Nonparametric segmentation of 2D curves

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Abstract

There is a considerable interest in the computer vision and computer graphics community to generate a suitable symbolic description of 2D curves. This paper presents a new approach to generate a compact description by a nonparametric segmentation algorithm. Important is that no thresholds are required to determine the segmentation, which best describes a 2D curve. The result is a symbolic description by a set of features of different order. The emphasis is on the used significance measure referring to the limits of an acceptable interpretableness of the reachable segmentation results. The proposed algorithm has a number of interesting properties: (1) independence *of the segmentation from any parameters, (2)* invariance *to geometric transformations, (3)* simplicity*, and (4)* efficiency *of the segmentation algorithm.*

1 Introduction

Segmentation of 2D curves is a common problem in computer vision and computer graphics. Numerous techniques have been proposed for generating a generalized description of a curve, mainly based on localization of points at which the 2D curve can be segmented, after which a 2D feature, mostly a straight line is fitted to the 2D curve between adjacent points. One of the most popular and effective algorithms is the algorithm simultaneously proposed by Duda and Hart [3] and Douglas and Peuker [2]. The problem thereby is the use of a threshold to control the recursive process of segmentation by determination the maximum distance between the curve and the approximation. The threshold is subjective and depends on a concrete application.

It was shown, that the approximation of a curve by a set of features of *different* order can be more significant than the approximation by only one typ of features (e. g. [16] and [17]). However, the determination of higher order representations is difficult to realize because of the increased complexity to approximate the curve by such a feature, the increased number of parameters and the ill-conditioned nature of the problem. Additionally most of the known methods to approximate straight lines cannot easily be extended to other feature types because of the specific nature of those methods. In [6] an approach is sketched to split a curve recursively. The algorithm works nonparametric in contrast to alternative approaches (e. g. [8, 9] and [12]). The segmentation is realized by comparing so-called *significance measures* (*SM*s), which are defined as the ratio of the maximum deviation of the curve from the feature primitive that describes the curve divided by the length of the feature primitive. That means the accepted deviation increases with the length of the feature primitive. Note, the *SM* bases on a pseudo-psychological model of perceptual significance.

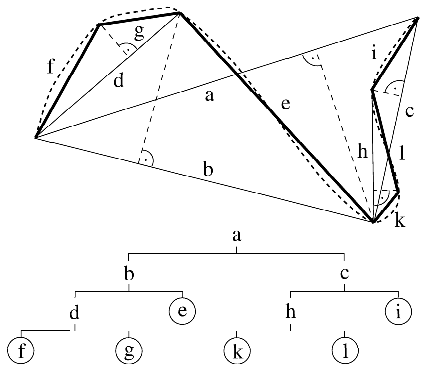

Fig. 1. Segmentation of a 2D curve by straight lines

The curve is splited into two at the point of maximum deviation between the approximation and the curve. In fig. 1 the recursive process is sketched. The final representation is the set of straight lines $\{f, g, e, k, l, i\}$. The approach proposed in this paper follows an extension of this idea of segmentation described in [10]. The authors proposed a technique for segmenting 2D curves into straight lines and elliptical arcs, with the disadvantage of *complete recursive subdivision* to form the binary tree. Then the whole tree is traversed ascending to select the best presentation of the curve by merging adjacent features and approximating the resulting part of the curve by a straight line or an elliptical arc. The approach proposed in this paper overcomes the *complete* recursive subdivision by using a very simple strategy to judge the actual curve segmentation with a decision to continue or to terminate the process.

2 Overview of the method

Let a curve $\mathcal{A} = \{a_i = (x_i, y_i), i \in [1, n]\}$ be given containing n connected points. The goal of the segmentation is to generate a set of approximations describing the curve best, whereby the set of possible approximations $\mathcal{H}_T$ consists of the two feature primitives: straight line l and elliptical arc e.

The curve is approximated simultaneously by the set of the two feature primitives. Each approximation is splited at the points of maximum deviation pos_T, $T = \{\textit{straight line } l, \textit{elliptical arc } e\}$. The resulting curve segments are also approximated by the set of possible feature primitives. Now, the decision, if the recursive process will be continued at this stage or terminated, bases on the significance measures, calculated for every approximation. Before in the following section the developed segmentation algorithm is described in details the substantial basics are sketched to approximate a straight line and an elliptical arc. The equations can be deduced from the general equation $\mathbf{F}(\mathbf{z}, \mathbf{a}) = \mathbf{z} \cdot \mathbf{a}$, with $\mathbf{z} = [A\ B\ C\ D\ E\ F]$ and $\mathbf{a} = \left[x^2\ xy\ y^2\ x\ y\ 1\right]^T$. The distance of a point a_i to the curve $\mathbf{F}(\mathbf{z}, \mathbf{a}) = 0$ is called algebraic distance $\mathbf{F}(\mathbf{z}, \mathbf{a}_i) = d_i$. A possible approach to determine an approximation is to minimize the sum of the algebraic distances $d_i\ \forall\, i \in [1, n]$ in the sense of least squares:

$$\Delta = \sum_{i=1}^{n} \mathbf{F}(\mathbf{z}, \mathbf{a}_i)^2 \rightarrow Min. \tag{1}$$

Note, a constraint must be formulated for the parameter vector $\mathbf{z}$ to avoid the trivial solution $\mathbf{z} = [0\ 0\ 0\ 0\ 0\ 0]$ and to guarantee, that any multiple of a solution $\mathbf{a}$ represents the same approximation.

