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RELIABILITY AND DISCERNABILITY OF EXTENDED GAUSS-MARKOV MODELS [1)]

Summary

The reliability of estimated parameters can be described by the sensitivity of detecting errors in the mathematical model and by the influence of nondetectable model errors on the parameters. The reliability theory of Baarda is extended in two directions: a) the alternative hypothesis may depend on a parameter vector. This leads to reliability measures which depend on the direction of the parameter vector. b) The ability to distinguish two alternative hypothesis can be described by the probability of choosing a wrong alternative. This error of III. type at the same time can serve for an evaluation of a nonstochastic correlation coefficient.

1. Introduction

1.1 Recently the methods of evaluating mathematical models in geodesy and photogrammetry by means of statistical testing procedures have been developed intensively. The evaluation of the adjustment results is meant to be objectified as much as possible in order to enable automatic preparation of decisions.

The first step is done by the mathematical formulation of problems occuring in practice and in science, i. e. hypothesis are stated and optimal tests are kept ready to check them. The sensitivity of the tests to distinguish the nullhypothesis, the model on which the evaluation is based, from the different alternative hypothesis then only depends on the design of the experiment, i. e. the design matrix and the assumed stochastical properties of the observations, and it can therefore serve for an optimization to reach the highest possible separability of the null- and the alternative hypothesis.

Finally the quality of the estimated parameters mainly depends on the effect of non-detected model errors on the result.

1.2 Baarda's fundamental studies on parameter estimation and reliability deal with this problem area. The concept of his theory is essentially based on two important ideas:

1) The alternative hypothesis is parameterized. Thus there is in fact a set of alternatives depending on one parameter. This idea gives way to far-reaching generalizations of the theory and leads to the second idea.
2) He does not inquire the probability by which the nullhypothesis and the alternative hypothesis can be separated, i. e. the power of the test as this is done in statistics: but he presets a required lower bound of the power of the test and then derives a (in general) lower bound for the parameter. Thus a statement on the least distance of the null- and the alternative hypothesis, which can just be proved, is obtained.

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1) English version of the German manuscript, extended by section 3.6

Motivation for the development of this theory was the gross error detection problem together with the definition of the controllability of observations and the determination of the effects of nondetectable gross errors onto the coordinates of geodetic nets. The formulation of the theory was kept general enough to treat also systematic errors in the observations.

1.3 On the other hand the theory contains several serious restrictions:

a) The observations are assumed to be normally distributed. Thus models for deviations from the normal distribution cannot be treated . In practical application this restriction does not lead to serious difficulties, except in cases e. g. where an extremely asymetric distribution is to be expected.

b) The covariance matrix of the observations is assumed to be known. This concerns the structure of the matrix, i. e. the weight coefficients as well as the variance factor σ_o^2. While errors in the stochastical model hardly influence the estimation of the parameters they have a direct influence on all measures of precision and reliability. Thus the hitherto unsolved problem of the evaluation of quality measures is touched.

c) The alternative hypothesis may depend on one parameter. Thus only single gross errors or a single systematic error can be treated. This restriction is the farthest reaching one and has caused doubts whether the theory is apt for practice.

c) As the theory is based on a single alternative hypothesis, several alternatives have to be considered successively one after the other. Therefore no information on the separability of different alternative hypothesis, e. g. on the locatability of gross errors, is obtainable.

Meanwhile several studies have reduced these restrictions. This primarily concernes the assumption that the variance factor σ_o^2 should be known. If σ_o^2 is unknown, a t-test can be used (cf. Krüger, 1976). The τ-statistic proposed by Thompson (1934) and Pope (1975) is functionally dependent on the t-statistic, thus both tests are fully equivalent (cf. Heck, 1980). Using the t-test leads to a small modification of the theory, namely a change of the non-centrality parameter $\lambda_o = \delta_o^2$ (cf. Förstner, 1980). The extension of the theory on alternative hypothesis, depending on more than one parameter up to now has been applied only to two-dimensional problems or only to the controllability of points, i. e. the determinability of additional parameters (cf. Mierlo, 1980; Pelzer, 1980; Koch, 1981; Stefanovic, 1978). At last numerous publications, especially in the field of photogrammetric point determination, the separability of additional parameters was treated with using the correlation coefficients of the test-statistics (cf. Grün, 1978; Jacobsen, 1980; Mauelshagen, 1977). Though the correlation coefficient is decisive for the discernability of different alternative hypothesis, still objective criteria are missing. V. Mierlo's studies on the socalled wrong alarm in deformation measurement application (1975, 1979) are the first ones showing a possibility of describing the separability by the probability of making a wrong decision between the alternatives.

1.4 This study tries to find a statistically founded evaluation of the separability of two alternatives with the help of the correlation coefficient of the corresponding test statistics. Thus the notion of reliability is extended and the evaluation of a nonstochastical correlation coefficient becomes possible.

The study is first based on the assumptions, that both alternative hypothesis are one dimensional and are tested independently using an optimal test.

An extension towards two alternatives depending on more than one parameter is possible and uses the eigenvalues of the covariance matrix of the standardized estimators for the parameters.

The case, that the alternatives are jointly tested, is also discussed. This leads to the more dimensional test wellknown from the general testing theory. This test can also be interpreted as a test of one alternative depending on a parameter vector $\underline{s}$, rather than a scalar parameter. The measures for the sensitivity of the test and for the reliability of the result then become functions of the parameter vector $\underline{s}$.

Thus the above mentioned restrictions 1 c) and d), namely, that only one dimensional alternatives can be treated and that informations on the discernability are available, are repealed.

1.5 The questions posed in the preceeding section shall be explained by an example.

The deformation of the cantilever of fig. 1, which is fixed at point 1, is to be determined by measuring the heights of 5 points. It is expected that the cantilever may sink compared with the given point A, that it may incline round point 1 or it may bend off (cf. fig. 1 a, b, c). The symmetrical design allows a variation of the location of points 2 and 4, which both have the distance a from point 3.

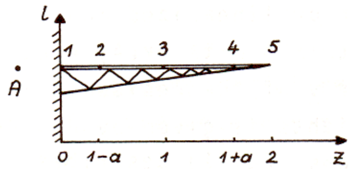
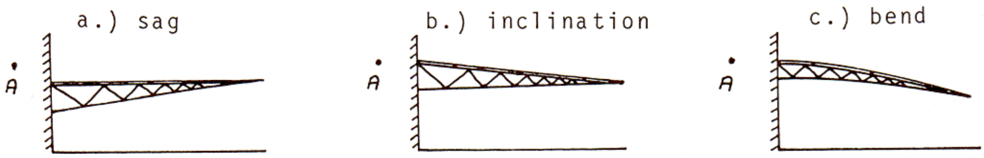


Fig. 1
Cantilever with
reference point A



Let us first assume, that we have only to discern sag and bend on one side and gross errors and deformations on the other side. We find out that the distance a should be approximately 0,7 to reach a high separability of gross errors and deformations. But just detectable deformations can only be separated with 70 % probability, i. e. with probability 30 % we will conclude that the cantilever is deformed, though in reality it is sunk.

Therefore we assume, that for reasons of the stability of the cantilever only the bend has to be examined. The estimation of the auxiliary parameters s_1 and s_2 for the sag and the inclination can be tested against the null hypothesis using a two dimensional test.

But then we are interested in the influence of nondetected sags or inclinations onto the determination of the bend. This influence describes the (external) reliability of the estimated bend and depends on the relation, i. e. the ratio of the auxiliary parameters s_1 and s_2.

We will treat first the theory for this second problem dealing with the test of more-dimensional alternative hypothesis, as it immediately follows from the already existing theory. Developping a criterium for the separability of two alternatives the geometrical interpretation of the test (according to Tienstra, 1956) will be used.

2. On the evaluation of parameter estimation using multi-dimensional tests

2.1 The mathematical model

Let be

$$\underline{l} + \underline{v} = \underline{A}\,\underline{\hat{x}} + \underline{a}_o , \qquad \underline{P} \tag{1}$$

the linear or linearized model with the n×1 vector $\underline{l}$ of observations, the n×u design matrix A with rk(A) = u, the u×1 vector $\underline{x}$ of the unknown parameters, the n×1 vector $\underline{v}$ of the residuals and a n×1 vector $\underline{a}_o$ of constants. The n×n matrix $\underline{P}$ of the weights is given by $\underline{P} = \sigma_o^2\,\underline{C}^{-1}$, where $\underline{C}$ is the covariance matrix of the observations. Eq.(1) describes the classical Gauß-Markov-model.

The null hypothesis ($\sim$ = true value)

$$H_o: \quad E(\underline{l}|H_o) = \underline{A}\,\underline{\tilde{x}} \tag{2}$$

is to be tested against the alternative hypothesis

$$H_a: \quad E(\underline{l}|H_a) = \underline{A}\,\underline{\tilde{x}} + \underline{\widetilde{\nabla e}} \tag{3}$$

with

$$\underline{\widetilde{\nabla e}} = \underline{\widetilde{\nabla e}}(\underline{s}) = \underline{H}\,\underline{\widetilde{\nabla s}} = \underline{H}\,\underline{\tilde{s}}\,\nabla(\underline{s}) , \qquad |\underline{\tilde{s}}| = 1 \tag{4}$$

In this formula the n×1 vector $\underline{\widetilde{\nabla e}}$ contains the true influence of the p parameters $\widetilde{\nabla s}_i$ onto the observations $\underline{l}$. The n×p matrix $\underline{H}$ is assumed to be given, rk($\underline{H}$) = p. It describes the space of the multi-dimensional alternative hypothesis H_a.

Eq. (4) contains a second notation for the causing parameter vector $\underline{\widetilde{\nabla s}}$. It is more convenient in the following discussion. The vector is subdivided into the direction $\underline{s}$ with $|\underline{s}| = 1$ and the length $\nabla(\underline{s})$. The length thus is a function of the direction. The condition $|\underline{s}| = 1$ is not necessary, but only serves for a better illustration; for deviations could be absorbed by $\nabla(\underline{s})$.

2.2 Estimation and testing of parameters

From eq. (1) we obtain the least squares estimation

$$\underline{\hat{x}} = (\underline{A}'\, \underline{P}\, \underline{A})^{-1}\, \underline{A}'\, \underline{P}\, \underline{l} \tag{5}$$

for the unknowns $\underline{x}$. The corresponding sum of the squared residuals is

$$\Omega = (\underline{l} - \underline{A}\, \underline{\hat{x}})'\, \underline{P}\, (\underline{l} - \underline{A}\, \underline{\hat{x}}) \quad . \tag{6}$$

If the alternative is H_a is true, the estimation is biased by

$$\underline{\widetilde{\nabla x}} = (\underline{A}'\, \underline{P}\, \underline{A})^{-1}\, \underline{A}'\, \underline{P}\, \underline{H}\, \underline{\widetilde{\nabla s}} \tag{7}$$

For testing the alternative H_a the adjustment can be subdivided in two steps.

