

# DETERMINATION OF THE ADDITIVE NOISE VARIANCE IN OBSERVED AUTO-REGRESSIVE PROCESSES USING VARIANCE COMPONENT ESTIMATION TECHNIQUE

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Abstract: The paper discusses the determination of the variances  $\sigma_e^2$  and  $\sigma_n^2$  in an observed autoregressive process  $y_i = x_i + n_i$ , with  $x_i = \sum a_k x_{i-k} + e_i$ . It is shown, that approximating the estimated Fourier powerspectrum  $\hat{P}_y(u)$  by a least squares fit  $E(\hat{P}_y(u)) = |H(u)|^2 \sigma_e^2 + \sigma_n^2$  is identical with the variance component estimation solution in the spatial domain. The statistical and numerical properties of the procedure are analysed showing the versatility of the approach.

## 1. Introduction

1.1 Autoregressive (AR)models are widely used for describing the statistical behaviour of one- and two-dimensional randomly varying discrete functions. Examples are time series and digital images. This study was motivated by the application of AR-models in the analysis and prediction of heights of topographic surfaces. Preliminary investigations (Jäckle, 1984) suggest that AR-models are suited to describe the behaviour of the slopes or the curvature in terrain profiles. The results of this paper may be used to derive an optimal, possibly adaptive, filter in the presence of observational errors.

1.2 The model for a stationary AR(p)-process (stochastical variables are underscored)

$$\underline{x}_i = \sum_{k=1}^p a_k \underline{x}_{i-k} + \underline{e}_i \quad (1)$$

of order  $p$  is fully described by the  $p$  coefficients  $a_k$  and the variance  $\sigma_e^2$  of the driving process ( $\underline{e}_i$ ) which in most cases is assumed to be white Gaussian noise. Stationarity is achieved by choosing  $\sigma_x^2$  in a way that  $\sigma_{x_i}^2 = \sigma_{x_j}^2$  for all  $i$  and  $j$ . Usually only the observed process ( $\underline{y}_i$ )

$$y_i = x_i + n_i \quad (2)$$

is available, where the  $n_i$  are observational errors. Then in addition to  $\alpha_k$  and  $\sigma_e^2$  the additive noise variance  $\sigma_n^2$  is unknown. For a reconstruction of the sequence  $(x_i)$  from the observed sequence  $(y_i)$ , e.g. by using a Wiener Filter or equivalently by least squares techniques, the power spectra of  $x_i$  and  $n_i$ , i. e. the parameters  $\alpha_k$  and both variances have to be known. The standard techniques for estimating the parameters  $\alpha_k$  (cf. e.g. Box/Jenkins 1976 [1]) however neglect the effect of the observational noise  $n_i$  or assume both variances or at least  $\sigma_n^2$  to be known (cf. e.g. Yum and Park 1983 [24]). A joint estimation of all parameters is desirable.

According to an idea of R.L. Kashyap, which became known to the author after finishing the manuscript, the variances can be derived from a nonlinear equation system (Kashyap and Rao 1976 [11], ch.2h) which is based on the representation of an observed AR-process by an autoregressive moving average (ARMA) process. It will be of great interest to compare this approach with the following one which in addition to the estimates of the variances also offers criteria for their evaluation. The estimation, however, needs not necessarily be accomplished in one step but may be achieved in an iterative manner, by alternatively estimating the process coefficients  $\alpha_k$  and the variances  $\sigma_e^2$  and  $\sigma_n^2$ .

1.3 This paper discusses the determination of the variances of the driving and the observation process assuming the process coefficients to be known. The procedure could be part of an iterative algorithm for the joint estimation of all unknowns or used in cases where the process coefficients are known from experience. As the transfer function  $H(u)$  of the AR-process only depends on the process parameters, the proposed estimation procedure will immediately yield an estimate for the signal to noise ratio of the observed signal and allows a proper reconstruction of the process  $(x_i)$ .

