now have no procedure for locating gross errors or even means for deciding whether an observed value is wrong or not. This can be achieved with statistical tests which are provided in the next section.

2. HYPOTHESIS TESTING FOR BLUNDER DETECTION

This section provides means for deciding whether an observation can be regarded as being erroneous. The statistical test takes into account the local geometry and is optimal. The power of the test, i.e., the ability to separate the null and the alternative hypothesis can be used to evaluate the decision which is based on the statistical test and can also be used to tell how large the difference between the hypotheses at least must be that erroneous decisions are made only with a small probability. We will apply these measures for evaluating the controllability of the observations and the sensitivity of the result with respect to undetectable errors in Sections 3.

2.1. Statistical Test

Statistical tests want to answer the question whether a given null-hypothesis has to be rejected in favor of a given alternative hypothesis. In our context a statistical test must tell whether the residuals give an indication that a certain prespecified observation is erroneous or not.

Thus statistical tests are not developed for blunder localization though they are, however, used for that purpose by taking the maximum test statistic of several alternatives as an indicator for the erroneous observation, i.e., the most likely alternative. This theoretically can be justified. We will concentrate on the more simple problem of assessing a single observation with a statistical test.

The test is based on a test statistic w_i which is a function of the observed values x. The probability density function, specifically the expectation and the variance of the test statistic, has to be known for the case where the null hypothesis H_0 is true.

Good test statistics use all information, i.e., are sufficient and lead to the best distinction between the null and alternative hypothesis, i.e., are efficient. The best test statistic for blunder detection is

$$w_i = \frac{-h_i' \Sigma^{-1} \hat{e}}{\sqrt{h_i' \Sigma^{-1} \Sigma_{\hat{e}\hat{e}} \Sigma^{-1} h_i}} \tag{2.1}$$

which for uncorrelated observations reduces to the *standardized residual*

$$\mathbf{w}_i = \frac{-\hat{\mathbf{e}}_i}{\sigma_{\hat{e}_i}} = \frac{\hat{\nabla}\mathbf{x}_i}{\sigma_{\hat{\nabla}x_i}} = \frac{-\hat{\mathbf{e}}_i}{\sigma_{x_i}\sqrt{r_i}} \sim N(0,1) \tag{2.2}$$

With w_i we also test the estimated size $\hat{\nabla}x_i$ of the observation and via r_i take the geometry into account. Provided the observations are normally distributed $\mathbf{w}_i$ follows a standard normal distribution $N(0, 1)$ if H_0 holds. Even if the observations are not exactly normally distributed, the residuals, because of the central limit theorem, are nearly normally distributed.

The test now consists in comparing the (absolute value of the) test statistic with a *critical value* k which in principle can be freely chosen, e.g., $k = 3$. The situation is shown in Figure 6 for the case that no error is present. The region $(-k, +k)$ is the *acceptance region* A, which is symmetric with respect to 0 because we do assume positive and negative blunders to occur equally likely. If $w_i \in A$ then we have no reason for rejecting H_0 and we accept H_0, i.e., assume the observation to be error-free. The region outside A is the *rejection region* R. If $w_i \in R$ then we reject H_0 in favor of the alternative hypothesis H_a, i.e., we assume the observation x_i to be erroneous.

EXAMPLES. E7. $\hat{e}_i = 1$ cm, $r_i = 0.3$, $\sigma_{x_i} = 1$ cm $\rightarrow |w_i| = 1.83 < 3 = k$ or

$$\hat{e}_i < 3 \cdot \sigma_{\hat{e}_i} = 3 \cdot \sigma_{x_i}\sqrt{r_i} = 1.64 \text{ cm}.$$

The observation is accepted.

E8. $\hat{e}_i = 5\ \mu$m, $r_i = 0.09$, $\sigma_{x_i} = 5\ \mu$m $\rightarrow |w_i| = 3.33 > 3 = k$ or

$$\hat{e}_i > 3 \cdot \sigma_{\hat{e}_i} = 4.5\ \mu\text{m}.$$

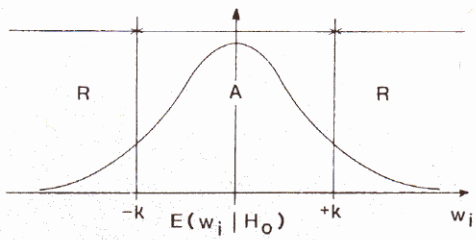


FIG. 6. Acceptance and rejection region for statistical test with normally distributed test statistic w_i for blunder detection; positive and negative blunders are assumed to be equally likely.

The observation will be rejected. The critical value of 4.5 μm for the residual $\hat{e}_i$ is smaller than the standard deviation σ_{x_i} of the observation!