In the first step one determines estimates $\underline{\widehat{\nabla s}}$ for the additional parameters $\underline{\nabla s}$ in the extended model

$$\underline{l} + \underline{v} = \underline{A}\, \underline{\hat{x}} + \underline{H}\, \underline{\widehat{\nabla s}} \tag{8}$$

with their weight matrix

$$\underline{P}_{ss} = \underline{Q}^{-1}_{\hat{s}\hat{s}} = \underline{H}'\, \underline{P}\, \underline{H} - \underline{H}'\, \underline{P}\, \underline{A}\, (\underline{A}'\, \underline{P}\, \underline{A})^{-1}\, \underline{A}'\, \underline{P}\, \underline{H} = \underline{H}'\, \underline{P}\, \underline{Q}_{vv}\, \underline{P}\, \underline{H} \tag{9}$$

($\underline{Q}_{vv}$ cf. eq. (29)) one obtains

$$\underline{\widehat{\nabla s}} = \underline{P}_{ss}(\underline{H}'\, \underline{P}\, \underline{l} - \underline{H}'\, \underline{P}\, \underline{A}\, (\underline{A}'\, \underline{P}\, \underline{A})^{-1}\, \underline{A}'\, \underline{P}\, \underline{l}) \tag{10}$$

The corresponding sum of squared residuals is

$$\Omega_1 = (\underline{l} - \underline{A}\, \underline{\hat{x}} - \underline{H}\, \underline{\widehat{\nabla s}})'\, \underline{P}\, (\underline{l} - \underline{A}\, \underline{\hat{x}} - \underline{H}\, \underline{\widehat{\nabla s}}) \quad . \tag{11}$$

The test of the alternative H_a is achieved by the adjustment of the second step:

$$E(\underline{\widehat{\nabla s}}) = \underline{0} \qquad \text{or} . \quad \underline{\widehat{\nabla s}} + \underline{v}_s = \underline{0}\ , \qquad \underline{P}_{ss} \quad . \tag{12}$$

Backsubstitution of the estimation $\underline{\widehat{\nabla s}}$ into the (here not given) normal equations of the model eq. (8) results in the same estimators $\underline{\hat{x}}$ for the unknowns x as eq. (5).

The sum of the squared residuals of the second step is

$$\Omega_2 = \underline{\widehat{\nabla s}}'\, \underline{P}_{ss}\, \underline{\widehat{\nabla s}} \quad . \tag{13}$$

As the steps are independent one obtains with eqs. (6) and (11)

$$\Omega = \Omega_1 + \Omega_2 \quad . \tag{14}$$

For the test of the parameters $\underline{\widehat{\nabla s}}$ one can use the test statistic ($r = n-u$)

$$T = \frac{\Omega_2/p}{\Omega_2/(r-p)} \sim F'(p,\, r-p,\, \delta^2) \quad . \tag{15}$$

It follows a noncentral Fisher distribution with p and r-p degrees of freedom. The noncentrality parameter $\lambda = \delta^2$

$$\delta^2(\underline{s}) = \underline{\widetilde{\nabla s}}'\, \underline{P}_{ss}\, \underline{\widetilde{\nabla s}} \,/\, \sigma_o^2 = \tilde{\nabla}^2(\underline{s})\, \underline{s}'\, \underline{P}_{ss}\, \underline{s} \,/\, \sigma_o^2 \tag{16}$$

depends on the geometry ($\underline{P}_{ss}$) and on the true value $\underline{\widetilde{\nabla s}} = \tilde{\nabla}(\underline{s}) \cdot \underline{s}$ of the vector $\underline{\nabla s}$. If the null hypothesis is true, T follows a central Fisher-distribution. For a preset significance level $S = 1-\alpha$ the null hypothesis H_o will be rejected if $T > F(\alpha, p, n-p)$.

2.3 Determinability of the parameters $\widehat{\nabla \underline{s}}$

Even if the test eq. (15) does not lead to the rejection of the null hypothesis model errors may stay undetected. The probability of this type II error is

$$1 - \beta(\underline{s}) = P(T < F(\alpha, p, r-p) \,|\, H_a(\delta(\underline{s}))) \qquad (17)$$

According to Baarda, now a lower bound $\nabla_o \underline{s}$ for the parameter $\widetilde{\nabla \underline{s}}$ can be determined, which can be detected just with a given probability $\beta(\underline{s})$, which in general may depend on the direction of $\underline{s}$. From α and $\beta_o(\underline{s})$ follows a lower bound

$$\delta_o^2(\underline{s}) = \delta^2(\alpha, \beta_o(\underline{s})) \qquad (18)$$

for the noncentrality parameter $\delta^2(\underline{s})$. From eq. (16) one obtains a lower bound $\nabla_o(\underline{s})$ for the length $\nabla(\underline{s})$ of the parameter $\underline{\nabla s}$

$$\nabla_o(\underline{s}) = \sigma_o \; \delta_o(\underline{s}) \;/\; \sqrt{\underline{s}' \; \underline{P}_{ss} \; \underline{s}} \quad . \qquad (19)$$

With eq. (4)

$$\boxed{\underline{\nabla_o s} = \nabla_o(\underline{s}) \; \underline{s} = \sigma_o \; \delta_o(\underline{s}) \;/\; \sqrt{\underline{s}' \; \underline{P}_{ss} \; \underline{s}} \; \underline{s}} \qquad (20)$$

circumscribes an area of vectors $\underline{\nabla s}$, which cannot be discovered by the test eq. (15). Only model errors $\underline{\nabla s}$ with $\nabla(\underline{s}) > \nabla_o(\underline{s})$ can be found by the test with a probability $\beta > \beta_o(\underline{s})$. If $\delta_o(\underline{s}) = \delta_o$ is chosen independently from $\underline{s}$ eq. (20) describes the ellipse of boundary values.

Remark: As the preset lower bound for the power $\beta_o(\underline{s})$ of the test can be chosen dependent on the direction of the vector $\underline{\nabla s}$, different alternatives can be distinguished according to their importance, one reason might be the different effect of non-detected deformations.

2.4 The reliability of the estimator $\hat{\underline{x}}$

The effect of non-detectable model errors (eq.(20)) onto the estimated unknowns $\hat{\underline{x}}$ (eq. (5)) is

$$\underline{\nabla_o x}(\underline{s}) = (\underline{A}' \; \underline{P} \; \underline{A})^{-1} \; \underline{A}' \; \underline{P} \; \underline{H} \; \underline{s} \; \nabla_o(\underline{s}) \quad . \qquad (21)$$

In eq. (21) $\nabla_o x_j(\underline{s})$ is the range depending on the direction $\underline{s}$ up to which $\hat{x}_j$ might be falsified by undetected model errors. To acquire a simpler formula we determine the length $\bar{\delta}_o(\underline{s})$ of this influence vector

$$\bar{\delta}_o(\underline{s}) = |\underline{\nabla_o x}(\underline{s})| = \sqrt{\underline{\nabla_o x}'(\underline{s}) \; \underline{A}' \; \underline{P} \; \underline{A} \; \underline{\nabla_o x}(\underline{s})} \;/\; \sigma_o \qquad (22)$$

If we define

$$\underline{\bar{P}}_{ss} = \underline{\bar{Q}}_{ss}^{-1} = \underline{H}' \; \underline{P} \; \underline{H} \qquad (23)$$

being the weight matrix of the parameters $\widehat{\nabla \underline{s}}$ in an adjustment without unknowns $\hat{\underline{x}}$, then using eq. (9) we obtain a formula suited for practical application

$$\boxed{\bar{\delta}_o(\underline{s}) = \delta_o(\underline{s}) \sqrt{\frac{\underline{s}' (\underline{\bar{P}}_{ss} - \underline{P}_{ss}) \; \underline{s}}{\underline{s}' \; \underline{P}_{ss} \; \underline{s}}}} \qquad (24)$$

The values $\underline{s}'\ \overline{P}_{ss}\ \underline{s}$ and $\underline{s}'\ \underline{P}_{ss}\ \underline{s}$ in the numerator of eq. (24) have the dimension of a weight (cf. the above discussion after eq. (4)). The weight $\underline{s}'\overline{\underline{P}}_{ss}\ \underline{s}$ of the adjusted parameters $\widehat{\nabla s}$ in an adjustment with fixed $\underline{x}$ is larger than in an adjustment in which also $\underline{x}$ is estimated. The loss of precision of $\widehat{\nabla s}$ just is needed for the determination of $\hat{\underline{x}}$ (cf. eq.(9)).

In total analogy to the onedimensional case here too the influence $\underline{\nabla}_o f(\underline{s})$ of a non-detectable model error $\underline{\nabla}_o \underline{s}$ on an arbitrary function $f = \underline{e}'\ \hat{\underline{x}}$ of the unknown parameters $\hat{\underline{x}}$ can be determined using the sensitivity parameter $\overline{\delta}_o(\underline{s})$:

$$\nabla_o f(\underline{s}) = \underline{e}'\ \underline{\nabla}_o \underline{x}(\underline{s}) = \underline{e}' \cdot (\underline{A}'\ \underline{P}\ \underline{A})^{-1} \cdot (\underline{A}'\ \underline{P}\ \underline{H}\ \underline{s}) \cdot \nabla_o(\underline{s})$$

and with Cauchy-Schwarz's inequality:

$$\leq \sqrt{e'\ (A'\ P\ A)^{-1}\ e}\ \sqrt{s'\ H'\ P\ A\ (A'\ P\ A)^{-1}\ A'\ P\ H\ s} \cdot \sigma_o\ \delta_o(\underline{s}) \ / \sqrt{s'\ P_{ss}\ s}$$

follows

$$\boxed{\nabla_o f(\underline{s}) \leq \overline{\delta}_o(\underline{s}) \cdot \sigma_f} \qquad (25)$$

where σ_f is the standard deviation of the function f.

In case $\delta_o(\underline{s}) = \delta_o$ the area, given by eq. (24), is the quotient of two ellipses given in polar coordinates, and therefore no ellipse in general (cf. example section 2). Function $\overline{\delta}_o(\underline{s})$ is the multi-dimensional extension of Baarda's measure $\delta_{oi}^2 = \lambda_{oi}$ for the external reliability.

2.5 Simplified evaluation of the external reliability

To acquire a simplified evaluation of $\overline{\delta}_o(\underline{s})$ we determine the directions $\underline{t}_i$ of $\underline{s}$, in which small changes of $\underline{s}$ do not lead to a change of $\overline{\delta}_o(\underline{s})$. These are, among others, the vectors $\underline{\nabla}_o \underline{t}_1$ and $\underline{\nabla}_o \underline{t}_p$, which have the largest and the smallest effect onto the parameters $\hat{\underline{x}}$ resp.

In the special case $\delta_o(\underline{s}) = \delta_o$ one obtains the vectors $\underline{t}_i$ as the solutions of the general eigenvalue problem:

$$(\overline{P}_{ss} - \underline{P}_{ss})\ \underline{t} = (\overline{\delta}_o^2/\delta_o^2)\ \underline{P}_{ss}\ \underline{t} \quad . \qquad (26)$$

The generally complicated figure $\overline{\delta}_o(s)$ can thus be interpreted by means of p vectors $\underline{t}$, which generally do not coincide with the unit vectors $\underline{e}_i = (0, 0,\ldots, 1,\ldots, 0)^T$. The vectors $\underline{t}$ linearily depend on the unit vectors $\underline{e}_i$ and together with the values $\delta_o(\underline{t}_i)$ fully describe the external reliability of the system with respect to the assumed alternative hypothesis.

2.6 Example on external reliability

The example treated in the introduction is supposed to illustrate this kind of evaluation.

The 5 points are assumed to have distances of 1 m. The measured heights l_i $(i = 1,\ldots, 5)$ at the positions $z_i = i - 1$ are to conceive a parabolie bend $z^2 \cdot x$

(x unknown). Moreover a sag ∇s_1 and an inclination $z_i \cdot \nabla s_2$ are expected. Thus the extended model according to eq. (8) is

$$l_i + v_i = z_i^2 \cdot \hat{x} + \widehat{\nabla s}_1 + z_i \cdot \widehat{\nabla s}_2 \quad .$$

It leads to the design matrices

$$\underline{A}' = (0\ 1\ 4\ 9\ 16) \quad \text{und} \quad \underline{H}' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \quad .$$

The observations are assumed to be of equal precision and uncorrelated ($\underline{P} = \underline{I}$). For the analysis we need the matrices

$$\underline{\overline{P}}_{ss} = \begin{pmatrix} 5 & 10 \\ 10 & 30 \end{pmatrix} \quad \text{und} \quad \underline{P}_{ss} = \frac{1}{177} \begin{pmatrix} 435 & 270 \\ 270 & 310 \end{pmatrix} = \begin{pmatrix} 2.542 & 1.525 \\ 1.525 & 1.751 \end{pmatrix} \quad .$$

In Fig. 2 the boundary ellipse $\underline{\nabla}_o s/\sigma_o$ according to eq. (20) is shown. The high correlation coefficient -73% and the nearly equally good determinability of the parameters $\widehat{\nabla s}_1$ and $\widehat{\nabla s}_2$ leads to an ellipse, whose large semiaxis fairly coincides with the diagonal. The high dependence of the determinability on the direction clearly can be recognized; the ratio of the semiaxis is about 2.6.