1.4 We will first derive estimates for the variances  $\sigma_e^2$  and  $\sigma_n^2$  based on the power spectrum of the observed process  $y_i$  and then show that the resulting equation system is identical to that obtained by a set up in the spatial domain using variance component estimation technique and assuming the process  $(y_i)$  to be periodic. The 3rd section analyses the estimability or determinability and the identifiability or discernability of the variances for a doubly integrated white noise process which is already in use for height interpolation in photogrammetry (Ebner 1979 [3]). The 4th section discusses the numerical effort for the variance component estimation and the versatility of the approach.

## 2. Estimation of the variances of the driving and the observing process

### 2.1 The power spectrum of observed AR-Processes

The power spectrum of the AR-process eq. (1) only depends on the parameters  $\alpha_k$  and the variance  $\sigma_e^2$  of the driving process  $(e_i)$ . It is given by (cf. e.g. Lückner 1980 [17], p. 52., Fuller 1976 [7], p. 144, (4.3.8)).

$$P_x(u) = T(u) \cdot \sigma_e^2 \quad (3a)$$

with

$$T(u) = \frac{1}{\left| 1 - \sum_{k=1}^p a_k e^{-j2\pi uk} \right|^2} \quad (3b)$$

With  $(n_i)$  being white noise and independent of  $(e_i)$  the power spectrum of  $(y_i)$  is immediately

$$P_y(u) = P_x(u) + P_n(u) = T(u) \sigma_e^2 + \sigma_n^2. \quad (4)$$

$T(u)$  is the squared transfer function of the system yielding  $(x_i)$  from  $(e_i)$ .

The total variance  $\int P_y(u) du$  of the process is spread over the frequencies  $u$  and consists of two components. As  $T(u)$ , depending on  $a_k$  only, is assumed to be known, each value of the power spectrum is a linear function of the two unknown variance components  $\sigma_e^2$  and  $\sigma_n^2$ .

Example 1: The non-stationary AR(2) process with  $a_1 = +2$  and  $a_2 = -1$  can be used to describe the heights in a terrain profile (cf. Ebner 1979 [3]). It is a doubly integrated white noise process. The power spectrum of an observed profile is then given by eq.(4) using eq.(3b) with the above coefficients:

$$T(u) = \frac{1}{16 \sin^4 \pi u}. \quad (5)$$

The Wiener or least squares filter for estimating  $(x_i)$  from  $(y_i)$  is (cf. e.g. Castleman 1979 [2], cf. also Link 1983 [16]):

$$H(u) = \frac{P_x(u)}{P_x(u) + P_n(u)} = \frac{1}{1 + \frac{\sigma_n^2}{\sigma_e^2} \cdot 16 \sin^4 \pi u} \quad (6)$$

and only depends on the ratio  $\sigma_n^2/\sigma_e^2$  between the two unknown variances. We will refer to this ratio later in conjunction with the least squares solution in the spatial domain. ■

## 2.2 Estimating Variance Components from the Power Spectrum $P_y(u)$

The power spectrum  $P_y(u)$  may be estimated from  $(y_i)$  in various ways (cf. the review Kay/Marple 1981 [12]) and leads to an estimate  $\hat{P}_y(u)$ . If the process  $(y_i)$  is periodic and Gaussian  $\hat{P}_y(u)$  reasonably can be estimated using a discrete Fourier transformation. Then the elements of  $\hat{P}_y(u)$  are independently (cf. Papoulis 1965 [19], p.368)  $\chi^2_2$ -distributed with variance  $V(\hat{P}_y(u)) = P_y^2(u)$ , as they are derived from the normally distributed complex amplitude spectrum of  $(y_i)$ . This leads to the following variance component model:

$$E(\hat{P}_y(u)) = T(u) \sigma_e^2 + \sigma_n^2 ; \quad V(\hat{P}_y(u)) = P_y^2(u) \quad (7)$$

Eq. (7) is linear in the variance components and reflects the "Expectation-Dispersion-Correspondence" of Pukelsheim (1976, [20]), here applied to a linear model in the spectral domain.