E6 (continued). $\hat{e}_1 = \hat{e}_3 = -12\ \mu\text{m}$, $\hat{e}_2 = 24\ \mu\text{m}$

$$r_1 = r_3 = \tfrac{1}{6}, \qquad r_2 = \tfrac{2}{3}$$
$$\rightarrow w_1 = w_2 = w_3 = 2.93 < 3 = k.$$

The statistical test suggests accepting the observations. The decision, however, is doubtful, as the standardized residual is only slightly smaller than the critical value. If we would choose $k = 2.5$ then all three observations are suggested to be erroneous. But because the test statistics are identical a localization is not possible, which truely reflects reality. If we are not certain whether to accept or reject the observations we can look at the influence factors $\bar{\delta}_{0i}$ which are 6.6, 2.1, and 6.6. If one of the observations x_1 or x_3 are erroneous the coordinates may be distorted up to 6.6 times their standard deviation, which seems to be too large. Thus one should reject this point!

2.2. Evaluation of the Test

The evaluation of the decision which is based on the statistical test depends on

(1) whether the test statistic belongs to the acceptance region or to the rejection region;

(2) whether the observation is erroneous or not.

In order to investigate the decision we determine the influence of an error ∇x_i in x_i onto the test statistic w_i. This influence is

$$\delta_i = \nabla w_i = \frac{-\nabla \hat{e}_i}{\sigma_{\hat{e}_i}} = \frac{-\nabla \hat{e}_i}{\sigma_{x_i}\sqrt{r_i}} = \frac{\nabla x_i}{\sigma_{x_i}}\sqrt{r_i} \tag{2.3}$$

and leads to a shift of the probability density function of w_i by δ_i. Thus we have

$$\begin{aligned} w_i|H_0 &\sim N(0,1) \\ w_i|H_{a_i} &\sim N(\delta_i,1). \end{aligned} \tag{2.4}$$

From Eq. (2.3) we can conclude that the weaker the geometry or the smaller the accuracy the smaller is the influence of a blunder onto the test statistic w_i, thus the more difficult it will be to discern between H_0 and H_a. $N(\delta_i,1)$ is the *noncentral normal distribution* with *noncentrality parameter* δ_i.

Figure 7 shows the density function of the test statistic w_i for three hypotheses:

$\mathbf{H}_0$ (null hypothesis). The observation x_i is error-free. The expectation of the test statistic is

$$E(\mathbf{w}_i|H_0) = 0.$$

$\mathbf{H}_{a_1}$ (1st alternative hypothesis). The observation x_i is erroneous by $\nabla_1 x_i$ which leads to a shift of the density function of $\mathbf{w}_i$ by

$$E(\mathbf{w}_i|H_{a_1}) = \delta_1.$$

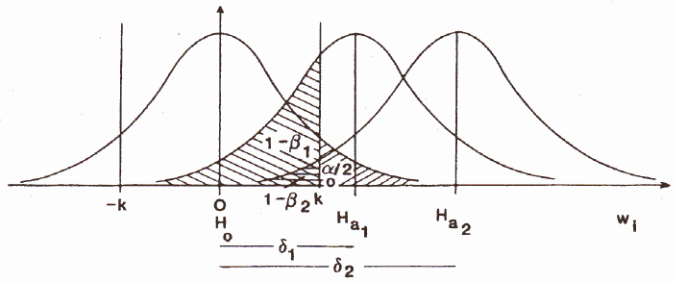


FIG. 7. Probability density function of the test statistic w_i for three hypotheses, significance number α_0. Left: null hypothesis, no gross error; middle: 1st alternative, small gross error, noncentrality parameter δ_1, power β_1; probability for type II error $1 - \beta_1$; right: 2nd alternative, larger gross error, noncentrality parameter $\delta_2 > \delta_1$, power $\beta_2 > \beta_1$ probability for type II error $1 - \beta_2 < 1 - \beta_1$.

$\mathbf{H}_{a_2}$ (2nd alternative hypothesis). The observation x_i has a larger blunder of size $\nabla_2 x_i$ which leads to a shift of the density function of w_i by

$$E(\mathbf{w}_i|H_{a_2}) = \delta_2.$$

The probability of correctly accepting the observation, i.e., the probability that w_i lies in the range between $-k$ and $+k$ and H_0 is true, is the area below the leftmost curve between $-k$ and $+k$. It is the *significance level* S_0 *of the test*

$$S_0 = P(\mathbf{w}_i \in A|H_0) = 1 - \alpha_0. \tag{2.5}$$

From k one can derive S_0 and vice versa using the distribution function $\phi(z)$ of w_i, that is, the integral of the density function of $\mathbf{w}_i$,

$$\phi(z) = P(\mathbf{w}_i < z) \tag{2.6}$$

which can be found in statistical tables. $\phi(z)$ is monotonously increasing between 0 and 1 with the point $\phi(0) = 0.5$ as center of symmetry.