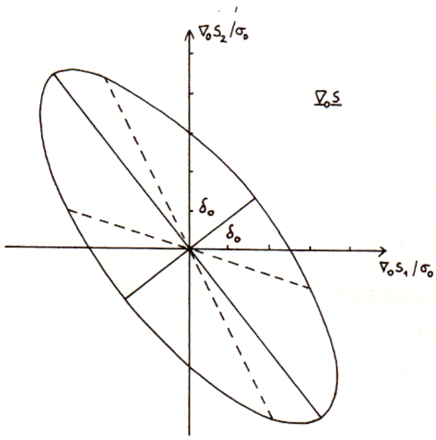


Fig. 2

Boundary ellipse $\underline{\nabla}_o \underline{s}$ for $\underline{\nabla s}$. Parameters lying within the ellipse are not detectable.

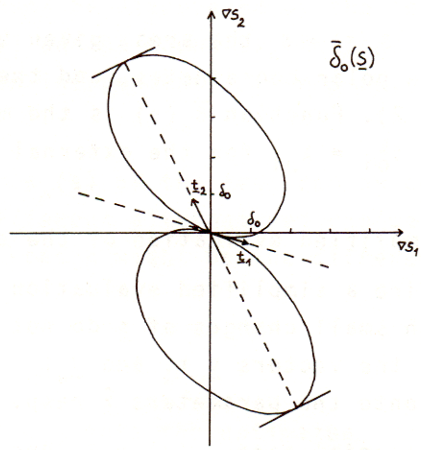


Fig. 3

Sensitivity area $\overline{\delta}_o(\underline{s})$. The influence of of nondetectable model errors $\underline{\nabla}_o \underline{s}$ onto $\hat{x}$ is inside the range of $\overline{\delta}_o(s) \cdot \sigma_x$.

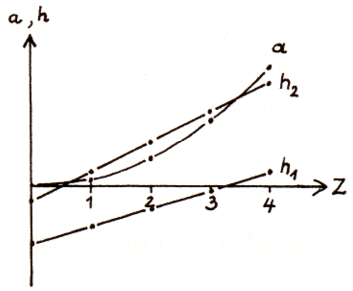


Fig. 4

New transformed parameters h_1 and h_2 for the evaluation of $\hat{x}$.

Fig. 3 shows the effect $\overline{\delta}_o(\underline{s})$ of nondetectable model errors onto the determination of the bend $\hat{x}$. Obviously the form of $\overline{\delta}_o(\underline{s})$ differs from an ellipse. Moreover, model errors with $\widetilde{\nabla s}_1/\widetilde{\nabla s}_2 = -10/3$ (i. e. $\underline{t}_1 = (10, -3)/\sqrt{109}$) do not influence the estimate $\hat{x}$ at all, whereas model errors with $\widetilde{\nabla s}_1/\widetilde{\nabla s}_2 = -1/2$ (i. e. $\underline{t}_2 = (1, -2)/\sqrt{5}$) have a very strong effect on $\hat{x}$ as $\bar{\delta}_o(\underline{t}_2) = 5\,\sigma_o$. In this case in eq. (25) even the sign of equality is valid, i. e. Fig. 3 reflects the real influence of nondetectable model errors and not only an upper bound.

The diagram Fig. 4 shows the new parameters $\widehat{\nabla t}_1$ and $\widehat{\nabla t}_2$, derived from the eigenvectors $\underline{t}_i$. The corresponding design matrix $\underline{\bar{H}}' = \begin{pmatrix} -10 & -7 & -4 & -1 & +2 \\ -1 & +1 & +3 & +5 & +7 \end{pmatrix}$ shows that the deformations $\widehat{\nabla t}_1$ and $\widehat{\nabla t}_2$ and also $\widehat{\nabla t}_1$ and $\hat{x}$ are orthogonal.

The solution of the general eigenvalue problem eq. (26) thus leads to a partial orthogonalization of the design matrix $(\underline{A}\ \underline{H})$. Hence, the evaluation of the external reliability is simplified, as only a few parameters are needed.

This simplification, however, only is possible, if the physical interpretability of the alternative hypothesis is not important. Otherwise an orthogonalization of the parameters is not wanted, as the separation of the physical effects is intended. The separation can be difficult or even impossible under given conditions and then leads to high correlations between the parameters of the different alternative hypothesis. One has no longer to do with testing a single moredimensional alternative but rather with the selection of one out of several - possibly moredimensional - alternatives.

In the following chapter the case of the separation of two onedimensional alternative hypothesis' will be investigated and then will be extended towards two moredimensional alternative hypothesis'.

3. On the separability of alternative hypothesis

3.1 Set up and test of alternative hypothesis

Given two alternative hypothesis each depending on one parameter:

$$H_{ai}\colon E(\underline{l}\,|\,H_{ai}) = E(\underline{l}\,|\,H_o) + \underline{h}_i\,\widetilde{\nabla s}_i\,; \qquad i = 1, 2 \quad . \tag{27}$$

The optimal test for the independent evaluation uses the test statistics

$$w_i = \underline{h}_i'\,\underline{P}\,\underline{v}\,/\,(\sigma_o\sqrt{\underline{h}_i'\,\underline{P}\,\underline{Q}_{vv}\,\underline{P}\,\underline{h}_i})\ ; \qquad i = 1, 2 \tag{28}$$

with

$$\underline{Q}_{vv} = \underline{Q} - \underline{A}\,(\underline{A}'\,\underline{P}\,\underline{A})^{-1}\,\underline{A}'\ , \tag{29}$$

if σ_o is given.

For a preset significance level $1-\alpha_o$ H_{ai} will be rejected if $|w_i| > k(\alpha_o)$. If both hypothesis are tested and both test statistics exceed the critical value in practice one will reject H_o in favour of the alternative whose test statistic is the larger one, in case one expects the test statistics to influence each other.

In this case we must be ready to come to a wrong decision, i. e. that we reject the really proper alternative and accept the other. This type of wrong decision is called a type III error according to standard terminology (cf. Hawkes, 1980). The problem is similar to the one of classification, where one has to choose between several classes based on a measured feature and where the quality of the decision is described by the probability of choosing a wrong class. The approach to analyse the phenomen of "false alarm" proposed by v. Mierlo (1975) is different to the one used upon, as there a procedure of two independent steps is used.

3.2 Erroneous decisions choosing one of two alternatives.

The possibilities of choosing between two alternatives are shown in tab. 1 together with the corresponding probabilites for the case that both alternatives are not true at the same time.

Table 1 Decisions when testing two alternative hypothesis (choosing one of two)

		result of the test		
		H_o $\|w_1\|<k,\ \|w_2\|<k$	H_{a1} $\|w_1\|>k,\ \|w_1\|>\|w_2\|$	H_{a2} $\|w_2\|>k,\ \|w_2\|>\|w_1\|$
unknown reality	H_o	correct decision $1-\alpha'_{oo}$	type I error α'_{o1}	type I error α'_{o2}
	H_{a1}	type II error $1-\beta'_{1o}$	correct decision β'_{11}	type III error γ'_{12}
	H_{a2}	type II error $1-\beta'_{2o}$	type III error γ'_{21}	correct decision β'_{22}

The notation for the probabilities for I. and II. type errors are adapted. The probabilities for type III errors are designated with γ'. The prime ' marks the joint test in contrast to the separate test, which usually is applied. γ'_{12} is the probability of choosing H_{a2} though H_{a1} is right.

For the determination of the probabilities α_{ij}, β_{ij} and γ_{ij} we use a geometric interpretation of the test (cf. Tienstra, 1956).

3.3 Geometric interpretation of the test

The test statistics eq. (28) are normally distributed with

$$\sigma_{w_i} = 1 \quad . \tag{30}$$

The correlation coefficient of the test statistics is

$$\rho_{12} = \frac{\underline{h}_1' \, \underline{P} \, \underline{Q}_{vv} \, \underline{P} \, \underline{h}_2}{\sqrt{\underline{h}_1' \, \underline{P} \, \underline{Q}_{vv} \, \underline{P} \, \underline{h}_1} \, \sqrt{\underline{h}_2' \, \underline{P} \, \underline{Q}_{vv} \, \underline{P} \, \underline{h}_2}} = \cos(\underline{h}_1, \underline{h}_2) = \cos \varepsilon_{12} \tag{31}$$

and can be interpreted as the cosine of the angle between the vectors $\underline{h}_1$ and $\underline{h}_2$ with respect to the metric $\underline{P}\,\underline{Q}_{vv}\,\underline{P}$. Analogously with $\underline{v} = \underline{Q}_{vv}\,\underline{P}\,\underline{v}$ one obtains the test statistics w_i of eq. (28):

$$w_i = \frac{\underline{h}_i' \, \underline{P} \, \underline{Q}_{vv} \, \underline{P} \, \underline{v}}{\sigma_o \sqrt{\underline{h}_i' \, \underline{P} \, \underline{Q}_{vv} \, \underline{P} \, \underline{h}_i}} = \frac{\sqrt{\underline{v}' \, \underline{P} \, \underline{v}}}{\sigma_o} \cos(\underline{h}_i, \underline{v}) = |\underline{v}| \cos \varepsilon_i , \tag{32}$$

where ε_i are the angles between the residual vector $\underline{v}$ and the vectors $\underline{h}_i$.

Fig. 5 shows the relations according to eq. (30)-(32) in the plane of the vectors $\underline{h}_1$ and $\underline{h}_2$. The test statistic w_i is the length of the projection of the vector $\underline{v} = \hat{\underline{l}} - \underline{l}$ having length $|\underline{v}| = \sqrt{\underline{v}'P\underline{v}} / \sigma_o$ on the vector $\underline{h}_1$. The scale of the axis is standardized to 1. The angle between the axis $\underline{h}_1$ and $\underline{h}_2$ is ε_{12}. If the null-hypothesis is true, $\nabla s_1 = \nabla s_2 = 0$ and one obtains an unbiased result $\hat{\underline{l}}$ for the adjusted observations. The joint probability density of w_1 and w_2 is given by

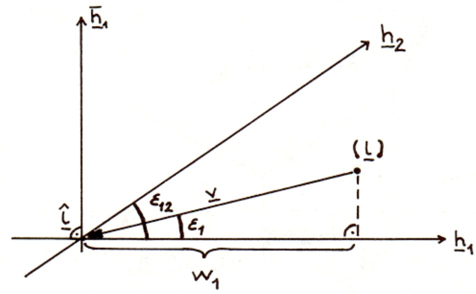


Fig. 5

Geometrical interpretation of the test statistic w_1

$$\phi(w_1, \overline{w}_1) = \frac{1}{2\pi} \exp(-(w_1^2 + \overline{w}_1^2)/2) \tag{33}$$

in the orthogonal coordinate system $\underline{h}_1$ and $\overline{\underline{h}}_2 \perp \underline{h}_1$ of this projection (cf. Baarda, 1968).

For the decision about H_{a1} one only uses w_1. The acceptance area $A = \{\underline{l} \,|\, |w_1| < k\}$ is the stripe orthogonal to the h_1-axis shown in fig. 6. From the one dimensional marginal distribution one derives the probabilites in the one dimensional case. The joint test is shown in fig. 7 and 8.