With approximate values  $\sigma_e^2(0)$  and  $\sigma_n^2(0)$  one can now derive estimates  $\hat{\phi}_e = \hat{\sigma}_e^2 / \sigma_e^2(0)$  and  $\hat{\phi}_n = \hat{\sigma}_n^2 / \sigma_n^2(0)$  for the variance factors using weighted least squares technique. The normal equations are

$$S \hat{\phi} = \underline{\omega} \quad (8a)$$

$$\begin{bmatrix} \sum_u \frac{T^2(u) \cdot \sigma_e^4}{P_y^2(u)} & \sum_u \frac{T(u) \sigma_e^2 \sigma_n^2}{P_y^2(u)} \\ \sum_u \frac{T(u) \cdot \sigma_e^2 \sigma_n^2}{P_y^2(u)} & \sum_u \frac{\sigma_n^4}{P_y^2(u)} \end{bmatrix} \begin{bmatrix} \hat{\phi}_e \\ \hat{\phi}_n \end{bmatrix} = \begin{bmatrix} \sum_u \frac{T(u) \cdot \sigma_e^2 \cdot \hat{P}_y(u)}{P_y^2(u)} \\ \sum_u \frac{\sigma_n^2 \cdot \hat{P}_y(u)}{P_y^2(u)} \end{bmatrix} \quad (8b)$$

As  $P_y(u)$  is unknown but needed for determining the weights

$$w(u) = 1 / V(\hat{P}_y(u)) \quad (9)$$

it may be substituted by  $\hat{P}_y(u)$ . Then eq. (8) becomes nonlinear in the unknowns and has to be solved iteratively (cf. Schaffrin 1983 [23]) by setting

$$\hat{\phi}_i^z(v+1) = \hat{\phi}_i^z(v) \cdot \hat{\phi}_i(v) ; \quad i = e, n ; \quad v = 0, 1, 2, \dots \quad (10)$$

The equation system eq. (8) reduces to that given by Pagano (1974 [18]) if the weights  $w(u)$  (eq.(9)) are neglected and the initial values for the variances are set to 1.

### 2.3 Estimating Variance Components from the Process ( $y_i$ )

The derivation in the spatial domain is more extensive than in the spectral domain. We start from the AR-model eqs. (1) and (2) and rewrite it in the form of a linear Gauss-Markov model (GMM) with  $x_i = E(\underline{x}_i)$ .

$$0 = E(\underline{e}_i) = -x_i + \sum_{k=1}^p a_k x_{i-k} ; \quad V(\underline{e}) = \sigma_e^2 \quad i = p+1, m \quad (11)$$

$$E(\underline{y}_i) = x_i ; \quad V(\underline{y}) = \sigma_n^2 \quad i = 1, m \quad (12)$$

The expectation of the  $m-p$  prediction errors  $\underline{e}_i$  are zero. According to Helmert (1924, [9]), we now are able to change our model. We treat the expected values  $E(\underline{e}_i)$  as fiction observations with value 0 and variance  $\sigma_e^2$ . It can be shown that the estimators and their precision are identical in both models. Together with the  $m$  observed values

$\underline{y}_i$  we have  $2m-p$  observations for the unknown  $x_i$  of the AR-process, leading to a redundancy of  $r=m-p$ . In general not all  $x_i$  need to be observed, moreover they need not be observed directly, allowing irregular gaps between the observations, or observations at arbitrary points  $t$  between two grid points  $t_i$  and  $t_{i+1}$ .

In this context we restrict the process  $\underline{y}_i$  to be periodic making a comparison with the previous results possible. Then additional  $p$  equations in the form of eq.(11) are available extending the range of the parameter  $i(i=1,m)$  and thus increasing the redundancy to  $r=m$ . The complete GMM for the estimation then reads as:

$$E(\underline{L}) = A \underline{x} ; \quad V(\underline{L}) = C = \sum_{i=1}^2 \phi_i Q_i \quad (13)$$

with

$$\underline{L} = \begin{pmatrix} 0 \\ \underline{y} \end{pmatrix}; \quad \begin{matrix} 0 = (a_i) \\ \underline{y} = (\underline{y}_i) \end{matrix} \quad A = \begin{pmatrix} A_1 \\ I \end{pmatrix}; \quad Q_1 = \begin{pmatrix} \sigma_e^2 I & 0 \\ 0 & 0 \end{pmatrix}; \quad Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_n^2 I \end{pmatrix}; \quad \phi_i = \sigma_i^2 / \sigma_i^2(0)$$

and

$$A_1 = \begin{bmatrix} -1 & a_1 & a_2 & \dots & a_k & \dots & 0 \\ 0 & -1 & a_1 & \dots & a_{k-1} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_1 & a_2 & \dots & a_k & 0 & \dots & -1 \end{bmatrix} = \text{circ} \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ a_k \\ \vdots \\ a_1 \end{bmatrix}$$