2.1 Approximation of straight lines

The general straight line is described implicitly by $Dx + Ey + F = 0$. To avoid the trivial solution the used constraint is $D^2 + E^2 = 1$. The advantage of this constraint is that the algebraic distance is identically to the Euclidean distance. Thus, the condition of *invariance* of the approximation in relation to geometrical transformations is fulfilled. And, as a_1 and a_n are situated on the straight line[1], the solution is simple.

2.2 Approximation of elliptical arcs

The general elliptical arc is described implicitly by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Due to the assumption concerning a_1 and a_n two parameters, here D and E, can be determined directly.

$$E = -\frac{Ax_n^2 + Bx_ny_n + Cy_n^2 + Dx_n + F}{y_n}$$

$$D = \frac{A\left(x_n^2y_1 - x_1^2y_n\right) + B\left(x_ny_ny_1 - x_1y_ny_1\right) + C\left(y_n^2y_1 - y_ny_1^2\right) + F\left(y_1 - y_n\right)}{x_1y_n - x_ny_1}$$

To avoid the trivial solution for the remaining parameters the formulation of a constraint is necessary. Many authors suggest $||\mathbf{z}||^2 = 1$ or $F = -1$, Bookstein [1] proposes $A^2 + \frac{1}{2}B + C^2 = 1$. Note, the use of such a constraint do not cause necessarily something the approximation of elliptical arcs. But, if the constraint

$$(\mathbf{z}')^T \mathbf{C}\mathbf{z}' = AC - 4B = 1 \tag{2}$$

with $\mathbf{z}' = [A\ B\ C\ F]$ is used the numerical solution of the nonlinear equation system describes the parameter set of the elliptical arc with a restriction, which is optimal to the points $a_i \in \mathcal{A}$, $i \in [1, n]$. The approximation of an elliptical arc is guaranteed. Δ is defined by

$$\Delta = \sum_{\substack{a_i \in \mathcal{A} \\ i \in [1,n]}} \Big[A\left(x_i^2 + a_{10}x_i - a_{01}y_i\right) + B\left(x_iy_i + b_{10}x_i - b_{01}y_i\right) + C\left(y_i^2 + c_{10}x_i - c_{01}y_i\right) + F\left(1 + f_{10}x_i - f_{01}y_i\right)\Big]^2 \rightarrow Minimum \tag{3}$$

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[1] The assumption that a_1 and a_n are situated on the approximation is motivated by the objective that the set of approximated features, that means the symbolic description of a curve, is connected.

with

$$a_{10} = \frac{x_n^2 y_1 - x_1^2 y_n}{x_1 y_n - x_n y_1}\,, \qquad b_{10} = \frac{x_n y_1 y_n - x_1 y_1 y_n}{x_1 y_n - x_n y_1}\,, \qquad c_{10} = \frac{y_1 y_n^2 - y_1^2 y_n}{x_1 y_n - x_n y_1}$$

$$a_{01} = \frac{x_1^2 x_n - x_1 x_n^2}{x_1 y_n - x_n y_1}\,, \qquad b_{01} = \frac{x_1 x_n y_1 - x_1 x_n y_n}{x_1 y_n - x_n y_1}\,, \qquad c_{01} = \frac{x_n y_1^2 - x_1 y_n^2}{x_1 y_n - x_n y_1}$$

$$f_{10} = \frac{y_1 - y_n}{x_1 y_n - x_n y_1}\,, \qquad f_{01} = \frac{x_n - x_1}{x_1 y_n - x_n y_1}$$

Note, the Euclidean distance of a point a_i from an elliptical arc cannot be determined with direct methods. Thus an algebraic distance is used as approximation. In [1] and [14] it was shown that the algebraic distance with the secondary condition (eq. 2) is a) *similar* to the Euclidean distance and thus b) *invariant* in relation to geometrical transformations.
In [1] and [4] a very simple approach is proposed to solve the resulting equation system. The minimization is realized by solving the rank-deficient generalized eigenvalue problem: $\mathbf{B}\mathbf{z}' = \lambda \mathbf{C}\mathbf{z}'$ with the matrix $\mathbf{B} = \mathbf{U}^T\mathbf{U}$, the matrix