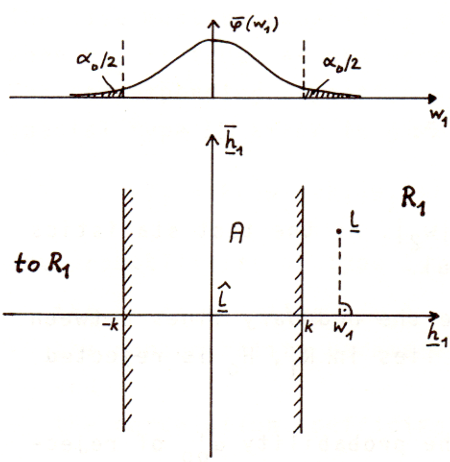


Fig. 6

Two dimensional diagram of the acceptance area A of the single test against H_{a1}. From the marginal density $\overline{\phi}(w_1)$ one derives the probability α_o for the type I error in the one dimensional case.

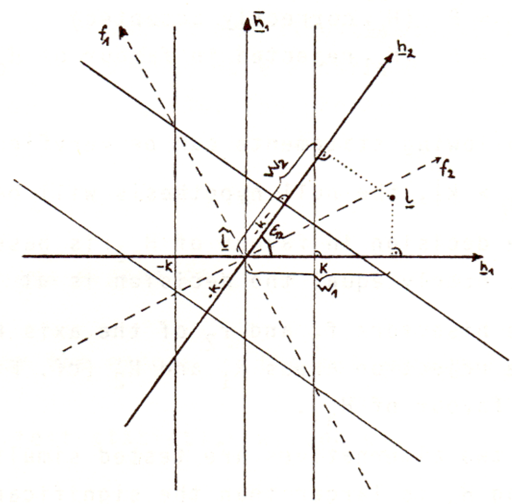


Fig. 7

On the geometric interpretation of the combined test against two alternatives H_{a1} and H_{a2}.

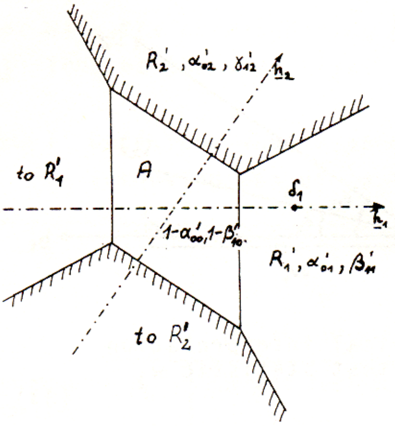


Fig. 8a

Acceptance and rejection areas A' and R' for a joint test of two alternatives

$1-\alpha'_{00}$ = P (H_0 correctly accepted)

α'_{01} = P (H_0 rejected in favour of H_{a1})

α'_{02} = P (H_0 rejected in favour of H_{a2})

$1-\beta'_{10}$ = P (H_{a1} rejected in favour of H_0)

β'_{11} = P (H_{a1} correctly accepted)

γ'_{12} = P (H_{a1} rejected in favour of H_{a2})

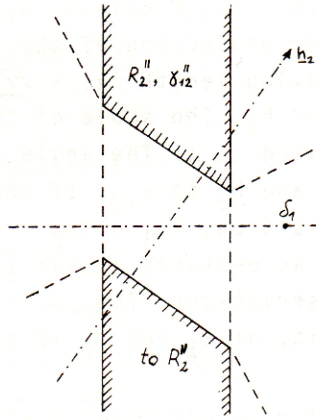


Fig. 8b

Rejection area R_2'' for the joint test of two alternatives

γ''_{12} = P (H_{a2} incorrectly accepted without hint H_{a1} being true)

The following statements can be veryfied using fig. 7:

- $|w_1 > k|$, the null hypothesis will be rejected.
- The decision in favour of H_{a1} is based on $|w_1| > |w_2|$. As the test statistics are nearly equal the decision is at least doubtful.
- The bisectors f_1 and f_2 of the axis h_1 and h_2 are the boundary lines between the rejection areas R_1' and R_2' (cf. Fig. 8). If $\underline{1}$ lies in R_i', H_0 is rejected in favour of H_{ai}.
- As two alternatives are tested simultaneously, the probability α'_{00} of rejecting H_0 is larger than the significance number $\alpha_0 < \alpha'_{01} + \alpha'_{02} = \alpha'_{00}$. The acceptance region A' is smaller than in the one dimensional case (cf. fig. 6 and 8).
- For the same reason the probability of accepting H_{a1}, say, when testing both H_{a1} and H_{a2}, is smaller than in the one dimensional case: $\beta_1 > \beta'_{11}$.
- In case one uses the same critical value k for both single tests the probabilites of erroneously rejecting an alternative are equal: $\gamma'_{12} = \gamma'_{21}$.

3.4 Probabilities for type I, II and III errors

We are now prepared to determine the probabilites of erroneous decisions. The probability for a type I error is given by the value under the density eq.(33) over the rejection areas R_1' and R_2' . E. g.

$$\alpha_{01}' = P(|w_1|>k, |w_1|>w_2||H_o) = \iint_{R_1'} \phi \, df \qquad (34)$$

is the probability of incorrectly rejecting H_o in favour of H_{a1}. For the determination of the probabilities for type II and III errors we assume H_{a1} to be correct and that the error $\widehat{\nabla s}_1$ causes a shift of the density function $\phi(w_1, \overline{w}_2)$ by the amount δ_1. The noncentral density then is given by

$$\phi_1(w_1, \overline{w}_1, \delta_1) = \frac{1}{2\pi} \exp(-((w_1 - \delta_1)^2 + \overline{w}_1^2) / 2) \quad . \qquad (35)$$

The probability $1 - \beta_{1o}'$ of incorrectly accepting H_o, though H_{a1} holds can be determined from

$$1 - \beta_{1o}' = P(|w_1|<k, |w_2|<k|H_{a1}(\delta_1)) = \iint_{A'} \phi_1 \, df \quad . \qquad (36)$$

The probability γ_{12}' of choosing H_{a2} instead of H_{a1} is

$$\gamma_{12}' = P(|w_2|>k, |w_2|>|w_1||H_{a1}(\delta_1)) = \iint_{R_2'} \phi_1 \, df \quad . \qquad (37)$$

The case must be expecially emphasized in which $|w_1|<k$ and simultaneously $|w_2|>k$ when H_{a1} is true, as one erroneously is sure of having made a correct decision. The corresponding rejection area $\overline{R}_2''$ is shown in fig. 8b. The probability of this special type III error is given by

$$\gamma_{12}'' = P(|w_1|<k, |w_2|>k|H_{a1}) = \iint_{\overline{R}_2''} \phi_1 \, df \quad . \qquad (38)$$

The probabilities of type I, II and II errors depend on

- the critical value $k(\alpha_o)$
- the noncentrality parameter δ caused by the errors in the mathematical model and
- the correlation coefficient ρ_{12} of the test statistics w_1 and w_2.

We are first interested in the power

$$\beta' = \beta'(\alpha_o, \delta, \rho) \quad . \qquad (39)$$

of the combined test. For a proper evaluation, however, the probabilites $\gamma_{12}' = \gamma_{21}' = \gamma'$ for a type III error are important:

$$\gamma' = \gamma'(\alpha_o, \delta, \rho) \quad . \qquad (40)$$

In case of very difficult and costly decisions we need the probabilities $\gamma_{12}'' = \gamma_{21}'' = \gamma''$ choosing the wrong alternative and being sure to have decided correctly.

In the appendix the functions β', γ' and γ'' are tabulated in dependence of ρ and δ. Fig. 9a shows $\gamma'(\alpha_o = 100\%, \rho, \delta)$. This is the case where we do not test

H_o but solely decide between H_{a1} and H_{a2}. In this special case $\beta' = 1 - \gamma'_{12}$ and $\gamma''_{12} = 1 - \alpha_{oo} = 0$. If $\delta = 0$ one obtains $\gamma' = 0.5$ as expected. The probabilities of type III errors decrease with increasing distance δ between H_a and H_o, thus being always less than 50 %.

If the correlation coefficient is larger than 0.9 and if a model error leads to $\delta = 4$, in 19 of 100 cases one chooses the wrong alternative ($\gamma' = 0.19$). On the other side, if one requires a lower bound for the separability of 95 % ($\gamma' > 0.95$) for model errors with $\delta = 3$ the correlation coefficient must not be larger than the upper bound $\nabla_o \rho = 0.4$. Finally, if a correlation coefficient of 0.9 is given and one requires a separability of greater 90 % then the distance δ between null and alternative hypothesis must be larger than 5.8.

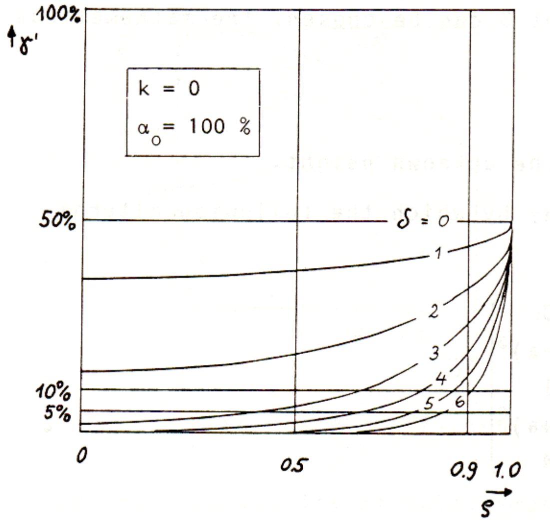
Fig. 9b and c show the probabilities γ' and γ'' of type III errors for $\alpha_o = 5\,\%$, i. e. $k = 1.96$. The comparison of fig. 9a and 9b demonstrates that the probabilities γ' are smaller in case b); this is because type I errors prevent a wrong decision for one of the two alternatives. Fig. 9c is superelevated by a factor 10, for the seemingly secure but erroneous decisions only occur with probabilities $\gamma'' < 5\,\% = \alpha_o$. But all the same in 1 % of all cases this type of wrong decision may occur if $\delta \approx 3.5$. Finally fig. 9d shows, that the power of the joint test decreases rapidly for correlation coefficients larger 0.9. Even model errors with $\delta = 6$ are detectable with a probability $< 80\,\%$ if $\rho = 0.95$.

For the evaluation of the reliability one may instead of the power β of the single test also use the power β' (eq.(39)) of the combined test, which again leads to a lower bound δ_o for the noncentrality parameter. This bound can be used to determine a lower bound $\nabla_o s_1$ of the parameter ∇s_1 which can be just localized by the combined test with a preset probability β'_o. This lower bound will be much larger than in the one dimensional case if high correlations are present.

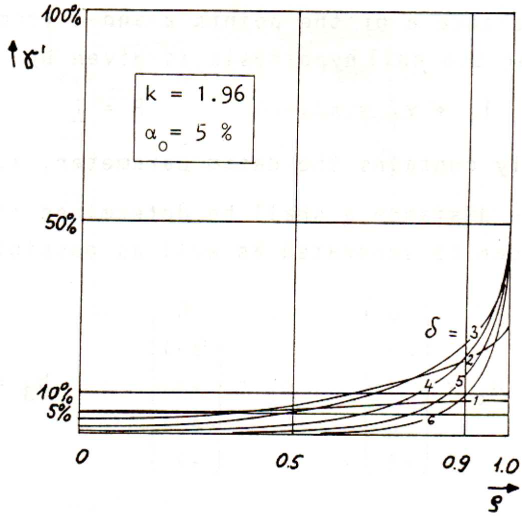
However, it seems to be more advantageous to use the probabilities $1 - \gamma'$ (or $1 - \gamma''$) to avoid a type III error for an evaluation. This probability might be regarded as a measure of the separability of the design with respect to the two alternatives in concern. The separability decreases with increasing correlation coefficient, i. e. with decreasing angle ε_{12}. Thus, if we - in analogy to the method for evaluating the determinability of the parameters - in addition to α_o and β_o preset a minimum probability $1 - \gamma'_o$ for the separability between two alternative hypothesis we obtain an upper bound $\nabla_o \rho$ for the correlation coefficient δ

$$\boxed{\nabla_o \rho = \nabla_o \rho(\alpha_o, \beta_o, \gamma'_o) = \nabla_o \rho(\alpha_o, \delta_o, \gamma'_o)} \qquad (41)$$

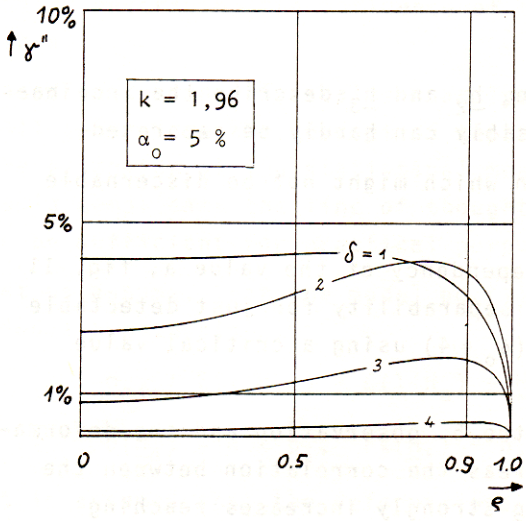
Correlation coefficients $|\rho| > \nabla_o \rho$ lead to a worse, i. e. lower separability than $1 - \gamma'_o$.