The complement to 1 of the significance level is the *significance number* α_0. We obtain

$$S_0 = 2\phi(k) - 1 \qquad \text{and} \qquad \alpha_0 = 2\{1 - \phi(k)\} \tag{2.7a}$$

or

$$k = \phi^{-1}\{(1 + S_0)/2\} \qquad \text{and} \qquad k = \phi^{-1}(1 - \alpha_0/2). \tag{2.7b}$$

The probability of correctly rejecting the observation, i.e., the probability that $\mathbf{w}_i$ lies in the rejection region when H_{a_p} $(p = 1, 2, \ldots)$ is true, is the *power* β_p *of the test with respect to the alternative* H_{a_p}

$$\beta_p = P(\mathbf{w}_i \in R|H_{a_p}). \tag{2.8}$$

Thus the significance level is the probability of correctly accepting the observation while the power of the test is the probability of *correctly rejecting* the observation.

TABLE 1
Decisions during Testing

	$\lvert w_i \rvert < k$	$\lvert w_i \rvert > k$
H_0 true	Accept H_0 Significance level S_0 Correct decision	Reject H_0 Significance number α_0 Type I error
H_{a_p} true	Accept H_0 $1 - \beta_p$ Type II error	Reject H_0 Power β_p Correct decision

As the size of the residuals and therefore also the test statistic are influenced not only by the assumed gross but also by random errors there are four possible decisions. They are shown in Table 1 together with their probabilities.

Besides the two correct decisions mentioned above we have two types of erroneous decisions:

Type I Error. Though H_0 is true, i.e., the observation is error-free, one may conclude from $|w_i| > k$ that the observation contains an error. The probability for this Type I error is the significance number α_0. As with the rejection of H_0 the superfluous search for the error sources starts, the value α_0 also is called the producer's risk and is usually chosen small, e.g., 0.01 or 0.001, which corresponds to critical values $k = 2.58$ or 3.29, respectively.

Type II Error. Though H_{a_p} is true, i.e., the observation is erroneous one may conclude from $|w_i| < k$ that the observation is correct. The probability $1 - \beta_p$ of this Type II error depends not only on the critical value k but also on the size of the gross error, or, generally speaking, on the distance of the alternative hypothesis from the null hypothesis. This can clearly be seen from the two right curves in Fig. 7. The complement of the power β_p, the value $1 - \beta_p$ also is called the consumer's risk. As α_0 it should be small.

The probability β_p of detecting an error which leads to δ_p can be calculated from

$$\begin{aligned}\beta_p &= P\left(\mathbf{w}_i > k | H_{a_p}\right) + P\left(\mathbf{w}_k < -k | H_{a_p}\right) \\ &= \phi\left(\delta_p - k\right) + \left[1 - \phi\left(\delta_p + k\right)\right]. \end{aligned} \tag{2.9a}$$

The first term corresponds to the area below the second or third curve in Fig. 7 for values $w_i > k$. The second term is usually very small if $\delta_p > 0$ and corresponds to the area below the second or third curve for values $w_i < -k$.

If δ_p is larger than k then the term in brackets can be neglected and we obtain

$$\beta_p \approx \phi\left(\delta_p - k\right). \tag{2.9b}$$

If we replace δ_p by $|w_i|$ in this formula we can derive the probability with which we are able to detect an error of the estimated size in the actually tested observation.

TABLE 2
Significance Number α and Critical Value k

k	α (%)
1.96	5
2.58	1
3.0	0.27
3.29	0.1

TABLE 3
Significance Level $1 - \alpha$ and Power β for Given $\delta = 4$

$1 - \alpha$ (%)	k	β (%)
99.9	3.29	76
99.7	3.0	84
99.0	2.58	93
95.0	1.96	98

The following three examples show the relative dependency of the values k, α, β, and δ and demonstrate that they reflect common sense.

(a) The probability α of making a Type I error, i.e., of erroneously rejecting a good observation becomes smaller with increasing critical value k (see Table 2).

(b) The ability to find an error is larger if the critical value is chosen smaller (see Table 3).

(c) The power β_p of the test, i.e., the probability of finding a gross error increases with its size represented by δ_p. On the other hand the probability $1 - \beta_p$ of not finding the error decreases for larger errors. The *power function* $\beta(\delta)$ is shown in Fig. 8 (also see Table 4).

Specifically, errors which have the size of the critical value k can only be detected with a probability of 50%.

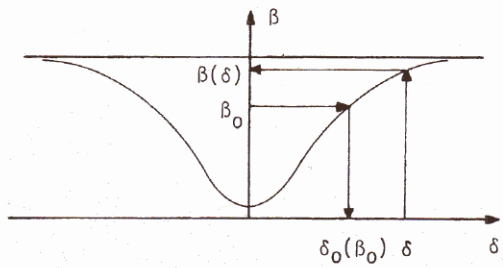


FIG. 8. Power function $\beta(\delta)$: (1) power of test $\beta(\delta)$ in dependency of the size of the error, i.e., the distance δ (noncentrality parameter) between null and alternative hypothesis; (2) lower bound $\delta_0(\beta_0)$ for noncentrality parameter derived from required minimum probability β_0 for the separation of the hypothesis, i.e., for the detection of gross errors (according to Baarda [1, 2]).