$$\mathbf{U} = \begin{bmatrix} x_1^2 + a_{10}x_1 - a_{01}y_1 & x_2^2 + a_{10}x_2 - a_{01}y_2 & \dots & x_n^2 + a_{10}x_n - a_{01}y_n \\ x_1y_1 + b_{10}x_1 - b_{01}y_1 & x_2y_2 + b_{10}x_2 - b_{01}y_2 & \dots & x_ny_n + b_{10}x_n - b_{01}y_n \\ y_1^2 + c_{10}x_1 - c_{01}y_1 & y_2^2 + c_{10}x_2 - c_{01}y_2 & \dots & y_n^2 + c_{10}x_n - c_{01}y_n \\ 1 + f_{10}x_1 - f_{01}y_1 & 1 + f_{10}x_2 - f_{01}y_2 & \dots & 1 + f_{10}x_n - f_{01}y_n \end{bmatrix}^T$$

and the matrix $\mathbf{C}$, which describes the constraint (eq. 2) respecting the matrix form $(\mathbf{z}')^T\,\mathbf{C}\mathbf{z}' = 1$. The solution of the eigensystem gives four eigenvalues. Each of the corresponding eigenvectors is a valid solution, but only one eigenvalue is $\lambda_i > 0$. The corresponding eigenvector is the solution for the approximation of an elliptical arc to the curve.

3 Segmentation algorithm

The principle structure of the algorithm is characterized as follows:

1. Approximate the curve by a straight line and an elliptical arc and calculate for every approximation the significance measure *SM*

$$SM_T = \frac{dev_{\max}(\mathcal{A}, \mathcal{H}_T)}{length\,(\mathcal{H}_T)} = \frac{\max\limits_{a_i \in \mathcal{A}} \left(\min\limits_{h_j \in \mathcal{H}_T} \| \,\mathbf{x}\,(a_i) - \mathbf{x}\,(h_j)\, \|^2 \right)}{length\,(\mathcal{H}_T)} \tag{4}$$

   with

$$\mathbf{x}\,(a_i) = \begin{pmatrix} x(a_i) \\ y(a_i) \end{pmatrix}\,, \quad x(a_i), y(a_i) \in \mathbb{N}$$

$$\mathbf{x}\,(h_j) = \begin{pmatrix} x(h_j) \\ y(h_j) \end{pmatrix}\,, \quad x(h_j), y(h_j) \in \mathbb{R}$$

2. Split the curve into either two or three at the points pos_T of maximum deviation.
$$pos_T = \arg\max_{a_i \in \mathcal{A}} \left(\min_{h_j \in \mathcal{H}_T} \| \mathbf{x}(a_i) - \mathbf{x}(h_j) \|^2 \right) \tag{5}$$
Normally, there are two different points pos_T, $T \in \{l, e\}$ and the curve is splited into three. If $pos_l = pos_e$ the curve is splited into two.
3. Approximate every new curve segment by straight line and elliptical arc. It can deduced from fig. 2, that generally 12 different decompositions of the curve (dotted line) are possible.
4. Calculate for every approximation the significance measure SM
5. Use the calculated significance measures as criterion for the acceptance of the current segmentation.
   5.1 if the current segmentation is accepted select the combination of approximations, which describes the given curve segment best and continue with step 2,
   5.2 else store the last accepted segmentation and terminate the process.

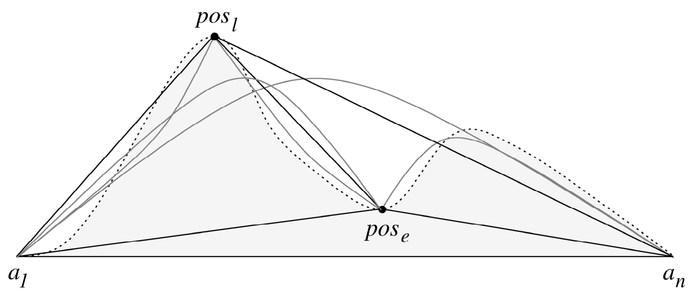

Fig. 2. Set of all approximations for a curve

With respect to step 5 the decision if the process of segmentation is continued or terminated is based on the evaluation of the significance measures of two successive levels. Thereby, the used strategy to evaluate the current segmentation at level $[p]$ in context to the previous segmentation at level $[p-1]$ is of special interest. In the following it is emphasis on the formulation of a suitable strategy, which supports the selection of the best combination in the sense of an optimality criterion.
Some possible strategies are sketched. A current segmentation for the curve is accepted, if

(A) the lowest significance measure
$$SM^{[p]}_{\min} = \min_{(D,T)} \left({}_D SM^{[p]}_T \right)$$
is smaller than $SM^{[p-1]}$. Then the combination with the lowest and the second lowest SM is selected. Note, the index T determines the feature typ (straight line l or elliptical arc e) and the index D determines the curve

segment ($D = 1$, if (pos_1, pos_l) or (pos_1, pos_e), $D = 2$, if (pos_l, pos_n) or (pos_e, pos_n) and $D = 0$, if (pos_l, pos_e) or (pos_e, pos_l)).

(B) the lowest, arithmetically averaged significance measure

$$\overline{SM}^{[p]}_{\min} = \min_{i=1,\ldots,12} \left(\frac{1}{card(D,T)_i} \sum_{(D,T)_i} {}_D SM^{[p]}_T \right)$$

is smaller than $SM^{[p-1]}$. The segmentation is continued with the curve segments of the selected combination.