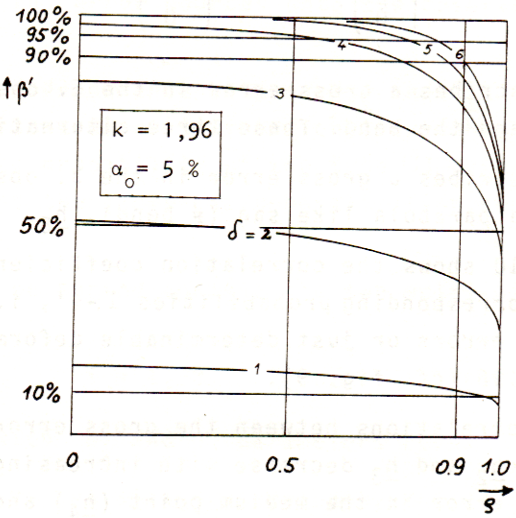
a) type III error γ', $\alpha_o = 100\,\%$



b) type III error γ', $\alpha_o = 5\,\%$



c) type III error γ'', $\alpha_o = 5\,\%$



d) power of the test β', $\alpha_o = 5\,\%$

Fig. 9 Probabilities γ', γ'' and β' in dependency of the non-centrality parameter δ and the correlation coefficient ρ

3.5 Example for the separability of different alternative hypothesis

The above mentioned example is now supposed to illustrate the optimization of a design with respect to the separability between different alternative hypothesis.

For the determination of deformations at the cantilever of fig. 1 the heights of 5 points are supposed to be measured. In contrast to the example in sect. 2.6

the distance a of the points 2 and 4 from point 3 can be chosen. The mathematical model of the null hypothesis is given by

$$l_i + v_i = x \,, \qquad \underline{P} = \underline{I}$$

and only contains the datum parameter, i. e. the unknown height.

Now the distance a shall be determined in a way by which the following alternatives can be separated as well as possible:

$$\underline{h}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \underline{h}_2 = \begin{pmatrix} 0 \\ a-1 \\ -1 \\ -a-1 \\ -2 \end{pmatrix} \qquad \underline{h}_3 = \begin{pmatrix} 0 \\ -(1-a)^2 \\ -1 \\ -(1+a)^2 \\ -4 \end{pmatrix}$$

and

$$\underline{h}_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{h}_5 = \begin{pmatrix} 0 \\ a^2-1 \\ -1 \\ a^2-1 \\ 0 \end{pmatrix}$$

$\underline{h}_1$ describes a gross error in the 5. observation, $\underline{h}_2$ and $\underline{h}_3$ describe the inclination and the bend. These three alternatives possibly can hardly be separated.

$\underline{h}_4$ describes a gross error in the 3. observation which might not be discernable from a parabola like sag ($\neq$ bend), $\underline{h}_5$.

Fig. 10 shows the correlation coefficients in dependency of the value a, fig. 11 the corresponding probabilities $1-\gamma'$, i. e. the separability for just detectable gross errors or just determinable deformations ($\delta_o = 4$) using a critical value $k = 1.96$ (cf. fig. 9).

The correlations between the gross error $\underline{h}_1$ in the 5. observation and the deformations $\underline{h}_2$ and $\underline{h}_3$ decrease with increasing a, whereas the correlation between the gross error in the medium point ($\underline{h}_4$) and the sag strongly increases reaching 100 % for a = 1. Thus a distance a = 0.68 seems to be optimal for the separation of gross and systematic errors. The probability $\gamma' = 8\,\%$ of committing a type III error, i. e. supposing a deformation though in reality a gross error is present is sufficiently small.

The separability of the inclination $\underline{h}_2$ and the bend $\underline{h}_3$ however with $1-\gamma'_{23} < 75\,\%$ is not sufficient. A denser point distribution does not change the separability really if one starts from just detectable deformations, i. e. if δ_o is kept constant. However, because of the higher precision of the parameter estimation fixing the lower bound $\nabla_o s = \sigma_o\, \delta_o \cdot \sqrt{q_{\hat{s}\hat{s}}}$ enables to choose a larger δ_o which increases the separability. For instance, if one uses 5 groups of 4 independent points, the separability increases from 68 % to 80 %.

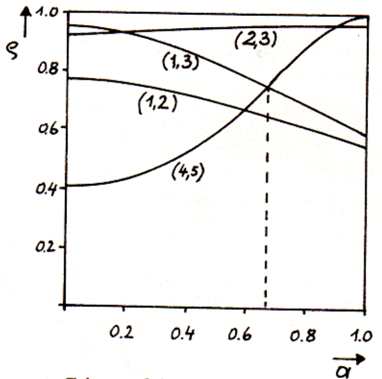


Fig. 10
Correlation coefficient ρ_{ij}

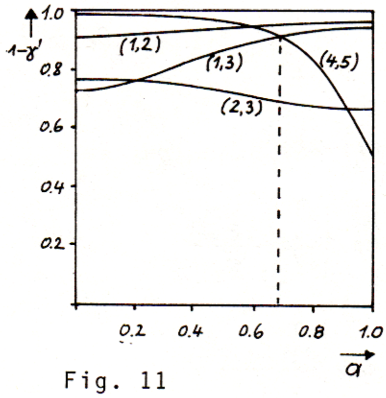


Fig. 11
Separability $1-\gamma'$

3.6 On the separability of multidimensional hypothesis

In sec. 3.4 a measure for the separability of two one dimensional hypothesis was introduced. It was based on the correlation coefficient ρ_{ij} of the test statistics $\underline{w}_1$ and $\underline{w}_j$ corresponding to the hypothesis H_{a1} and H_{a2}. The geometric interpretation of the test showed ρ_{ij} to be the cosine of the angle between the vectors $\underline{h}_i$ and $\underline{h}_j$ describing the influence of the original error sources onto the observations.

Based on this purely analytic analysis of the dependence between H_{a1} and H_{a2} we will generalize the concept of separability towards multidimensional hypothesis. The extension of the statistical measure, the probability of the type III error, would complicate the line of thought, whereas the geometric generalization seems to be sufficient for practical purposes.

Let us assume, that the model eq.(1) has to be tested against the two multi-dimensional hypothesis

$$H_{ai} : E(\underline{l}|H_{ai}) = E(\underline{l}|H_o) + \underline{H}_i \widetilde{\nabla_i \underline{s}} \qquad i = 1, 2 \tag{42}$$

$$o(\underline{H}_i) = n \times p_i, \quad rk(\underline{H}_i) = p_i, \qquad \text{with } p_1 \text{ and } p_2 \text{ parameters resp.}$$

It is assumed that the deformations $\underline{H}_i \; \nabla_i \underline{s}$ are not already modeled by the unknowns $\underline{x}$, i. e. that the colimn spaces $col(\underline{A})$ and $col(\underline{H}_i)$ have no common elements. Instead of evaluating

$$\rho_{12} = \rho(\underline{w}_1, w_2) = \cos(\underline{h}_1, \underline{h}_2) \ ,$$

we now use the maximum value

$$\rho_{12} = \max_A \cos(\underline{h}_1, \underline{h}_2) \qquad A = \{(\underline{h}_1, h_2);\ \begin{array}{l}\underline{h}_1 \in col(H_1) \\ \underline{h}_2 \in col(H_2)\}\end{array} \tag{43}$$

of the cosine of the angle ε_{12} between two vectors $\underline{h}_1$ and $\underline{h}_2$ contained in the column spaces $col(\underline{H}_1)$ and $col(\underline{H}_2)$ resp.. This corresponds to the smallest angle ε_{12} between the vectors $\underline{h}_1$ and $\underline{h}_2$, one describing a specific influence $\underline{H}_1 \underline{s}_1$ of H_{a1}, the other a specific influence $\underline{H}_2 \underline{s}_2$ of H_{a2}. The value ρ_{12} furtheron still can be interpreted as a correlation coefficient, namely the maximum correlation between the test statistics $\underline{w}_1$ and $\underline{w}_2$ from eq.(32) but now $\underline{h}_1$ and $\underline{h}_2$ varying over <u>all</u> vectors fromed by $\underline{h}_1 = \underline{H}_1 \underline{s}_1$ and $\underline{h}_2 = \underline{H}_2 \underline{s}_2$.

With the generally nonsquare matrices $\underline{P}_{ij} = \underline{H}_i^T \, \underline{P} \, \underline{Q}_{vv} \, \underline{P} \, \underline{H}_j$ $(i,j \in \{1,2\})$ (cf. eq.(9)), the standardization $\overline{\underline{s}}_i = \underline{P}_{ii}^{1/2} \, \underline{s}_i$ and the norm $|\underline{s}_i| = \sqrt{\underline{s}_i^T \underline{P}_{ii} \underline{s}_i}$ we obtain for the square ρ_{12}^2 :

$$\rho_{12}^2 = \max_A \cos^2(\underline{h}_1, \underline{h}_2)$$

$$= \max \left[\frac{\underline{s}_1^T \underline{P}_{12} \underline{s}_2}{|\underline{s}_1||\underline{s}_2|}\right]^2 \qquad \text{with } \overline{\underline{P}}_{12} = \underline{P}_{11}^{-1/2} \, \underline{P}_{12} \, \underline{P}_{22}^{-1/2}$$

$$= \max \left[\frac{\overline{\underline{s}}_1^T \overline{\underline{P}}_{12} \overline{\underline{s}}_2}{|\overline{\underline{s}}_1||\overline{\underline{s}}_2|}\right]^2 \qquad \text{(similar to Zurmühl, sect. 13.6)}$$

$$= \max \left[\lambda(\overline{\underline{P}}_{12})\right]^2 \qquad \text{(cf. Schaffrin, et al., p. 285)}$$

$$= \max \lambda(\overline{\underline{P}}_{12} \cdot \overline{\underline{P}}_{21}) = \max \lambda(\overline{\underline{P}}_{21} \cdot \overline{\underline{P}}_{12})$$

thus: $$\rho_{12}^2 = \max \lambda(P_{21} \, P_{11}^{-1} \, P_{12} \, P_{22}^{-1}) \qquad (44)$$

where $\lambda(\underline{A})$ denotes the eigenvalue of $\underline{A}$. The derivation of eq.(44) uses the singular value decomposition of the matrix $\overline{\underline{P}}_{12} = \underline{C}_1 \, \Lambda \, \underline{C}_2^T$ where $\underline{C}_i^T \underline{C}_i = \underline{I}_{r_p}$, $r_p = rk(\overline{\underline{P}}_{12})$. The eigenvalues λ of $\overline{\underline{P}}_{12}$ can be obtained from the eigenvalues λ^2 of $\overline{\underline{P}}_{12} \overline{\underline{P}}_{21}$ or of $\overline{\underline{P}}_{21} \overline{\underline{P}}_{12}$ (cf. Schaffrin, Grafarend, Schmitt, 1977, Anhang 1, p.285) or from the similar matrix $\underline{P}_{21} \underline{P}_{11}^{-1} \underline{P}_{12} \underline{P}_{22}^{-1}$.