TABLE 4
Noncentrality Parameter δ_p and Power β_p for $k = 3$

δ_p	β_p (%)	$1 - \beta_p$ (%)
3	50	50
4	84	16
5	97.7	2.3
6	99.9	0.1

EXAMPLES. E7 (continued). Using Eq. (2.9a) and replacing δ_p by $|w_i|$ we obtain

$$|w_i| = 1.83 \rightarrow \beta_i = \phi(1.83 - 3) + [1 - \phi(1.83 + 3)]$$
$$= 0.121 \qquad + 1 - 0.99999 \approx 0.121.$$

The small error can only be found with a very low probability.

E8 (continued). $|w_i| = 3.33 \rightarrow \beta_i \approx \phi(3.33 - 3) = 0.629$. The error can be found with a not too high probability. If we have 100 cases with an error of this size only in 63% of these cases could we expect to find the blunder.

2.3. Lower Bound for the Distance of H_0 and H_a

If we are given a test, a significance level and the two hypotheses we can derive the power $\beta = \beta(\delta, \alpha_0)$ of the test using Eq. (2.9a). The value δ can be termed the distance between the null hypothesis and the alternative hypothesis. As the size of the errors in general are not known, we cannot determine this parameter δ and thus are not able to use this equation.

The basic idea of W. Baarda was to invert the relation between the power of the test and the noncentrality parameter and use

$$\delta = \delta(\beta, \alpha_0). \tag{2.10}$$

For a preset power of the test equation (2.10) gives the distance between H_0 and H_a, i.e., the size of the error which can be detected with that power. If we now require that an alternative, i.e., an error has to be detected with a probability β which is larger than a prespecified bound β_0 then the distance δ between H_0 and H_a has to be larger than a bound δ_0. This *lower bound* δ_0 for the noncentrality parameter δ can be derived from (cf. Fig. 8)

$$\beta > \beta_0 \rightarrow \delta > \delta_0$$
$$\delta_0 = \delta_0(\beta_0, \alpha_0). \tag{2.11}$$

It depends on

—the used test, specifically the distribution of the test statistic

—the significance level $S_0 = 1 - \alpha_0$ or the preset probability α_0 of making a Type I error

—the required minimum power of the test, i.e., the required minimum probability of correctly detecting an error.

TABLE 5

Lower Bounds δ_0 for Noncentrality Parameter δ in Dependency of the Significance Number α_0 and the Required Minimum Power β_0 of the Test

	α_0	0.01%	0.1%	1%	5%
β_0	k	3.72	3.29	2.58	1.96
50		3.72	3.29	2.58	1.96
70		4.41	3.82	3.10	2.48
80		4.73	4.13	3.42	2.80
90		5.17	4.57	3.86	3.24
95		5.54	4.94	4.22	3.61
99		6.22	5.62	4.90	4.29
99.9		6.98	6.38	5.67	5.05

If we now use the approximation equation (2.9b) for the power of the test this lower bound δ_0 can be computed from

$$\delta_0(\beta_0, \alpha_0) \approx \phi^{-1}(1 - \alpha_0/2) + \phi^{-1}(\beta_0) = k + \phi^{-1}(\beta_0). \tag{2.12}$$

Because for values $\beta_0 > 0.5$ the function $\phi^{-1}(\beta_0)$ is positive, the minimum bound δ_0 for the noncentrality parameter δ is always larger than the critical value k if one requires a power of at least 50%. But this also means that the distance δ between H_0 and H_a must be—possibly much—larger than the critical value k if one with a high probability wants to correctly reject H_0 when H_a actually is true. Some values for δ_0 are listed in Table 5.

It is the merit of W. Baarda to make the power of the testing procedure applicable. He proposed to keep

$$\delta_0 = \text{const.} \qquad \text{for all mensuration designs}$$

in order to be able to compare different designs. A constant value δ_0 means that the testing procedures have to be able to separate the hypothesis of a minimum distance which is common for all applications. This leaves freedom in balancing the producer's and the consumer's risks α_0 and $1 - \beta_0$, respectively.

The next section applies this line of thought to the error detection problem.

3. RELIABILITY ANALYSIS WITH RESPECT TO GROSS ERRORS

In the first section we have discussed the evaluation of observational values using the geometry of the design. The second section provided the means for error detection and their assessment. We now want to combine both concepts to accomplish our final aim: the reliability analysis of a design with respect to gross errors. According to Baarda one distinguishes between internal and external reliability. For measuring the degree of internal reliability we will derive lower bounds for just detectable errors which can also be seen as a measure for the controllability of the observations. From the lower bounds for detectable errors we directly arrive at the sensitivity or the robustness of the result, which measures the external reliability and essentially is the influence of nondetectable gross errors onto the estimated parameters.