(C) the lowest, weighted averaged significance measure

$$\tilde{SM}^{[p]}_{\min} = \min_{i=1,\ldots,12} \left(\frac{\sum_{(D,T)_i} \left(w_T \cdot_D SM^{[p]}_T \right)}{\sum_{(k_i)} p_T} \right)$$

is smaller than $SM^{[p-1]}$. The segmentation is continued with the curve segments of the selected combination.

Extensive experiments led to the prediction that the "quality" of segmentation is only secondarily influenced by the selected strategy. Derived from this prediction strategy (A) is selected due its simplicity.

4 Experiments

The experimental investigations are concentrated on the problem of evaluating the proposed approach of segmentation a curve. Firstly, the performance of this approach is characterized by a number of criteria such as geometry and topological distortions. And secondly, the performance of the used significance measure is examined regarding the usefulness to evaluate approximations of different feature types.

4.1 Performance characterization

One of the most practical interest in assessing segmentation algorithms has been restricted to quantifying the deviation from the approximation to the curve. The two most common measures are the compression ratio and the accuracy. Firstly, Sarkar[11] combined the two measures as a ratio, producing a normalized figure of merit. Similar formulations were given by Held *et.al* [5] and Rosin and West [10]. Lowe [7] assumed, that the accepted deviation between curve and approximation increases. Another possibility (e. g. Rosin and West [10]) is to sum the normalized maximum deviation from each approximated segment. Ventura and Chen [15] assessed their algorithms with respect to a reference segmentation of an *optimal* algorithm. The advantage is that it enables to compare approximations with different numbers of segments. But remaining problems are, how to get the reference segmentation of an optimal algorithm and the quantification of the error.

This short overview shows, that the formulation of suitable criteria for the comparison with other approaches for segmenting a curve (e. g. [7, 8, 10] and [13]) becomes however somewhat more complex. The reason for it can be derived on one hand from the different approaches, on the other hand factors like the context which can be considered, the concrete task and the subjectivity influence of the selection of the criteria as well as their evaluation.

A substantial feature of a first group, the classical approaches, to segmentation is the integration of heuristics during the definition of concrete, for this type of segmentation necessary thresholds. In contrast to this the approach proposed here needs no control parameters. So an appropriate comparison becomes difficult on this basis, since for the determination of the necessary thresholds frequently complex heuristics are used.

The second group of approaches reduce itself to the approach proposed in [6]. The substantial features are the calculation of the significance measure for each approximation and their suitable comparison at different approximation levels. On the basis of this general structure the segmentation here represents a special variation and an appropriate comparison of the procedures appears realizable.

The formulation of the criteria for the performance characterization follows as far as possible the features defined by Rosin and West [10]:

(a) the *number* of approximated segments $\mathcal{H}_T$ of the curve $\mathcal{A}$

$$card\,(\mathcal{H}_T) = \sum_{m=1}^{M} 1 \tag{6}$$

(b) the *averaged accuracy* of all approximated segments

$$\overline{ISE} = \frac{1}{M} \sum_{m=1}^{M} (ISE)_m = \frac{1}{M} \sum_{m=1}^{M} \Big(dev_{\max}\,(\mathcal{A}, \mathcal{H}_T) \Big)^2 \tag{7}$$

(c) the *minimal accuracy* of all approximated segments, that means the maximal deviation of all (accepted) approximations to the curve

$$ISE_{\min} = \max_{m=1,\ldots,M} ISE_m = \max_{m=1,\ldots,M} \Big(dev_{\max}\,(\mathcal{A}, \mathcal{H}_T) \Big)^2 \tag{8}$$

(d) the *averaged significance measure*

$$\overline{SM} = \frac{1}{M} \sum_{m=1}^{M} (SM)_m \tag{9}$$

and the two ratios of compactness FOM_1 and FOM_2

(e) the ratio between number of approximated segments $card\,(\mathcal{H}_T)$ and the number of all connected points of the curve $card\,(\mathcal{A})$ multiplied by the averaged accuracy $\overline{ISE}$

$$FOM_1 = \frac{card\,(\mathcal{H}_T)}{card\,(\mathcal{A})} \cdot \overline{ISE} \tag{10}$$

(e) the ratio between the number of parameters of all approximated segments and the number of all connected points of the curve $card\,(\mathcal{A})$ multiplied by the averaged accuracy $\overline{ISE}$

$$FOM_2 = \frac{\sum_{\mathcal{H}_T} card\,(\mathcal{P}_T)}{card\,(\mathcal{A})} \cdot \overline{ISE} \tag{11}$$

The second ratio bases on the concept of minimum description length.