The evaluation of the separability, based on the eigenvalues of $\underline{P}_{21} \underline{P}_{11}^{-1} \underline{P}_{12} \underline{P}_{22}^{-1}$, is done in two steps:

a) If at least one eigenvalue equals 1 then there exists at least one vector $\underline{h}$ which is contained in both column spaces $col(\underline{H}_1)$ and $col(\underline{H}_2)$; i. e. there is at least one model error which is common to both alternative hypothesis. If p_o eigenvalues equal 1 then there is a group of p_o additional parameters common to H_{a1} and H_{a2}. They of course prevent a separation of the two alternatives. But they provide a first important insight into the relation between H_{a1} and H_{a2}.

b) The second step of the evaluation consists in constructing 3 parameter groups (cf. fig. 12):

   1) p_o common parameters $\underline{s}_o$ with influence $\underline{H}_{12} \underline{s}_o$ onto the observations. The matrix $\underline{H}_{12}$ has to fullfill the conditions $col(\underline{H}_{12}) \subset col(\underline{H}_1)$ and $col(\underline{H}_{12}) \subset col(\underline{H}_2)$, thus $col(\underline{H}_{12}) = col(\underline{H}_1) \cap col(\underline{H}_2)$. It can be obtained from $\underline{H}_{12} = \underline{H}_1 \, \underline{P}_{11}^{-1/2} \cdot \underline{C}_{1o} = \underline{H}_2 \, \underline{P}_{22}^{-1/2} \cdot \underline{C}_{2o}$, where $\underline{C}_{1o}$ and $\underline{C}_{2o}$ contain those p_o eigenvectors of $\overline{\underline{P}}_{12} = \underline{C}_1 \, \Lambda \, \underline{C}_2^T$ which correspond to the eigenvalues $\lambda_i = 1$.

   2) $p_1 - p_o$ non common parameters $\overline{\underline{s}}_1$ only described by H_{a1} with influence $\overline{\underline{H}}_1 \overline{\underline{s}}_1$ onto the observations. The parameters $\overline{\underline{s}}_1$ are orthogonal to $\underline{s}_o$, thus $\underline{H}_{12}^T \overline{\underline{H}}_1 = 0$, and $\overline{\underline{H}}_1$ has to fullfill the condition $col(\overline{\underline{H}}_1) \subset col(\underline{H}_1)$.

   3) $p_2 - p_o$ non common parameters $\overline{\underline{s}}_2$ only described by H_{a2} with influence $\overline{\underline{H}}_2 \overline{\underline{s}}_2$ onto the observations similar to $\overline{\underline{s}}_1 \left(\underline{H}_{12}^T \overline{\underline{H}}_2 = 0, col(\overline{\underline{H}}_2) \subset col(\underline{H}_2)\right)$.

The groups of the non common parameters $\overline{\underline{s}}_1$ and $\overline{\underline{s}}_2$ can now be evaluated by applying eq. (44) to the new set of non overlapping alternative hypothesis

$$\overline{H}_{ai}: E(\underline{l}|\overline{H}_{ai}) = E(\underline{l}|H_o) + \underline{\overline{H}}_i \, \underline{\widetilde{\nabla s}}_i \quad , \quad i = 1, 2 \qquad (45)$$

The situation is sketched in fig. 12 for $\underline{H}_1 = (\underline{h}_{11}, \underline{h}_{12})$ and $\underline{H}_2 = (\underline{h}_{21}, \underline{h}_{22})$, the corresponding column spaces represented by planes. The column space of $\underline{H}_{12}$ is the intersecting line. The maximum correlation coefficient in this case is identical with the cosine of the angle ε_{12} between the two planes.

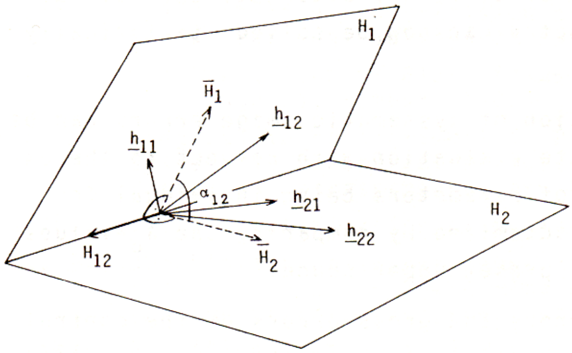


Fig. 12

Geometrical representation of the column spaces of two 2-dimensional alternatives

$\cos \alpha_{12} = \cos(\overline{\underline{H}}_1, \overline{\underline{H}}_2) = \rho_{12}$

Example

With the vectors $\underline{h}_i$ (i = 1,..., 5) from the last example we form the matrices

$$\underline{H}_1 = -(\underline{h}_1, 2\,\underline{h}_2, 4\,\underline{h}_3) \quad \text{and} \quad \underline{H}_2 = -(\underline{h}_4, 4\,\underline{h}_5)$$

using $a = \frac{1}{2}$ thus

$$\underline{H}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \quad \text{and} \quad \underline{H}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \\ 1 & 4 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$$

Thus we want to discern between a gross error in observation l_5, an inclination and a bend on one side and a gross error in l_3 and a sag on the other side.

The weight matrices P_{ij} are

$$\underline{P}_{11} = \begin{pmatrix} 4 & 10 & 50 \\ 10 & 50 & 200 \\ 50 & 200 & 870 \end{pmatrix} \quad ; \quad \underline{P}_{22} = \begin{pmatrix} 4 & 10 \\ 10 & 70 \end{pmatrix} \quad ; \quad \underline{P}_{12} = \begin{pmatrix} -1 & -10 \\ 0 & 0 \\ -10 & -70 \end{pmatrix}$$

The eigenvalues of

$$\underline{P}_{21}\, \underline{P}_{11}^{-1}\, \underline{P}_{21}\, \underline{P}_{22}^{-1} = \frac{1}{8} \begin{pmatrix} 1 & 1 \\ 0 & 8 \end{pmatrix}$$

are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{8}$.

As the maximum eigenvalue equals 1 there is a common parameter belonging to H_{a1} and H_{a2}, namely the parabolic term. The non common parameters lead to a maximum correlation coefficient of $\rho_{12} = \sqrt{\lambda_2} = 0.35$ corresponding to an angle $\varepsilon_{12} = 69.5^{\circ}$, indicating a high separability, between the gross error in l_5 and the inclination on one side and the gross error in l_3 on the other side.

4. Discussion

The evaluation of the reliability of a network design can be based on the proposed parametrization also applicable for more dimensional alternative hypothesis and allow statements about the separability of different alternatives. Several important problems of analytical photogrammetry can now be solved statistically rigorous:

a) Additional parameters for the compensation of systematic image errors can be chosen in a way which allows the separate evaluation with respect to the external reliability. This leads to sets of parameters being individual for each block. The selection can be fully automized by comparing the $\bar{\delta}_o$-values of the orthogonalized parameters with a preset upper bound.

b) The separability of systematic image errors and gross errors of the control points can be determined and used for an optimization of the control point pattern.

c) Experiments for the identification of physical causes of systematic errors can be designed with respect to the separability of the effects.

Both extensions of the theory can be completed and leave important problems for further research.

d) The evaluation of the stochastical model may be based on the reliability of the variance components.

e) The evaluation of the reliability of estimated covariance matrices can be based on a parametrization of the alternative hypothesis using a matrix instead of a parameter vector.

f) This vector needs not to be of finite dimension. For the evaluation of stochastic process one might use a description of the alternative based on a function.

In all cases one needs the central and non central distribution for the test statistics used for the evaluation. However, the transfer of the existing theoretical solution into practical procedures is at least as complicated as the set up of the theory but surely more important.

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A p p e n d i x

values of β' ≈ 1 only approximately

1. Errors of II. and III. type, k = 1.96

GAMMA'(RHO,DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0850	0.2530	0.4254	0.4896	0.4993	0.4999	0.4999	0.4998	0.4998	0.4998
0.99 \	0.0863	0.2366	0.3576	0.3812	0.3614	0.3357	0.3103	0.2858	0.2623	0.2398
0.98 \	0.0862	0.2260	0.3289	0.3330	0.3082	0.2742	0.2420	0.2119	0.1841	0.1587
0.97 \	0.0858	0.2174	0.3070	0.3061	0.2698	0.2312	0.1956	0.1636	0.1352	0.1103
0.96 \	0.0853	0.2099	0.2888	0.2802	0.2395	0.1981	0.1611	0.1239	0.1015	0.0786
0.95 \	0.0848	0.2030	0.2729	0.2583	0.2143	0.1714	0.1342	0.1030	0.0774	0.0569
0.94 \	0.0842	0.1967	0.2589	0.2392	0.1930	0.1493	0.1127	0.0829	0.0595	0.0416
0.93 \	0.0835	0.1909	0.2462	0.2224	0.1746	0.1308	0.0952	0.0672	0.0461	0.0307
0.92 \	0.0829	0.1854	0.2345	0.2073	0.1584	0.1151	0.0808	0.0548	0.0359	0.0228
0.91 \	0.0822	0.1802	0.2238	0.1937	0.1442	0.1015	0.0688	0.0448	0.0281	0.0169
0.90 \	0.0816	0.1752	0.2139	0.1814	0.1316	0.0899	0.0588	0.0368	0.0221	0.0127
0.80 \	0.0746	0.1353	0.1420	0.0998	0.0568	0.0289	0.0134	0.0057	0.0022	0.0008
0.70 \	0.0630	0.1064	0.0979	0.0581	0.0263	0.0101	0.0034	0.0010	0.0002	0.0001
0.60 \	0.0621	0.0841	0.0684	0.0347	0.0126	0.0036	0.0009	0.0002	0.0000	0.0000
0.50 \	0.0569	0.0668	0.0431	0.0209	0.0061	0.0013	0.0002	0.0000	0.0000	0.0000
0.40 \	0.0526	0.0534	0.0339	0.0127	0.0030	0.0005	0.0001	0.0000	0.0000	0.0000
0.30 \	0.0473	0.0435	0.0242	0.0078	0.0015	0.0002	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0469	0.0366	0.0179	0.0049	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0454	0.0326	0.0144	0.0034	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0450	0.0313	0.0133	0.0030	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000

GAMMA''(RHO,DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.99 \	0.0132	0.0188	0.0101	0.0020	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.98 \	0.0130	0.0245	0.0128	0.0025	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.97 \	0.0213	0.0283	0.0144	0.0027	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.96 \	0.0239	0.0310	0.0155	0.0029	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.95 \	0.0260	0.0331	0.0163	0.0030	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.94 \	0.0279	0.0347	0.0169	0.0031	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.93 \	0.0294	0.0361	0.0173	0.0031	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.92 \	0.0308	0.0372	0.0177	0.0032	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.91 \	0.0320	0.0380	0.0179	0.0032	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.90 \	0.0331	0.0388	0.0181	0.0032	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.80 \	0.0396	0.0411	0.0179	0.0030	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.70 \	0.0422	0.0396	0.0163	0.0027	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.60 \	0.0430	0.0367	0.0144	0.0023	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.50 \	0.0431	0.0335	0.0125	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.40 \	0.0427	0.0304	0.0108	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.30 \	0.0423	0.0278	0.0094	0.0014	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0419	0.0259	0.0084	0.0012	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0416	0.0247	0.0077	0.0011	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0416	0.0244	0.0076	0.0011	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