3.1. *Internal Reliability According to Baarda*

In the last section we have derived the lower bound δ_0 for the distance between H_0 and H_a which guarantees a required separability of the two hypotheses. The distance δ depends on the size of the gross error in the observation and can be derived using Eq. (2.3). We now want to derive a lower bound for just detectable gross errors. If we substitute the actual influence $\delta_i = \nabla w_i$ of the gross error ∇x_i onto the test statistic w_i by the lower bound δ_0 we can solve for the lower bound $\nabla_0 x_i$ for the gross error ∇x_i in the observation x_i and obtain the *boundary value*

$$\nabla_0 x_i = \sigma_{x_i} \frac{\delta_0}{\sqrt{r_i}} . \tag{3.1}$$

Gross errors ∇x_i larger than the boundary value $\nabla_0 x_i$ can be detected with a higher probability than β_0 if a significance level $1 - \alpha_0$ for the test with the test statistic w_i is used. Or in short: Gross errors larger than the boundary value can be detected; if they are smaller than the boundary value they are not detectable.

The boundary value $\nabla_0 x_i$ takes into account:

—the precision of the observation, described by the standard deviation σ_{x_i}

—the geometry of the design of the measuring procedure, described by the redundancy number r_i

—the significance level and the required minimum power, comprised in the lower bound δ_0 for the noncentrality parameter.

The smaller the redundancy number r_i of the observation, the larger a gross error has to be in order to be detectable. This is reasonable, because the residuals are smaller in this case. The boundary values draw a clear picture of the internal reliability of the design.

The case $r_i = 1$ only occurs if the true value of the observation is known, e.g., a distance between two given (and fixed) points in case one does not simultaneously estimate an unknown scale parameter. In this optimal case gross errors ∇x_i larger than $\delta_0 \cdot \sigma_{x_i}$ can be found already with a probability greater than β_0. If the significance level is $S_0 = 1 - \alpha_0 = 99.9\%$ then (cf. Table 5) gross errors smaller than $4.13 \cdot \sigma_{x_i}$ are not detectable with a probability larger than 80%. The redundancy number usually is much smaller than 1. Therefore gross errors have to be much larger than 4 times the standard deviation of the observation to be detectable. On the other hand, if an observation cannot be controlled by the other observations, then $r_i = \hat{e}_i = 0$ and a test of the observation is not possible. The boundary value is infinite.

EXAMPLES. Assuming $\delta_0 = 4$ we, e.g., obtain

E 9: distance	$\sigma_{x_i} = 1.0$ cm,	$r_i = 0.25$,	$\nabla_0 x_i = 8.0$ cm
E10: distance	$\sigma_{x_i} = 1.5$ cm,	$r_i = 0.7$,	$\nabla_0 x_i = 7.2$ cm
E11: image coordinate	$\sigma_{x_i} = 5\ \mu$m,	$r_i = \frac{1}{6}$,	$\nabla_0 x_i = 49\ \mu$m
E12: image coordinate	$\sigma_{x_i} = 3\ \mu$m,	$r_i = \frac{1}{12}$,	$\nabla_0 x_i = 42\ \mu$m.

A comparison of the boundary values for heterogeneous observations is not directly possible. But even if one evaluates the distances and image coordinates in the examples separately it is, e.g., questionable whether a boundary value of 42 μm is better than a boundary value of 49 μm. For, if one relates the boundary values to the measuring accuracy, the second image coordinate is less reliable, as their redundancy number is smaller than that of the first image coordinate.

We therefore, analogously to the normalizations in Section 1, split the boundary value into two parts, one referring to the precision and one referring to the reliability. Thus we arrive at the measure for the controllability of the observation (cf. Eq. (1.25a))

$$\delta'_{0i} = \frac{\nabla_0 x_i}{\sigma_{x_i}} = \frac{\delta_0}{\sqrt{r_i}}. \tag{3.2}$$

Thus δ'_{0i} is a factor for the standard deviation σ_{x_i} of the observation to obtain the boundary value $\nabla_0 x_i$. With the *controllability factor* δ'_{0i} we finally get (cf. Eq. (1.25b))

$$\nabla_0 x_i = \delta'_{0i} \cdot \sigma_{x_i}, \qquad \delta'_{0i} = \delta_0 / \sqrt{r_i}. \tag{3.3}$$

The controllability value δ'_{0i} thus is the factor by which a gross error at least has to be larger than the standard deviation of the observation to be detectable with a minimum probability of β_0.

The controllability factors in Example 5 are (a) 8.0, (b) 4.8, (c) 9.8, and (d) 14.0. The distance (b) is best controllable with $\nabla_0 x_i = 4.8\ \sigma_{x_i}$ while the image coordinate (d) is worst controllable with $\nabla_0 x_i = 14\ \sigma_{x_i}$.

After fixing the significance level S_0 and the minimum power of the test the controllability in a simple manner depends on the redundancy number r_i. Evaluating the controllability thus is the same as evaluating the geometry. This is the reason why Baarda required δ_0 to be chosen constant for all mensuration designs.