In table 1 the performance of the proposed approach to segment a curve is characterized by the features described above. For comparison the results of segmentation according to Rosin and West [10] and Lowe [7] are also entered into the table.

test image	*strategy*	$\mathcal{H}_T$ (*l*)	$\mathcal{H}_T$ (*e*)	$\overline{ISE}$	ISE_{min}	$\overline{SM}$	FOM_1	FOM_2
TENNIS	(A)	228	175	1.3962	6.4392	0.1235	33.3234	174.8801
	Rosin/West	297	58	1.3542	6.2534	0.1250	42.5968	165.9745
	Lowe	585	–	1.2521	9.9611	0.1286	50.9951	203.9805
		–	403	3.1251	50.9902	0.4052	40.3425	282.3977
ICCV32	(A)	287	198	1.6129	9.2980	0.2148	59.4853	306.6994
	Rosin/West	623	77	1.6270	7.5263	0.2509	72.7235	352.9167
	Lowe	572	–	1.6617	6.3881	0.2762	84.5815	338.3259
		–	180	3.1358	17.0294	0.1652	15.3900	107.7297
MUK_007	(A)	607	220	1.0457	9.2980	0.1814	110.9054	532.8120
	Rosin/West	742	167	1.0322	8.7345	0.1932	114.4157	543.6634
	Lowe	876	–	0.9953	6.3881	0.1825	121.9529	487.8117
		–	289	2.5785	28.2843	0.2946	36.7892	257.5241

Table 1. Calculated values for the features $card\,(\mathcal{H}_T)$ with (l): number of straight lines and (e): the number of elliptical arcs, $\overline{ISE}$, ISE_{min}, $\overline{SM}$, FOM_1 and FOM_2 for the three test images TENNIS, **ICCV32** and **MUK_007**

On the basis of the table 1 it can deduced that the segmentation of a 2D curve with the proposed approach is comparable to a segmentation according to [7] and [10]. The set of approximated segments describes the curve in an acceptable quality. The values for the features $\overline{SM}$, FOM_1 and FOM_2 are comparable, except the segmentation of a 2D curve by only elliptical arcs with Lowe's approach. The $\overline{ISE}$ and the ISE_{min} are features, where the values differ significantly.
The efficiency of the proposed approach with the strategy (A) is compared to the approaches by Rosin and West and by Lowe in table 2. Here, the total number of approximations (straight lines and elliptical arcs) to get the final symbolic description is measured.

test image / *strategy*	TENNIS (*l*)	TENNIS (*e*)	ICCV32 (*l*)	ICCV32 (*e*)	MUK_007 (*l*)	MUK_007 (*e*)
(A)	3844	3021	1655	1351	3319	1918
Rosin/West	8577	2537	6580	1832	4641	1281
Lowe	3078	–	1895	–	1852	–
	–	749	–	305	–	354

Table 2. Number of approximations to get the final symbolic description

It can be easily deduced, that using the proposed approach in the sum considerably less approximations are needed than using the approach by Rosin and West. This becomes very obvious, if the number of approximations of straight lines is compared. One reason for this is the complete decomposition of a curves by Rosin and West into a set of straight lines[2]. The total number of approximations of straight lines is sometimes higher and the total number of approximations of elliptical arcs is significantly higher than the total number derived by the approach of Lowe. Note: by the approach of Lowe segments are approximated by *only one feature primitive* (here: straight lines *or* elliptical arcs) in the same time.
In the figures 3, 4 and 5 the appropriate results of the segmentation of the iconic descriptions of the images TENNIS, ICCV32 and MUK_007 after the proposed approach, the approach of Rosin and West and the approach of Lowe (straight lines and elliptical arcs) are sketched.

4.2 Significance measure

The crucial measure to terminate or continue the curve segmentation is the significance measure. In [7] it was shown that the significance measure can be used for the evaluation of approximations of different feature types. The application of the significance measure in case of a simultaneous approximation of different feature types is verified in [10].
However, it is reasonable to assume from the results sketched in fig. 6, that for the used significance measure the limits of an acceptable interpretableness are visible. In the upper part of fig. 6b) the curve is approximated by only one elliptical arc instead of two elliptical arcs connected by one straight line. It can be assumed that the segmentation will be improved by applying an upper threshold for the used criterion for the acceptance *SM* of an actual segmentation. The result for applying such an upper threshold is sketched in fig. 6c).
The possibility to accept a suboptimal solution is higher than in the approaches to segment a curve sketched in Rosin and West [10] and Lowe [7]. The reason for this is the strategy connected with the curve decomposition. In contrast to the proposed approach the segmentation described in [10] and [7] works with only one current search direction and accordingly the probability is smaller that segmentation adapts to a local minimum.

[2] Note, the original software package of Rosin and West was used to generate comparable results. The complete package including the used images TENNIS and ICCV32 is available by **ftp://ftp.brunel.ac.uk/CompSci/Paul.Rosin/curves/**.