BETA'(RHO,DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0850	0.2580	0.4255	0.4897	0.4995	0.5001	0.5001	0.5002	0.5002	0.5002
0.99 \	0.0969	0.2982	0.5033	0.6001	0.6375	0.6643	0.6897	0.7142	0.7377	0.7602
0.98 \	0.1018	0.3145	0.5347	0.6438	0.6908	0.7257	0.7580	0.7881	0.8159	0.8413
0.97 \	0.1055	0.3269	0.5582	0.6760	0.7292	0.7688	0.8044	0.8364	0.8648	0.8897
0.96 \	0.1086	0.3371	0.5776	0.7020	0.7595	0.8019	0.8389	0.8711	0.8985	0.9214
0.95 \	0.1113	0.3461	0.5942	0.7240	0.7847	0.8286	0.8658	0.8970	0.9226	0.9431
0.94 \	0.1137	0.3540	0.6088	0.7432	0.8060	0.8506	0.8873	0.9171	0.9405	0.9584
0.93 \	0.1159	0.3612	0.6220	0.7601	0.8245	0.8692	0.9048	0.9328	0.9539	0.9693
0.92 \	0.1180	0.3678	0.6339	0.7752	0.8406	0.8849	0.9192	0.9452	0.9641	0.9772
0.91 \	0.1198	0.3739	0.6449	0.7888	0.8548	0.8984	0.9312	0.9552	0.9719	0.9831
0.90 \	0.1216	0.3795	0.6550	0.8012	0.8675	0.9101	0.9412	0.9632	0.9779	0.9873
0.80 \	0.1351	0.4218	0.7267	0.8826	0.9422	0.9711	0.9866	0.9943	0.9978	0.9992
0.70 \	0.1442	0.4492	0.7693	0.9239	0.9727	0.9899	0.9966	0.9990	0.9998	0.9999
0.60 \	0.1510	0.4686	0.7968	0.9470	0.9863	0.9962	0.9991	0.9998	1.0000	1.0000
0.50 \	0.1562	0.4827	0.8152	0.9602	0.9923	0.9979	0.9994	0.9999	1.0000	1.0000
0.40 \	0.1602	0.4930	0.8277	0.9679	0.9944	0.9974	0.9989	0.9998	1.0000	1.0000
0.30 \	0.1631	0.5004	0.8360	0.9723	0.9949	0.9961	0.9980	0.9996	1.0000	1.0000
0.20 \	0.1651	0.5054	0.8414	0.9749	0.9947	0.9948	0.9973	0.9995	1.0000	1.0000
0.10 \	0.1664	0.5084	0.8445	0.9763	0.9945	0.9942	0.9971	0.9996	1.0001	1.0001
0.00 \	0.1668	0.5095	0.8459	0.9772	0.9951	0.9945	0.9976	1.0003	1.0009	1.0010

2. Errors of II. and III. type, k = 2.56

GAMMA'(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0298	0.1439	0.3350	0.4625	0.4962	0.4997	0.4999	0.4998	0.4998	0.4998
0.99 \	0.0309	0.1341	0.2843	0.3613	0.3594	0.3356	0.3103	0.2858	0.2623	0.2398
0.98 \	0.0310	0.1285	0.2617	0.3203	0.3064	0.2742	0.2420	0.2119	0.1841	0.1587
0.97 \	0.0309	0.1237	0.2442	0.2899	0.2682	0.2311	0.1956	0.1636	0.1352	0.1103
0.96 \	0.0307	0.1194	0.2294	0.2652	0.2380	0.1980	0.1611	0.1289	0.1015	0.0786
0.95 \	0.0305	0.1153	0.2164	0.2442	0.2130	0.1713	0.1342	0.1030	0.0774	0.0569
0.94 \	0.0303	0.1116	0.2049	0.2259	0.1917	0.1493	0.1127	0.0829	0.0595	0.0416
0.93 \	0.0300	0.1080	0.1944	0.2097	0.1734	0.1308	0.0952	0.0672	0.0461	0.0307
0.92 \	0.0298	0.1046	0.1847	0.1952	0.1573	0.1150	0.0808	0.0548	0.0359	0.0228
0.91 \	0.0295	0.1014	0.1758	0.1822	0.1431	0.1015	0.0688	0.0448	0.0281	0.0169
0.90 \	0.0292	0.0983	0.1675	0.1703	0.1305	0.0898	0.0588	0.0368	0.0221	0.0127
0.80 \	0.0258	0.0731	0.1073	0.0920	0.0561	0.0289	0.0134	0.0057	0.0022	0.0008
0.70 \	0.0225	0.0545	0.0707	0.0522	0.0258	0.0100	0.0034	0.0010	0.0002	0.0001
0.60 \	0.0194	0.0404	0.0467	0.0300	0.0122	0.0036	0.0009	0.0002	0.0000	0.0000
0.50 \	0.0167	0.0297	0.0306	0.0173	0.0058	0.0013	0.0002	0.0000	0.0000	0.0000
0.40 \	0.0144	0.0216	0.0198	0.0099	0.0028	0.0005	0.0001	0.0000	0.0000	0.0000
0.30 \	0.0125	0.0156	0.0126	0.0055	0.0013	0.0002	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0112	0.0115	0.0080	0.0031	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0104	0.0092	0.0055	0.0019	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0101	0.0084	0.0048	0.0015	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000

GAMMA''(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.99 \	0.0061	0.0159	0.0156	0.0056	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000
0.98 \	0.0082	0.0207	0.0196	0.0069	0.0009	0.0000	0.0000	0.0000	0.0000	0.0000
0.97 \	0.0097	0.0237	0.0219	0.0076	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.96 \	0.0108	0.0258	0.0234	0.0079	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.95 \	0.0117	0.0274	0.0244	0.0082	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.94 \	0.0124	0.0286	0.0251	0.0083	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.93 \	0.0131	0.0295	0.0256	0.0084	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.92 \	0.0136	0.0301	0.0258	0.0084	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.91 \	0.0140	0.0306	0.0260	0.0084	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.90 \	0.0144	0.0310	0.0260	0.0083	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000
0.80 \	0.0162	0.0306	0.0237	0.0072	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000
0.70 \	0.0161	0.0271	0.0198	0.0058	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000
0.60 \	0.0152	0.0229	0.0158	0.0045	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000
0.50 \	0.0140	0.0187	0.0122	0.0034	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
0.40 \	0.0127	0.0149	0.0092	0.0024	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
0.30 \	0.0115	0.0118	0.0067	0.0017	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0106	0.0094	0.0049	0.0012	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0100	0.0080	0.0038	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0099	0.0075	0.0035	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

BETA'(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0298	0.1439	0.3351	0.4626	0.4964	0.5000	0.5001	0.5002	0.5002	0.5002
0.99 \	0.0348	0.1696	0.4013	0.5694	0.6340	0.6641	0.6897	0.7142	0.7377	0.7602
0.98 \	0.0368	0.1800	0.4279	0.6117	0.6871	0.7256	0.7580	0.7881	0.8159	0.8413
0.97 \	0.0383	0.1878	0.4478	0.6427	0.7254	0.7686	0.8044	0.8364	0.8648	0.8897
0.96 \	0.0396	0.1942	0.4641	0.6679	0.7557	0.8017	0.8389	0.8711	0.8985	0.9214
0.95 \	0.0407	0.1998	0.4780	0.6891	0.7807	0.8284	0.8658	0.8970	0.9226	0.9431
0.94 \	0.0417	0.2047	0.4903	0.7075	0.8020	0.8505	0.8873	0.9171	0.9405	0.9584
0.93 \	0.0426	0.2092	0.5012	0.7237	0.8203	0.8690	0.9048	0.9328	0.9539	0.9693
0.92 \	0.0434	0.2132	0.5111	0.7382	0.8364	0.8847	0.9192	0.9452	0.9641	0.9772
0.91 \	0.0441	0.2170	0.5202	0.7513	0.8505	0.8983	0.9312	0.9552	0.9719	0.9831
0.90 \	0.0448	0.2204	0.5285	0.7631	0.8631	0.9099	0.9412	0.9632	0.9779	0.9873
0.80 \	0.0499	0.2453	0.5864	0.8403	0.9374	0.9709	0.9866	0.9943	0.9978	0.9992
0.70 \	0.0532	0.2604	0.6191	0.8787	0.9675	0.9897	0.9966	0.9990	0.9998	0.9999
0.60 \	0.0554	0.2702	0.6391	0.8995	0.9810	0.9961	0.9991	0.9998	1.0000	1.0000
0.50 \	0.0569	0.2768	0.6516	0.9111	0.9872	0.9983	0.9994	0.9996	0.9998	1.0000
0.40 \	0.0579	0.2811	0.6594	0.9176	0.9901	0.9987	0.9984	0.9983	0.9993	0.9999
0.30 \	0.0586	0.2839	0.6641	0.9212	0.9914	0.9985	0.9968	0.9964	0.9985	0.9998
0.20 \	0.0590	0.2856	0.6669	0.9232	0.9920	0.9982	0.9955	0.9948	0.9978	0.9996
0.10 \	0.0592	0.2866	0.6634	0.9242	0.9923	0.9981	0.9948	0.9941	0.9975	0.9996
0.00 \	0.0593	0.2869	0.6639	0.9246	0.9926	0.9983	0.9949	0.9940	0.9975	0.9998

3. Errors of II. and III. type , k = 3.00

GAMMA'(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0114	0.0793	0.2500	0.4206	0.4885	0.4992	0.4995	0.4998	0.4998	0.4998
0.99 \	0.0120	0.0749	0.2140	0.3299	0.3541	0.3353	0.3103	0.2853	0.2623	0.2398
0.98 \	0.0121	0.0719	0.1971	0.2924	0.3019	0.2739	0.2420	0.2119	0.1841	0.1587
0.97 \	0.0121	0.0693	0.1838	0.2644	0.2642	0.2309	0.1956	0.1636	0.1352	0.1103
0.96 \	0.0120	0.0668	0.1724	0.2415	0.2343	0.1978	0.1611	0.1289	0.1015	0.0786
0.95 \	0.0119	0.0644	0.1623	0.2221	0.2095	0.1711	0.1342	0.1030	0.0774	0.0569
0.94 \	0.0118	0.0622	0.1533	0.2051	0.1885	0.1491	0.1127	0.0829	0.0595	0.0416
0.93 \	0.0117	0.0601	0.1450	0.1901	0.1703	0.1306	0.0952	0.0672	0.0461	0.0307
0.92 \	0.0116	0.0581	0.1374	0.1766	0.1545	0.1149	0.0808	0.0548	0.0359	0.0228
0.91 \	0.0115	0.0561	0.1304	0.1644	0.1405	0.1013	0.0688	0.0448	0.0281	0.0169
0.90 \	0.0113	0.0543	0.1239	0.1534	0.1280	0.0897	0.0588	0.0368	0.0221	0.0127
0.80 \	0.0098	0.0388	0.0764	0.0806	0.0544	0.0288	0.0134	0.0057	0.0022	0.0008
0.70 \	0.0082	0.0276	0.0479	0.0442	0.0246	0.0100	0.0034	0.0010	0.0002	0.0001
0.60 \	0.0067	0.0194	0.0299	0.0243	0.0114	0.0036	0.0009	0.0002	0.0000	0.0000
0.50 \	0.0055	0.0133	0.0183	0.0133	0.0053	0.0013	0.0002	0.0000	0.0000	0.0000
0.40 \	0.0045	0.0089	0.0109	0.0071	0.0024	0.0005	0.0001	0.0000	0.0000	0.0000
0.30 \	0.0037	0.0059	0.0063	0.0036	0.0011	0.0002	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0031	0.0039	0.0035	0.0018	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0028	0.0028	0.0021	0.0009	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0027	0.0024	0.0016	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