The boundary values also contain information on the test used and thus can be used to compare different tests. E.g., if one uses the common test statistic

$$w_i^* = -\hat{e}_i / \sigma_{x_i}, \tag{3.4}$$

which is not optimal and does not take the geometry into account, one obtains the boundary value

$$\nabla_0^* x_i = \sigma_{x_i} \frac{\delta_0}{r_i}. \tag{3.5}$$

It is larger than $\nabla_0 x_i$ by a factor $1/\sqrt{r_i}$. Especially if the geometry is weak ($r_i \ll 1$) the difference is significant. For $r_i = \frac{1}{9}$ we obtain $\nabla_0^* x_i = 3\ \nabla_0 x_i$. In Example E12 errors must be larger than 144 μm if one uses the usual test statistic Eq. (3.4), compared to 42 μm if one uses the standardized residual. These values have to be compared with the assumed measuring accuracy of 3 μm!

3.2. External Reliability According to Baarda

The last section discussed how large gross errors have to be in order to be detectable. Gross errors smaller than the boundary values $\nabla_0 x_i$ may stay undetected

and contaminate the result. We now determine the maximum influence of undetectable gross errors onto the estimates. We can use the relations derived in Section 1, where we analysed the effect of the observations onto the result.

If we substitute δ_i' in Eq. (1.47) by δ_{0i}' we obtain the *sensitivity or robustness factor* $\bar{\delta}_{0i}$

$$\bar{\delta}_{0i} = \delta_{0i}'\sqrt{\bar{u}_{k_i}} = \delta_0\sqrt{\frac{\bar{u}_{k_i}}{r_i}} = \delta_0\sqrt{\frac{1 - r_i - u_{t_i}}{r_i}} . \tag{3.6}$$

We can follow the same line of thought which led to Eq. (1.37). This now gives us the maximum influence of an undetectable error $\nabla x_i < \nabla_0 x_i$ in observation x_i onto an arbitrary function f of the estimated coordinates $\hat{k}$ or any other subset of the unknowns comprised in $\hat{k}$. If we substitute $\hat{\nabla}_i f$ by $\nabla_{0i} f$ and $\bar{\delta}_i$ by $\bar{\delta}_{0i}$ we finally obtain

$$\nabla_{0i} f \leq \bar{\delta}_{0i} \cdot \sigma_f = \sigma_f \cdot \delta_0 \cdot \sqrt{\bar{u}_{k_i}/r_i} . \tag{3.7}$$

Equation (3.7) shows the decisive role of the ratio $\bar{u}_{k_i}/r_i$ for the sensitivity or robustness of the result with respect to undetectable gross errors (cf. Section 1.6). The sensitivity or robustness factors $\bar{\delta}_{0i}$ measure the external reliability according to Baarda.

EXAMPLES (continued).

E 9: $\delta_{0i}' = 8.0$, $\bar{\delta}_{0i} = 6.9$
E10: $\delta_{0i}' = 4.8$, $\bar{\delta}_{0i} = 2.6$
E11: $\delta_{0i}' = 9.8$, $\bar{\delta}_{0i} = 8.9$
E12: $\delta_{0i}' = 14.0$, $\bar{\delta}_{0i} = 13.2$.

The sensitivity factors are calculated assuming $u_{t_i} = 0$ or $\bar{u}_{k_i} = 1 - r_i$. The values show that only the second observation (E10) can be accepted.

Let us now investigate the reliability, i.e., the controllability of the observations and the robustness of the estimated parameters in the first three examples. In all cases we have assumed $\delta_0 = 4$.

E1. *Template matching* (continued). The only unknown parameter in the matching example is the shift. We therefore can calculate the controllability factors, the boundary values for detectable gross errors, and the sensitivity factors from Eqs. (3.5) and (3.7) with $\bar{u}_{k_i} = 1 - r_i$. For the center pixels of the edge they are collected in the following table, assuming a standard deviation of $\sigma_x = 5$ gray levels for the measured gray values:

Slope	0	10	30	60	30	10	0
δ_{0i}'	4.00	4.04	4.36	6.67	4.36	4.04	4.00
$\nabla_0 x_i$ [gr./v.]	20	20	22	33	22	20	20
$\bar{\delta}_{0i}$	0	0.57	1.75	5.33	1.75	0.57	0

The controllability of the gray value of the center pixel is just acceptable. Under these assumptions errors in the gray value less than 33 gray levels, which, e.g., may be caused by an irregularity at the object cannot be detected. The effect onto the estimated shift is less than 5 times its standard deviation. Obviously the gray values with low slope have a negligible effect onto the shift and thus may be ommitted from the estimation procedure. In general very small sensitivity values $\bar{\delta}_{0i}$ suggest checking whether the observation is really necessary, or whether it is superfluous.

E2. *Absolute Orientation of a Single Camera* (continued). Also in this case we only deal with one group of unknown parameters, namely the 6 orientation parameters of the camera. With the assumed standard deviation $\sigma_x = 3$ μm we obtain the values of the following table:

	Pt. #	1	2	3	4	5	6	7
x	δ'_{0i}	5.30	6.81	5.30	5.21	5.05	4.95	5.05
	$\nabla_0 x_i$ [μm]	16	20	16	16	15	15	15
	$\bar{\delta}_{0i}$	3.48	5.51	3.48	3.65	3.08	2.92	3.08
y	δ'_{0i}	5.09	8.06	5.09	4.85	5.32	4.69	5.32
	$\nabla_0 x_i$ [μm]	15	24	15	15	16	14	16
	$\bar{\delta}_{0i}$	3.15	7.00	3.15	2.74	3.50	2.46	3.50

The design obviously is very homogenous, as could be expected. The y-coordinate of point 2 is least controllable. Gross errors less than 8 times the measuring precision are not detectable, approximately one pixel. The maximum effect ($\bar{\delta}_{0i} = 7$) of an undetectable gross error in this observation onto the result is just acceptable.