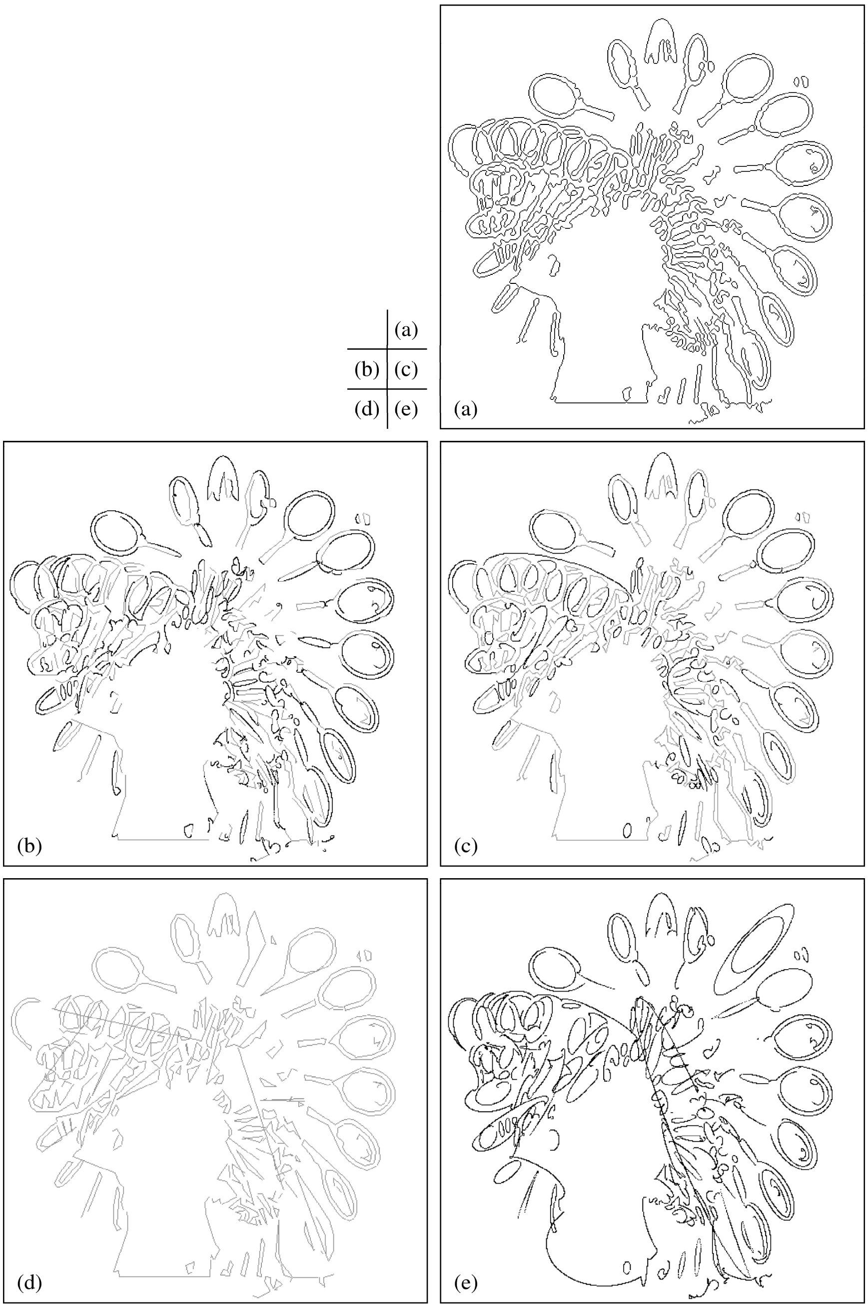

Fig. 3. Segmentation of the test image TENNIS, (a) iconic description, (b) new approach, (c) *Rosin/West*, (d) *Lowe* (only straight lines) and (e) *Lowe* (only elliptical arcs), gray: straight lines, black: elliptical arcs