The periodicity of  $(\underline{y}_i)$  is reflected in the circulant matrix  $A_1$  with kernel vector  $[-1 \ 0 \dots 0 \ a_k \dots a_1]^T$ . The filtered values  $\underline{x}_i$  can be estimated from the normal equation system

$$N \hat{\underline{x}} = \underline{h} \quad \text{with} \quad N = \sigma_e^{-2} A_1^T A_1 + \sigma_n^{-2} I ; \quad \underline{h} = \sigma_n^{-2} \underline{L} \quad (14)$$

if the variances  $\sigma_e^2$  and  $\sigma_n^2$ , or at least the ratio  $\sigma_n^2/\sigma_e^2$  is known.

The variances however can be estimated using variance component estimation technique (Helmert 1924 [9], Grafarend/d'Hone 1978 [8], Koch 1980 [14], Förstner 1979 [5]).

The variance factors  $\phi_e$  and  $\phi_n$  can be obtained from the equation system

$$S \hat{\underline{\phi}} = \underline{\omega} \quad (15)$$

with

$$S = (s_{ij}) = (\text{tr} (C^{-1} D Q_i C^{-1} D Q_j)) \quad , \quad \hat{\underline{\phi}}^T = (\hat{\phi}_e, \hat{\phi}_n)$$

$$\underline{\omega} = (\omega_i) = (\text{tr} (\underline{L}^T C^{-1} D Q_i C^{-1} D \underline{L})) \quad , \quad i = e, n$$

$$D = I - A(A^T C^{-1} A)^{-1} A^T C^{-1}$$

The elements  $s_{ij}$  in our case are (with  $Q_1 = \sigma_e^2 I$  and  $Q_2 = \sigma_n^2 I$ )

$$s_{11} = \text{tr} [(I - A_1 N^{-1} A_1^T Q_1^{-1})^2] \quad (16a)$$

$$s_{12} = \text{tr} [N^{-1} A_1^T Q_1^{-1} A_1 N^{-1} Q_2^{-1}] = s_{21} \quad (16b)$$

$$s_{22} = \text{tr} [(I - N^{-1} Q_2^{-1})^2] \quad (16c)$$

If we now exploit the special structure of the matrices  $A_1$ ,  $N$  and  $Q_i$ , all three being circulant matrices, we can further simplify the expressions.

With the unitarian matrix

$$F = (f_{il}) = \left( \frac{1}{\sqrt{m}} e^{-j2\pi il/m} \right) \quad (17)$$

the circulant matrices  $A_1$  and  $N$  can be diagonalized (cf. Klein 1976 [13], Fuller 1976 [7], p. 135):

$$F A_1 F^{-1} = \text{diag}(\lambda_u) \quad ; \quad F N F^{-1} = \text{diag}(\sigma_e^{-2} |\lambda_u|^2 + \sigma_n^{-2}) \quad ; \quad u = 1, n \quad (18)$$

Pre- and post-multiplying the matrices  $N$ ,  $A_1$  etc. in eq.(16) with  $F$  and  $F^{-1} = F^*$  does not change the values  $s_{ij}$  but allows to write the traces as sums of the eigenvalues. With  $\Lambda(u) = |\lambda_u|^2$  this finally leads to

$$s_{11} = \sum_u \left( 1 - \frac{\Lambda(u) \sigma_e^{-2}}{\Lambda(u) \sigma_e^{-2} + \sigma_n^{-2}} \right)^2 = \sum_u \frac{\sigma_n^{-4}}{(\Lambda(u) \sigma_e^{-2} + \sigma_n^{-2})^2}$$