E3. *Relative Orientation of Two Cameras* (continued). Up to now we have always analyzed the quality of the orientation parameters. We will now investigate the robustness of the reconstruction of an unknown object from two images with the only additional information being the scale factor, e.g., based on the distance between the two camera positions. Errors in the observation influence both, the orientation parameters and the object coordinates. We already know the redundancy numbers for the case of fixed object coordinates from the previous example (b). The loss in local redundancy for each image coordinate yields the contribution number $\bar{u}_{k_i}$ (cf. the Eq. (1.45b)). E.g., the redundancy number of the y coordinate of point 5 changes from 0.566 to 0.048 (cf. E3 above), resulting in $\bar{u}_{k_i} = 0.518$ and thus in a sensitivity factor of $\bar{\delta}_{0i} = 4 \cdot \sqrt{0.518/0.048} = 13.1$. Altogether we thus obtain the reliability values for the y-coordinates:

Pt. #	1	2	3	4	5	6	7
δ'_{0i}	10.3	10.0	10.3	9.30	18.3	7.86	18.3
$\nabla_0 x_i$ [μm]	31	30	31	28	55	24	55
$\bar{\delta}_{0i}$	7.06	2.93	7.06	6.54	13.1	5.37	13.1

The design is very weak due to low controllability of the 5th point. Blunders up to

55 μm or approximately 3 pixels are not detectable with a statistically founded test. With a conventional test errors up to 250 μm or 12 may stay undetected (cf. Fig. 4). The maximum influence onto the resultant coordinates is up to 13 times their standard deviation, which thus has no legitimacy as a quality measure if gross errors are likely to happen. As already observed earlier the x-coordinates are not controllable at all, as their redundancy numbers are zero. This is reflected in the controllability and sensitivity factors being infinite. On the other hand, the sensitivity factor of y-coordinate 2 is much smaller (2.93) than the controllability factor (10.0); this means that though relatively large gross errors may stay undetected their influence onto the result is acceptable. This is because a large part of the error is absorbed by the orientation parameters (cf. E3 above).

Nevertheless, at least three images or a sufficiently reliable object model are necessary to be able to reconstruct an object from images with a reasonable quality.

4. RELIABILITY ANALYSIS OF DATA AND DESIGN

The evaluation of data and design of a measuring process can be based on quite a number of measures which have been provided in and motivated by the previous sections. They are collected in Table 6. There is full analogy between the empirical and the theoretical evaluation measures. The empirical measures refer to the actual data, the theoretical values to the lower bounds of detectable errors. The test statistic w_i and the lower bound δ_0 for the standardized distance between H_0 and

TABLE 6
Measures for Evaluating the Reliability

Empirical	Theoretical
Test statistic	Standardized distance between H_0 and H_a
$w_i = \delta_i = \dfrac{-\hat{e}_i}{\sigma_{\hat{e}_i}}$	$\nabla_0 w_i = \delta_0$
Estimated size	Lower bound for detectable error
$\hat{\nabla} x_i = \dfrac{-\hat{e}_i}{r_i} = w_i \dfrac{\sigma_{x_i}}{\sqrt{r_i}}$	$\nabla_0 x_i = \delta_0 \dfrac{\sigma_{x_i}}{\sqrt{r_i}}$
Standardized size of error	Controllability factor
$\delta_i' = \dfrac{\hat{\nabla} x_i}{\sigma_{x_i}}$	$\delta_{0i}' = \dfrac{\nabla_0 x_i}{\sigma_{x_i}}$
Empirical sensitivity factor	Theoretical sensitivity factor
$\bar{\delta}_i = w_i \sqrt{\dfrac{\bar{u}_{k_i}}{r_i}} = \delta_i' \sqrt{\bar{u}_{k_i}}$	$\bar{\delta}_{0i} = \delta_0 \sqrt{\dfrac{\bar{u}_{k_i}}{r_i}} = \delta_{0i}' \sqrt{\bar{u}_{k_i}}$
Actual influence of observation	Maximum influence of undetectable error
$\hat{\nabla}_i f \leq \bar{\delta}_i \cdot \sigma_f$	$\nabla_{0i} f \leq \bar{\delta}_{0i} \cdot \sigma_f$

H_a are the only difference in corresponding formulas. The index 0 indicates the theoretical values.

We now want to discuss the relevance of the measures in different situations. We already have pointed out the interaction between the statistical test and the effect of nonsignificant errors and the problem of choosing the right criterium for eliminating observations.