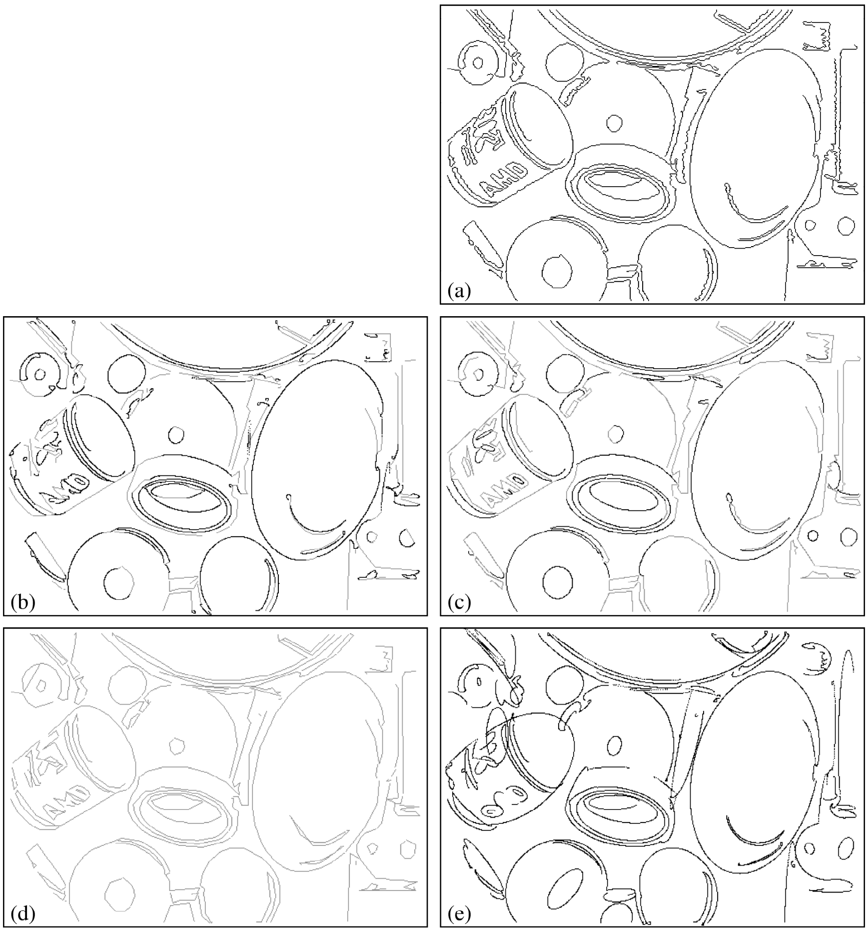

Fig. 4. Segmentation of the test image ICCV32, (a) iconic description, (b) new approach, (c) *Rosin/West*, (d) *Lowe* (only straight lines) and (e) *Lowe* (only elliptical arcs), gray: straight lines, black: elliptical arcs

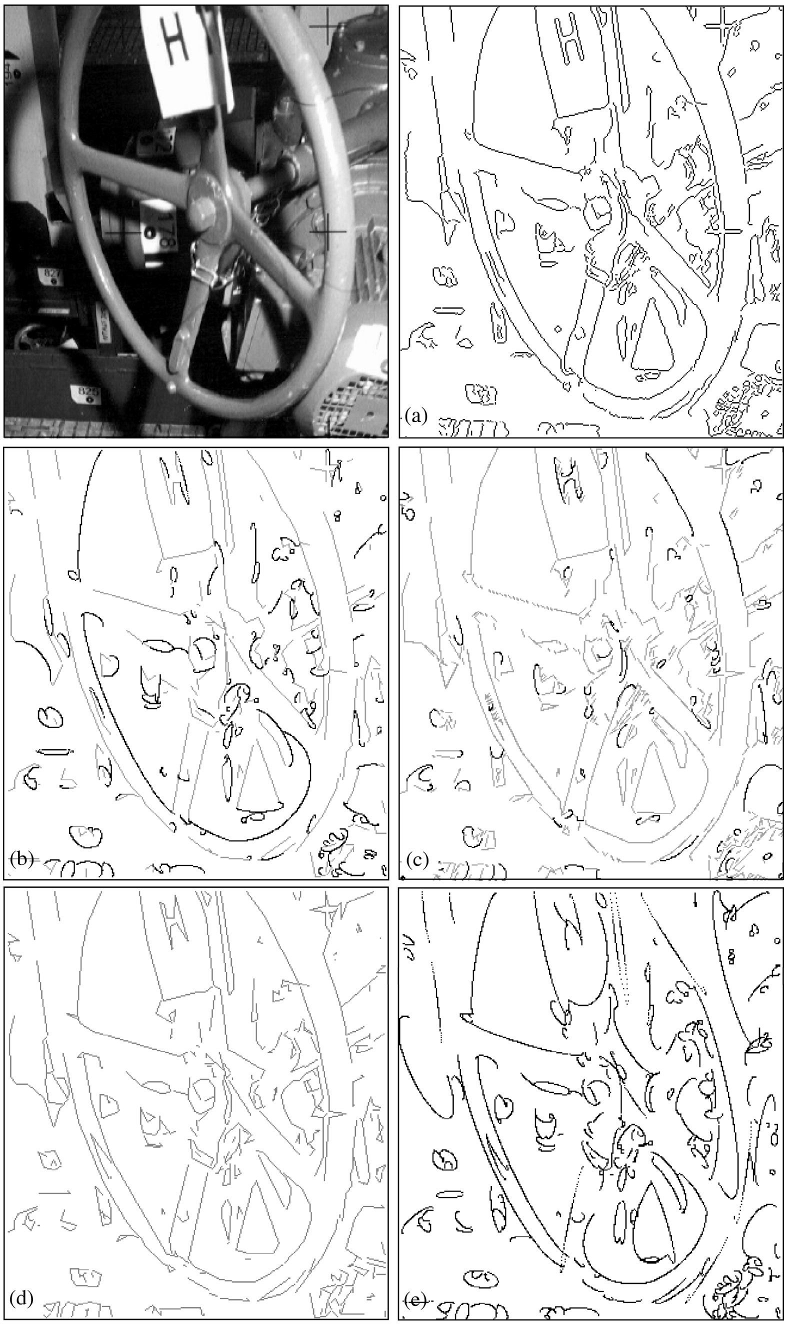

Fig. 5. Segmentation of the test image MUK_007, (a) iconic description, (b) new approach, (c) *Rosin/West*, (d) *Lowe* (only straight lines) and (e) *Lowe* (only elliptical arcs), gray: straight lines, black: elliptical arcs