or

$$s_{11} = \sum_u \frac{\sigma_e^4}{(\Lambda(u) \sigma_n^2 + \sigma_e^2)^2} = \sum_u \frac{\Lambda^{-1}(u) \sigma_e^4}{(\Lambda^{-1}(u) \sigma_e^2 + \sigma_n^2)^2} \quad (19a,b)$$

and analogously

$$s_{12} = \sum_u \frac{\Lambda(u) \sigma_n^2 \sigma_e^2}{(\Lambda(u) \sigma_n^2 + \sigma_e^2)^2} = \sum_u \frac{\Lambda^{-1}(u) \sigma_e^2 \sigma_n^2}{(\Lambda^{-1}(u) \sigma_e^2 + \sigma_n^2)^2} \quad (20a,b)$$

$$s_{22} = \sum_u \frac{\Lambda^2(u) \sigma_n^4}{(\Lambda(u) \sigma_n^2 + \sigma_e^2)^2} = \sum_u \frac{\sigma_n^4}{(\Lambda^{-1}(u) \sigma_e^2 + \sigma_n^2)^2} \quad (21a,b)$$

The right sides of eq.(15) can be treated similarly. But now we are able to show the equivalence of eq.(8) and (19)-(21): The transformation eq.(18) is identical to the discrete Fourier transformation, which also has been used for the estimation of  $P_y(u)$ . Therefore, if we substitute  $\Lambda^{-1}(u)$  by  $T(u)$  in eqs.(19b), (20b) and (21b) we immediately obtain eq.(8b). This enables us to use both, the spatial and the spectral, version of the estimation procedure to advantage.

**Example 2:** Figure 1 shows the graph of a terrain profile with 150 points derived from aerial photographs of scale 1:28000 using photogrammetric measuring device (Zeiss Planicomp 100). The precision of the operator was determined using the model eq.(13) based on the AR(2)-process from example 1 ( $\alpha_1 = 2$ ,  $\alpha_2 = -1$ ) (cf. Lindlohr 1982 [15]). The estimated standard error  $\hat{\sigma}_n$  was 0.45m. This is ca. 0,1 % of the flying height of 4300m over the terrain and in full agreement with photogrammetric experience. ■



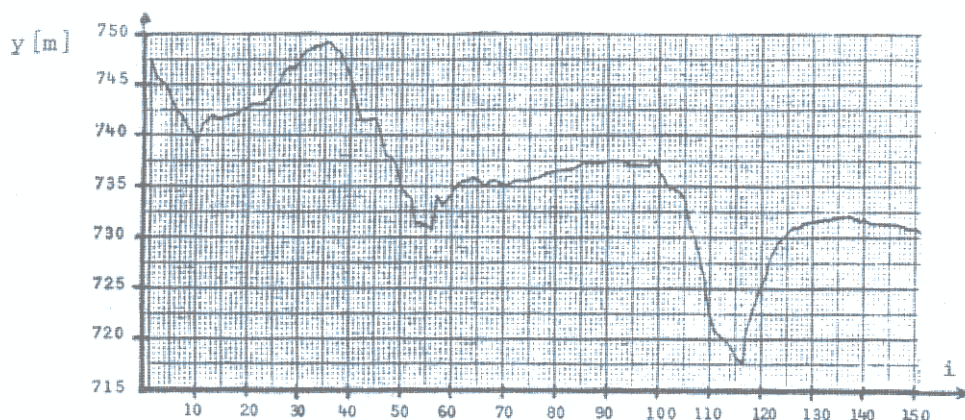


Fig. 1 Terrain profile, point distance 10 m

### 3. Evaluation of the Estimated Variances

#### 3.1 Determinability and Separability

The properties of various estimators for variance components have been analysed by Schaffrin (1983 [23]). The estimated variances from eq.(15) are best invariant quadratic estimators with minimum bias (BIQUAMBE). Though their distribution is not known, one can derive their variance under the assumption that  $(\underline{y}_i)$  is Gaussian (cf. Koch 1980 [14], p. 211):

$$V(\hat{\phi}_i) = 2 \cdot (S^{-1})_{ii} \rightarrow \hat{V}(\hat{\phi}_i) = 2 \cdot (S^{-1})_{ii} \hat{\sigma}_i^2 \quad (22)$$

They give an indication about the estimability or the determinability of the variances  $\sigma_\phi^2$  and  $\sigma_n^2$ . If the standard deviations  $\hat{\sigma}_{\phi_i}$  are below 0.2 the variances  $\sigma_i^2$  can be said to be well determinable, as they are accurate up to 20%.