There seems to be a clear distinction between two types of problems:

A. The evaluation process may aim at an explanation or an understanding of the error sources. Alternatives in this case represent models which loosely speaking have a meaning. Examples are well-defined errors, such as, a 1 m-error in a coordinate.

B. The evaluation process may aim at a check of the quality of the resultant coordinates or other geometrical or physical entities. In this case the alternatives are set up in order to detect and compensate for gross errors and to reach a result of high reliability. The nature of the errors is of secondary interest. The set of alternatives just has to be large enough not to leave any effects unconsidered.

A and B correspond to problems in natural and engineering science, respectively.

Table 7 contains the empirical and theoretical measures indicating which of them are decisive for the evaluation of data and design in the context of these two types of problems:

A. We detect and localize gross errors using the test statistic w_i and possibly base our correction of the observation onto the estimated size $\hat{\nabla} x_i$ of the gross error. (A correction, however, should only be performed, if the type of the error is explainable, otherwise one should remeasure.) A reliable detection requires small boundary values $\nabla_0 x_i$.

As a safe localization of gross errors only is possible using the test statistic w_i, this type of evaluation of the data always is necessary during the editing process.

B. The reliability of the resultant coordinates $\hat{k}$ or any other subset of the unknown parameters $\hat{y}$ can be fully described by the empirical and the theoretical sensitivity factors. Small empirical values $\bar{\delta}_i$ (say < 2 or 3) indicate that the actual

TABLE 7

Empirical and Theoretical Measures for Two Types of Evaluation Processes

Type of measure	Name of evaluation measure	Value	Eq.	Type of problem A	Type of problem B
Empirical	Test statistic	w_i	(2.1)/(2.2)	x	
	Estimated size	$\hat{\nabla} x_i$	(1.22)	x	
	Empirical sensitivity	$\bar{\delta}_i, \hat{\nabla} if$	(1.47)/(1.37)		x
Theoretical	Boundary values (controllability, internal reliability)	$\nabla_0 x_i$	(3.3)	x	
	Theoretical sensitivity (robustness, external reliability)	$\bar{\delta}_{0i}, \nabla_{0i} f$	(3.7)		x

Note. Aim A: explanation/understanding of errors; aim B: quality control of result.

model errors hardly have an effect on the result, even if they are large. Small theoretical values $\bar{\delta}_{0i}$ indicate that the design is robust with respect to undetectable model errors. The test statistics w_i do not seem to be of great value in this context, if the empirical sensitivity factors are satisfyingly small. In case the precision of the unknown parameters is not very homogeneous, the analysis should be based on the influence of $\nabla_i f$ and/or $\hat{\Delta}_{0i} f$ of the observations and/or the nondetectable errors, respectively, onto a specified function of the parameters.

5. SUMMARY AND CONCLUSIONS

The analysis of a mensuration process with respect to the quality of the result can be based on objective measures which take the geometry of the design and the used estimation and testing procedures into account. This is the central message of this paper. The evaluation of the design, based on measures for precision, controllability, and robustness can be used for planning purposes. The evaluation of the data can be based on a statistical test, estimates for the size of possible blunders and the influence of the observed values onto the result and simplifies the data editing phase. The examples demonstrate that the quality measures coincide with intuition. These quality measures may reveal weak points in design and data which otherwise would not have been detected. They thus seem to be suitable for automatic quality control in mensuration problems encountered in computer vision, at least for a better self-diagnosis of the estimation procedures.

One, however, has to keep the simplified assumptions in mind, which have been made in this presentation:

—The analysis assumed only one error to be present. The theory for tackling more than one error is available, also the more difficult problem of evaluating the separability or locatability of different errors. In a similar line of thought, one then arrives at requirements for the correlation between the residuals, in order to correctly locate a blunder with a prespecified probability [10]. The practical problem of data cleaning, however, can use the techniques of robust estimation to advantage which provide a suggestion for the hypotheses and thus a starting point for a further quality analysis:

—The analysis is based on the validity of the model. The concept, however, is also suitable to evaluate systematic errors, and already is formulated that way: in this case the h_i-vector in Eq. (1.8b) is a full vector, or a matrix describing the effect of the error source onto the observations. One then arrives at statements on the determinability and the sensitivity with respect to systematic errors [10].

—The analysis assumes the precision of the observed values to be known. The analysis can simply be adapted to the case where a F-test, based on the estimated variance factor, is used; the geometrical analysis stays unaltered. Individual errors in the standard deviations anyway cannot be separated from blunders. The estimation of the precision of groups of heterogeneous observations is possible, but not yet integrated into the concept [8, 10].

The analysis of the precision, as indicated in Section 1.1, has to take the influence of the chosen coordinate system into account. This problem has not been tackled here, but can rely on the theory of so-called S-systems (cf. [3, 18]).

In spite of these restrictions the application of these concepts would lead to conspicious insight into the geometry of the design of experiments. Theoretical and experimental studies will be necessary to come to strategies which lead to highly reliable results in mensuration problems in computer vision.

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