GAMMA''(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.99 \	0.0028	0.0113	0.0170	0.0095	0.0020	0.0002	0.0000	0.0000	0.0000	0.0000
0.98 \	0.0037	0.0146	0.0213	0.0116	0.0023	0.0002	0.0000	0.0000	0.0000	0.0000
0.97 \	0.0043	0.0166	0.0237	0.0126	0.0025	0.0002	0.0000	0.0000	0.0000	0.0000
0.96 \	0.0046	0.0180	0.0251	0.0132	0.0026	0.0002	0.0000	0.0000	0.0000	0.0000
0.95 \	0.0052	0.0189	0.0261	0.0135	0.0026	0.0002	0.0000	0.0000	0.0000	0.0000
0.94 \	0.0055	0.0196	0.0266	0.0136	0.0026	0.0002	0.0000	0.0000	0.0000	0.0000
0.93 \	0.0058	0.0201	0.0269	0.0136	0.0026	0.0002	0.0000	0.0000	0.0000	0.0000
0.92 \	0.0060	0.0205	0.0270	0.0135	0.0026	0.0002	0.0000	0.0000	0.0000	0.0000
0.91 \	0.0062	0.0207	0.0270	0.0134	0.0025	0.0002	0.0000	0.0000	0.0000	0.0000
0.90 \	0.0063	0.0208	0.0268	0.0132	0.0025	0.0002	0.0000	0.0000	0.0000	0.0000
0.80 \	0.0067	0.0194	0.0228	0.0106	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000
0.70 \	0.0064	0.0161	0.0177	0.0078	0.0014	0.0001	0.0000	0.0000	0.0000	0.0000
0.60 \	0.0057	0.0126	0.0130	0.0056	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000
0.50 \	0.0049	0.0095	0.0092	0.0038	0.0006	0.0000	0.0000	0.0000	0.0000	0.0000
0.40 \	0.0041	0.0069	0.0062	0.0025	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000
0.30 \	0.0035	0.0049	0.0040	0.0015	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0030	0.0034	0.0025	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0027	0.0026	0.0016	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0026	0.0023	0.0014	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

BETA'(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0114	0.0793	0.2500	0.4207	0.4887	0.4994	0.5001	0.5002	0.5002	0.5002
0.99 \	0.0135	0.0950	0.3030	0.5210	0.6251	0.6635	0.6897	0.7142	0.7377	0.7602
0.98 \	0.0144	0.1013	0.3242	0.5605	0.6777	0.7249	0.7580	0.7881	0.8159	0.8413
0.97 \	0.0151	0.1060	0.3399	0.5896	0.7156	0.7679	0.8043	0.8364	0.8648	0.8897
0.96 \	0.0156	0.1098	0.3523	0.6130	0.7456	0.8010	0.8389	0.8711	0.8985	0.9214
0.95 \	0.0161	0.1132	0.3637	0.6327	0.7703	0.8277	0.8658	0.8970	0.9226	0.9431
0.94 \	0.0165	0.1161	0.3733	0.6498	0.7913	0.8497	0.8873	0.9171	0.9405	0.9584
0.93 \	0.0168	0.1187	0.3819	0.6649	0.8095	0.8682	0.9048	0.9328	0.9539	0.9693
0.92 \	0.0172	0.1211	0.3896	0.6783	0.8253	0.8840	0.9192	0.9452	0.9641	0.9772
0.91 \	0.0175	0.1233	0.3966	0.6903	0.8393	0.8975	0.9312	0.9552	0.9719	0.9831
0.90 \	0.0178	0.1252	0.4030	0.7012	0.8517	0.9092	0.9412	0.9632	0.9779	0.9873
0.80 \	0.0198	0.1392	0.4464	0.7713	0.9247	0.9700	0.9865	0.9943	0.9978	0.9992
0.70 \	0.0210	0.1471	0.4697	0.8050	0.9540	0.9888	0.9966	0.9990	0.9998	0.9999
0.60 \	0.0217	0.1519	0.4831	0.8226	0.9668	0.9951	0.9991	0.9998	1.0000	1.0000
0.50 \	0.0222	0.1549	0.4908	0.8319	0.9726	0.9974	0.9997	0.9998	0.9997	0.9999
0.40 \	0.0224	0.1566	0.4953	0.8367	0.9752	0.9982	0.9997	0.9990	0.9984	0.9991
0.30 \	0.0226	0.1576	0.4977	0.8392	0.9764	0.9984	0.9993	0.9974	0.9960	0.9978
0.20 \	0.0227	0.1582	0.4990	0.8404	0.9769	0.9985	0.9991	0.9964	0.9942	0.9966
0.10 \	0.0227	0.1584	0.4996	0.8410	0.9771	0.9986	0.9990	0.9959	0.9935	0.9962
0.00 \	0.0228	0.1585	0.4997	0.8412	0.9772	0.9986	0.9991	0.9960	0.9934	0.9960

4. Errors of II. and III. type , k = 3.29

GAMMA'(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0055	0.0493	0.1929	0.3805	0.4781	0.4982	0.4998	0.4998	0.4998	0.4998
0.99 \	0.0059	0.0470	0.1663	0.2995	0.3469	0.3346	0.3103	0.2858	0.2623	0.2398
0.98 \	0.0059	0.0451	0.1532	0.2654	0.2956	0.2734	0.2419	0.2119	0.1841	0.1587
0.97 \	0.0059	0.0434	0.1427	0.2397	0.2586	0.2304	0.1956	0.1636	0.1352	0.1103
0.96 \	0.0059	0.0413	0.1336	0.2187	0.2292	0.1974	0.1611	0.1289	0.1015	0.0786
0.95 \	0.0058	0.0403	0.1256	0.2008	0.2049	0.1707	0.1342	0.1030	0.0774	0.0569
0.94 \	0.0058	0.0389	0.1184	0.1852	0.1842	0.1488	0.1127	0.0829	0.0595	0.0416
0.93 \	0.0057	0.0375	0.1118	0.1713	0.1663	0.1303	0.0952	0.0672	0.0461	0.0307
0.92 \	0.0057	0.0361	0.1057	0.1589	0.1507	0.1146	0.0807	0.0543	0.0359	0.0228
0.91 \	0.0056	0.0349	0.1000	0.1476	0.1369	0.1011	0.0683	0.0448	0.0281	0.0169
0.90 \	0.0055	0.0336	0.0948	0.1374	0.1246	0.0894	0.0588	0.0368	0.0221	0.0127
0.80 \	0.0047	0.0234	0.0566	0.0705	0.0524	0.0286	0.0134	0.0057	0.0022	0.0008
0.70 \	0.0038	0.0160	0.0342	0.0374	0.0233	0.0099	0.0033	0.0010	0.0002	0.0001
0.60 \	0.0030	0.0103	0.0204	0.0198	0.0105	0.0035	0.0009	0.0002	0.0000	0.0000
0.50 \	0.0024	0.0071	0.0118	0.0103	0.0047	0.0013	0.0002	0.0000	0.0000	0.0000
0.40 \	0.0019	0.0045	0.0066	0.0052	0.0021	0.0005	0.0001	0.0000	0.0000	0.0000
0.30 \	0.0015	0.0028	0.0036	0.0025	0.0009	0.0002	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0012	0.0017	0.0018	0.0011	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0010	0.0011	0.0010	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0010	0.0009	0.0007	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

GAMMA''(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.99 \	0.0015	0.0080	0.0163	0.0122	0.0034	0.0003	0.0000	0.0000	0.0000	0.0000
0.98 \	0.0020	0.0104	0.0203	0.0147	0.0040	0.0004	0.0000	0.0000	0.0000	0.0000
0.97 \	0.0023	0.0118	0.0224	0.0159	0.0042	0.0004	0.0000	0.0000	0.0000	0.0000
0.96 \	0.0026	0.0127	0.0237	0.0165	0.0043	0.0004	0.0000	0.0000	0.0000	0.0000
0.95 \	0.0028	0.0133	0.0244	0.0168	0.0043	0.0004	0.0000	0.0000	0.0000	0.0000
0.94 \	0.0029	0.0138	0.0248	0.0169	0.0043	0.0004	0.0000	0.0000	0.0000	0.0000
0.93 \	0.0030	0.0141	0.0250	0.0168	0.0043	0.0004	0.0000	0.0000	0.0000	0.0000
0.92 \	0.0031	0.0143	0.0250	0.0166	0.0042	0.0004	0.0000	0.0000	0.0000	0.0000
0.91 \	0.0032	0.0144	0.0248	0.0164	0.0041	0.0004	0.0000	0.0000	0.0000	0.0000
0.90 \	0.0033	0.0144	0.0246	0.0161	0.0040	0.0004	0.0000	0.0000	0.0000	0.0000
0.80 \	0.0034	0.0128	0.0199	0.0122	0.0029	0.0003	0.0000	0.0000	0.0000	0.0000
0.70 \	0.0031	0.0102	0.0146	0.0085	0.0020	0.0002	0.0000	0.0000	0.0000	0.0000
0.60 \	0.0026	0.0076	0.0101	0.0057	0.0013	0.0001	0.0000	0.0000	0.0000	0.0000
0.50 \	0.0022	0.0054	0.0067	0.0036	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000
0.40 \	0.0018	0.0037	0.0042	0.0021	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000
0.30 \	0.0014	0.0024	0.0025	0.0012	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
0.20 \	0.0012	0.0015	0.0014	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.10 \	0.0010	0.0011	0.0008	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.00 \	0.0010	0.0009	0.0006	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

BETA'(RHO/DELTA)

0.00 \	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1.00 \	0.0055	0.0493	0.1930	0.3806	0.4783	0.4984	0.5001	0.5002	0.5002	0.5002
0.99 \	0.0066	0.0596	0.2359	0.4738	0.6128	0.6623	0.6896	0.7142	0.7377	0.7602
0.98 \	0.0071	0.0638	0.2530	0.5105	0.6647	0.7237	0.7580	0.7881	0.8159	0.8413
0.97 \	0.0074	0.0669	0.2657	0.5373	0.7020	0.7666	0.8043	0.8364	0.8648	0.8897
0.96 \	0.0077	0.0694	0.2760	0.5590	0.7315	0.7997	0.8388	0.8710	0.8985	0.9214
0.95 \	0.0079	0.0715	0.2847	0.5772	0.7558	0.8263	0.8657	0.8970	0.9226	0.9431
0.94 \	0.0081	0.0734	0.2924	0.5929	0.7765	0.8483	0.8873	0.9171	0.9405	0.9584
0.93 \	0.0083	0.0751	0.2991	0.6067	0.7943	0.8668	0.9048	0.9328	0.9539	0.9693
0.92 \	0.0085	0.0766	0.3052	0.6189	0.8099	0.8825	0.9192	0.9452	0.9641	0.9772
0.91 \	0.0086	0.0780	0.3107	0.6299	0.8236	0.8960	0.9311	0.9552	0.9719	0.9831
0.90 \	0.0088	0.0793	0.3157	0.6398	0.8357	0.9076	0.9412	0.9632	0.9779	0.9873
0.80 \	0.0098	0.0880	0.3492	0.7028	0.9069	0.9683	0.9865	0.9943	0.9978	0.9992
0.70 \	0.0103	0.0927	0.3663	0.7323	0.9350	0.9869	0.9966	0.9990	0.9998	0.9999
0.60 \	0.0106	0.0953	0.3756	0.7470	0.9471	0.9932	0.9990	0.9998	1.0000	1.0000
0.50 \	0.0108	0.0969	0.3807	0.7544	0.9524	0.9954	0.9997	0.9999	0.9999	0.9998
0.40 \	0.0109	0.0977	0.3834	0.7581	0.9548	0.9962	0.9998	0.9996	0.9987	0.9982
0.30 \	0.0110	0.0982	0.3848	0.7599	0.9557	0.9965	0.9998	0.9991	0.9965	0.9952
0.20 \	0.0110	0.0984	0.3855	0.7607	0.9561	0.9966	0.9997	0.9985	0.9946	0.9931
0.10 \	0.0110	0.0985	0.3857	0.7610	0.9563	0.9966	0.9998	0.9986	0.9946	0.9922
0.00 \	0.0110	0.0985	0.3858	0.7611	0.9564	0.9967	0.9998	0.9984	0.9941	0.9923