The correlation

$$\rho_{12} = \frac{(S^{-1})_{12}}{\sqrt{(S^{-1})_{11} (S^{-1})_{22}}} \quad (23)$$

between the estimates on the other hand is a measure for the discernability or the identifiability. If the correlation coefficient  $\rho_{12}$  is less than, say, 75 % the variances are well discernable. Then with a high probability (of ca. 95%) one will not identify the observational noise as signal or vice versa. This measure is derived from testing multiple linear hypothesis (Förstner 1983 [6]) and seems to be useful here also.

#### 3.2 Analysis of an AR(2) process

We will now investigate, under which conditions the signal and the noise in an observed process are estimable or separable. In order to get an idea of the features of the estimation process the already mentioned AR-2 process, with  $\alpha_1 = 2$  and  $\alpha_2 = -1$ , has been analysed in detail.

Fig. 2 shows the variances and the correlation of the estimated variance components  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_n^2$  in dependency of their ratio  $\sigma_n^2/\sigma_e^2$ .

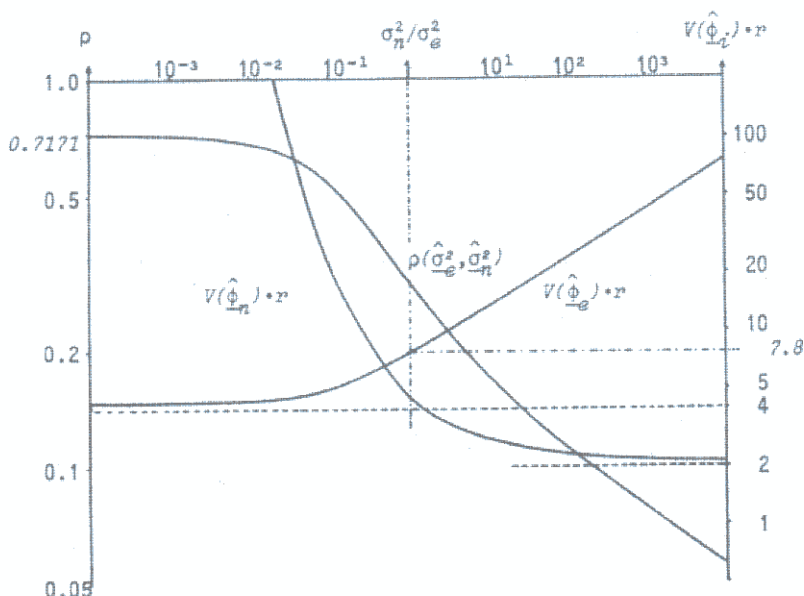


Fig. 2 Estimability and Separability of Variance Components  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_n^2$   
 $r$  = redundancy

Instead of  $V(\hat{\sigma}_e^2)$  the relative accuracy, i. e. the variances  $V(\hat{\phi}_e) \cdot r$  of the variance factors are given. They also depend on the redundancy  $r$  which for periodic sequences equals the number  $m$  of the observations. One can derive from fig. 2 that the estimated standard deviation of the additional noise variance  $\hat{\sigma}_n^2$  is  $\hat{\sigma}_n^2 \cdot \sqrt{7.8/100} = \hat{\sigma}_n^2 \cdot 0.28$  or 28%, if  $\sigma_n^2/\sigma_e^2 = 1$  and  $r=m=100$  observations are made.