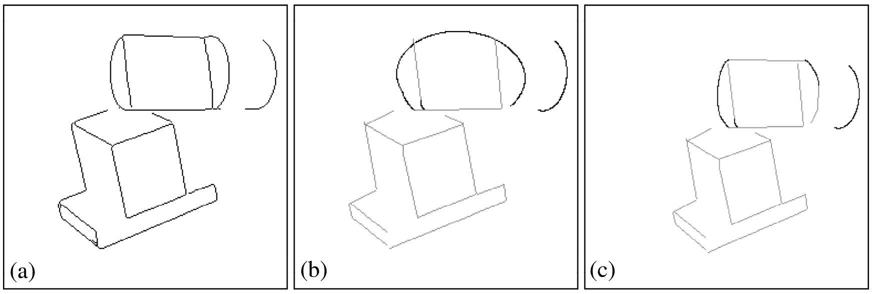

Fig. 6. Results of segmentation of 2D curves, (a) iconic description, (b) incorrect segmentation, (c) corrected segmentation (by an additional restriction for the used criterion SM), gray: straight lines, black: elliptical arcs

From these predictions it can be concluded that the incorrect segmentation of a curve documented exemplary in the fig. 6 can be attributed to the nonparametric formulation of curve segmentation.

5 Summary and Conclusions

On the basis of the results sketched in table 1, table 2 and the figures it can be derived that the proposed approach can be used very well for the segmentation of a curve.
The substantial qualitative difference to segmentation in accordance to [7] is the simultaneous approximation of straight lines *and* elliptical arcs to a curve and in accordance to [10] the modification of the strategy to be able to evaluate several possible segmentations of a curve and the associated formulation of a suitable rule to determine the best segmentation among those. The proposed approach overcomes the disadvantage of the approach by [10] of determining the complete binary tree. Here, *only the necessary tree* is determinated and so the new approach is more *efficient*. Note: the problem of merging adjacent features exists for all regarded approaches.
In summary: the proposed algorithm has a number of interesting properties: (1) *independence* of the segmentation from any parameters, (2) *invariance* to geometric transformations, (3) *simplicity*, and (4) *efficiency* of the segmentation algorithm.

References

1. F. L. Bookstein. Fitting conic sections to scattered data. *Computer Graphics and Image Processing*, 9:56–71, 1979.
2. D. H. Douglas and T. K. Peuker. Algorithms for the reduction of the number of points required to represent a digitized line or its caricature. *The Canadian Cartographer*, 10(2):112–122, 1973.

3. R. O. Duda and P. E. Hart. *Pattern Classification and Scene Analysis.* John Wiley and Sons, New York, 1973.
4. A. W. Fitzgibbon and R. B. Fisher. A buyer's guide to conic fitting. In *6th British Machine Vision Conference.* IEE, BMVA Press, 1995.
5. A. Held, K. Abe, and C. Arcelli. Towards a hierarchical contour description via dominant point detection. 24:942–949, 1994.
6. D. G. Lowe. *Perceptual Organization and Visual Recognition.* Kluwer Academic Publishers, Dortrecht-Boston-London, 1985.
7. D. G. Lowe. Three-dimensional object recognition from single two-dimensional images. *Artificial Intelligence*, 31:355–395, 1987.
8. T. Pavlidis. Curve fitting with conic splines. *ACM Transactions on Graphics*, 2(1):1–31, 1983.
9. S. Pollard and J. Porrill. Robust recovery of 3D ellipse data. In *British Machine Vision Conference*, pages 39–48. Spinger Verlag, 1992.
10. P. L. Rosin and G. A. W. West. Nonparametric segmentation of curves into various representations. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 17(12):1140–1153, December 1995.
11. D. Sarkar. A simple algorithm for detection of significant vertices for polygonal approximation of chain-coded curves. 14:959–964, 1993.
12. Y. Shirai. Recognition of real-world objects using edge cues. In A. R. Hanson and E. M. Riseman, editors, *Computer Vision Systems*, pages 353–362. Academic Press, New York, 1978.
13. Y. Shirai. *Three-Dimensional Computer Vision.* Springer Verlag, Berlin-Heidelberg-New York, 1987.
14. G. Taubin. Estimation of planar curves, surfaces and non-planar space curves defined by implicit equations, with applications to edge and range image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(11):1115–1138, November 1991.
15. J. A. Ventura and J. M. Chen. Segmentation of two-dimensional curve contours. 25:1129–1140, 1992.
16. G. A. W. West and P. L. Rosin. Techniques for segmenting image curves into meaningful descriptions. *Pattern Recognition*, 24(7):643–652, July 1991.
17. D. M. Wuenscher and K. L. Boyer. Robust contour decomposition using a constant curvature criterion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(1):41–51, January 1991.