The variances and the correlation of the variance components have been calculated from eq.(16) using simulated data. The used processes, with  $m=100$  observations each, were not periodic (Lindlohr 1982 [15]). Independently the values were derived from the theoretical power spectra, thus representing periodic processes. The sums in eq.(8) were replaced by integrals assuming a sufficiently large number  $m$  of observations. E.g. the elements  $s_{11}$  then reads as (cf. eq.(19a)):

$$s_{11} \left( \frac{\sigma_n^2}{\sigma_e^2} \right) = m \int_{-1/2}^{+1/2} \frac{T^2(u) \sigma_e^4}{P^2(u)} du = m \int_{-1/2}^{+1/2} \frac{du}{\left( \frac{\sigma_n^2}{\sigma_e^2} 16 \sin^4 \pi u + 1 \right)^2} \quad (24)$$

The integrals for  $s_{11}$ ,  $s_{12}$  and  $s_{22}$  were solved numerically with a HP 15C computer. The results of both calculations differed not more than 1%, except for some profiles with very low signal to noise ratio. This demonstrates the low influence of the border effects.



The correlation between the estimated variances never exceeds 75 %. The maximum value is  $\sqrt{18/35} = 0.7171$  and is reached for small observational errors ( $\sigma_n^2/\sigma_e^2 \rightarrow 0$ ). Thus signal and noise are always separable, the procedure will not really interpret noise as signal or vice versa.

The variances  $\sigma_e^2$  and  $\sigma_n^2$  are not always determinable, though. If the variances are of different order, i.e. the ratio  $\sigma_n^2/\sigma_e^2$  is very different from 1, then only the larger variance can be estimated with sufficient accuracy. In the extreme cases ( $\sigma_n^2 \rightarrow 0$ ,  $\sigma_e^2 \rightarrow 0$ ) the relative accuracy is  $4/x$  and  $2/x$  for the variances  $\sigma_e^2$  and  $\sigma_n^2$  resp.. The last value  $2/x$  is identical with the variance of the estimated variance factor  $\sigma_0^2$  of a least squares estimation. The additional noise variance  $\sigma_n^2$  obviously can only be determined if it is not much smaller than the variance  $\sigma_e^2$  of the driving process. On the other hand, even for strongly contaminated signals,  $\sigma_e^2$  is estimable, though with only moderate accuracy.

These results are representative for observed processes where the spectral properties of signal and noise are different, specifically if the power spectra differ in shape. If, in our case, the noise would have been correlated, e. g. according to an autoregressive scheme of order 1, the separability would have been much less, due to the similarity of the power spectra.

#### 4. Numerical Considerations

##### 4.1 Irregular Observations

Up to now we always have assumed that all signal values  $x_i$  have been observed. But the estimation of the variance components is also possible if the sequence of observations is irregular. This is of great practical importance as it increases the flexibility of the procedure.

If not all signal values are observed eq.(12) only is valid for the  $m_n$  observations  $y_i$ . Then, the special structure of  $A$  is lost. The normal equation matrix  $N$  in eq.(14) is still band limited, with band width  $p$ . The prediction of the  $x_i$  from the observed values  $y_i$  needs about  $mp^2/6$  operations (cf. Ebner et al. 1984 [4]).

On the other hand, the effort for calculating the elements  $s_{ij}$  for the variance estimation is prohibitive, as all elements of the inverse  $N^{-1}$  are needed, which requires appr.  $m^3/2$  operations. But it is possible to reduce the effort considerably, if one uses a slightly different iteration scheme to solve the nonlinear (cf. the text after eq.(8b)) equation system.

The following equation system (cf. Förstner 1979 [5])

$$\begin{bmatrix} \bar{s}_{11} & 0 \\ 0 & \bar{s}_{22} \end{bmatrix} \begin{bmatrix} \hat{\phi}_e \\ \hat{\phi}_n \end{bmatrix} = \begin{bmatrix} s_{11} + s_{12} & 0 \\ 0 & s_{21} + s_{22} \end{bmatrix} \begin{bmatrix} \hat{\phi}_e \\ \hat{\phi}_n \end{bmatrix} = \begin{bmatrix} \omega_e \\ \omega_n \end{bmatrix} \quad (25)$$

diagonalizes eq.(15) and after convergence leads to the same result, as  $\hat{\phi}_e$  and  $\hat{\phi}_n$  then equal 1 (cf. Schaffrin 1983 [23]). But the sums  $\bar{s}_{11}$  and  $\bar{s}_{12}$  are much easier to obtain. First observe, that

$$\bar{s}_{11} + \bar{s}_{22} = \text{tr } D = r = m_n - p, \quad (26)$$

the total redundancy of the system. Now,  $\bar{s}_{22}$  can be calculated from

$$\bar{s}_{22} = \text{tr} [(C - A N^{-1} A^T) Q_2^{-1}] = m_n - \text{tr } N^{-1} / \sigma_n^2 \quad (27)$$

where  $m_n$  is the number of observed values  $y_i$ , possibly not equal to  $m$ . The main effort now is to determine  $\text{tr } N^{-1}$ . But as  $N$  is band limited and only the elements of  $N^{-1}$  within the band are necessary, the number of operations is only  $mp^2/2$ , which is considerably less than  $m^3/2$ . Therefore with

$$\bar{s}_{11} = r - \bar{s}_{22} \quad (28)$$

from eq.(26) the solution of eq.(25) can directly be given:

$$\hat{\phi}_e = \frac{\omega_e}{r - \bar{s}_{22}} = \frac{\hat{a}^T \hat{a} / \sigma_e^2}{r - \bar{s}_{22}}; \text{ with the prediction errors } \hat{e} = A_1 \hat{x} \quad (29)$$

and

$$\hat{\phi}_n = \frac{\omega_n}{\bar{s}_{22}} = \frac{\hat{n}^T \hat{n} / \sigma_n^2}{\bar{s}_{22}}; \text{ with the residuals } \hat{n}_i = y_i - \hat{x}_i \quad (30)$$

and  $\bar{s}_{22}$  from eq.(27).

Thus the total effort for estimating the variance components especially the additional noise variance  $\sigma_n^2$  is appr. 3 times the effort for the prediction of  $(x_i)$  alone. (In this model the estimation of  $E(\underline{x}_i)$  equals the prediction of  $\underline{x}_i$ ; cf. Rao (1967 [21]).

The simplification has the disadvantage that the speed of convergence is reduced and the information about the separability is not available. The convergence can be increased by numerical methods, which are discussed by Schaffrin (1983 [23]) with special emphasis on variance component estimation. On the other hand, the correlation of the variance components may be approximated by the theoretical values (cf. 3.2).

#### 4.2 Regular Observations

If all values of the process  $(\underline{x}_i)$  are observed with no gaps one might distinguish two cases:

- For rather short sequences ( $m < 20$ ) the direct calculation according to eqs.(15) and (16) seems to be feasible, if the sparsity of  $A_1$  and the diagonality of the  $Q_i$  are used to advantage.
- For longer sequences the estimation of the prediction errors  $\hat{e}_i$  and the residuals  $\hat{n}_i$  could be achieved from eq.(14). The variance component estimation could reasonably neglect the border effects and the nonperiodicity of the sequence and calculate the elements  $s_{i,j}$  directly from eqs.(19)-(21), whereas the right sides could be derived from  $\hat{e}$  and  $\hat{n}$  (cf. eq.(29), (30)). In this case the additional effort for the variance estimation becomes negligible.

## 5. Discussion

The estimation of variances in observed autoregressive processes can be accomplished in a statistically rigorous manner from a single sequence of observations thus not needing more information than the classical identification procedures.

The interpretation of the variance component estimation procedure as weighted least squares solution for the composite power spectrum enables a simple theoretical analysis of the model in the spectral domain and at the same time numerical advantages. The solution in the spatial domain is very flexible allowing irregular gaps, indirect observations such as slopes or curvatures or observations between the grid points. An evaluation can be based on the variance covariance matrix of the estimated variances. Specifically, the estimability and the separability of the variance components can be derived, which for a special process have been discussed in detail, demonstrating the feasibility of the approach.

The method easily can be extended towards more general processes including autoregressive moving-average processes or vector valued processes. The solution in the spectral domain may even be used for processes with arbitrary power spectrum.

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