analysis. But algebraic methods require idealized situations and therefore cannot be used for online self diagnosis.

- *Monte Carlo simulations* for determining bias and covariance matrices are able to handle all types of models, errors and configurations and therefore are not limited to specific application[1] but have the disadvantage that a generalization of their results is limited.
- *Numerical determination* of bias and covariance matrices share advantages and disadvantages of algebraic and simulation techniques. They assume comparably small errors and smooth functions and require the specification of a certain representative geometric setup. However, their results have more predictive power than simulations, they are applicable for online diagnosis and also for quite general nonlinear problems. The lack of representativity and predictive power may be compensated by linking their results to algebraically achieved ones.

We want to demonstrate this combination of algebraic and numerical analysis by choosing the analysis of the interrelation of calibration and SfM as an example.

2 Theoretical Background

2.1 Statistical Tools

This section describes the necessary tools from statistics. Stochastical variables are underscored, $\underline{x}$, if this seems necessary for clarification; the covariance matrix of a vector $\underline{x}$ is given by $\mathsf{D}(\underline{x}) = \boldsymbol{\Sigma}_{xx}$; a general distribution of $\underline{x}$ with first and second moments $\boldsymbol{\mu}_x$ and $\boldsymbol{\Sigma}_{xx}$ is designated with $\underline{x} \sim M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$.

Propagation of True and Random Errors. Given a nonlinear function $\underline{y} = \boldsymbol{f}(\underline{x})$ of a stochastic vector $\underline{x} = \boldsymbol{\mu}_x + \underline{e}_x$ with distribution $\underline{x} \sim M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ the following holds for small deterministic or stochastic errors:

$$\boldsymbol{\Delta y} = \boldsymbol{J}_{yx}\boldsymbol{\Delta x} \qquad \underline{y} \sim M(\boldsymbol{f}(\boldsymbol{\mu}_x), \boldsymbol{J}_{yx}\boldsymbol{\Sigma}_{xx}\boldsymbol{J}_{yx}^{\mathsf{T}}) \tag{1}$$

with the Jacobian $\boldsymbol{J}_{yx} = (\partial y_i / \partial x_j)$. Observe that this error propagation of means and variances is independent on the actual distribution, thus does not require the stochastical variables to be normally distributed. Possible mutual dependencies between the input variables are taken into account by the covariances on the off diagonal elements of the covariance matrix.

Parameter Estimation. Parameter estimation aims at inferring the most likely values of unknown parameters, collected in a U-vector $\boldsymbol{\beta} = (\beta_j)$ from observations, collected in a N-vector $\boldsymbol{y} = (y_i)$. It is most easily set up if an explicit expression for the observations as a function of the unknown parameters is available, specifying the first moments $\mathsf{E}(\underline{y}) = \boldsymbol{f}(\boldsymbol{\beta})$. Together with the covariance matrix $\mathsf{D}(\boldsymbol{y}) = \boldsymbol{\Sigma}_{yy}$ it establishes the nonlinear model, which, when making errors $\boldsymbol{e}$ in the observations explicit by $\underline{y} = \mathsf{E}(\underline{y}) + \underline{e}$, reads as

$$\underline{y} = \mathsf{E}(\underline{y}) + \underline{e}; \qquad \underline{y} =\sim M(\boldsymbol{f}(\boldsymbol{\beta}), \boldsymbol{\Sigma}_{yy}) \tag{2}$$

[1] even are able to handle all types of gross errors

We obtain a linear model using approximate values $\beta^{(0)}$: $\beta = \beta^{(0)} + \Delta\beta$ and the Jacobian $A = (\partial y_i / \partial \beta_j)$:

$$\underline{\Delta y} = \mathsf{E}(\underline{\Delta y}) + \underline{e} = \underline{y} - f(\beta^{(0)}) \sim M(A\Delta\beta, \Sigma_{yy}) \tag{3}$$

It is assumed to hold also for the estimated quantities, thus we have for the first moments:

$$\Delta y = A\widehat{\Delta\beta} + \widehat{e} \tag{4}$$

The best linear unbiased estimate (BLUE, cf. e. g. [MA76]) for the unknown parameters is given by $\widehat{\beta} = \beta^{(0)} + \widehat{\Delta\beta}$ with

$$\widehat{\Delta\beta} = (A^{\mathsf{T}}\Sigma_{yy}^{-1}A)^{-1}A^{\mathsf{T}}\Sigma_{yy}^{-1}\Delta y \quad \text{with} \quad \Sigma_{\widehat{\beta}\widehat{\beta}} = N^{-1} \doteq (A^{\mathsf{T}}\Sigma_{yy}^{-1}A)^{-1} \tag{5}$$

Observe that the covariance matrix of the estimated parameters is equal to the inverse of the normal equation matrix N which is always available when estimating parameters using – possibly weighted – least squares. It can be derived by error propagation from eq. (5) using eq. (1).

Self Calibration and SfM. The general setup of *self calibration* from several images, containing SfM as a special case, leads to a natural partitioning of the unknowns $\beta^{\mathsf{T}} = (t^{\mathsf{T}}, k^{\mathsf{T}}, s^{\mathsf{T}})$ into transformation parameters t representing the orientation, motion or external parameters of all camera stations, parameters k for describing the object structure, e. g. by coordinates of 3D-points and additional *calibration parameters* s for describing the the internal orientation (cf. sect. 3.1) of the camera. Instead of $\Delta y = A\widehat{\Delta\beta} + \widehat{e}$ (cf. eq. (4)) the linear model now may be written as

$$\Delta y = B\widehat{\Delta t} + C\widehat{\Delta k} + H\widehat{\Delta s} + \widehat{e} \tag{6}$$

This setup allows to flexibly model all types of deviations from the pin hole model, e. g. the case where two cameras of a stereo image sequence have different characteristics which keeps constant over a certain time interval.

Obviously eqs. (2) to (6) can be used for both, self calibration and SfM. The geometry in SfM is usually weak, this is why the parameters $\widehat{\Delta s}$ are often calibrated off-line.

2.2 Effect of Calibration Errors

The model (6) is also used to investigate the effect of errors and inaccuracies in the calibration on the 3D–coordinates $\widehat{\Delta k}$ and/or motion parameters $\widehat{\Delta t}$.

The Effect of Bias in Calibration Parameters. Assume the model (6) holds in SfM for estimating the unknown parameters β. If the calibration of the camera is incorrect, the observations y are biased by $H\Delta s$. With eq. (5) the influence on the unknown parameters, actually only containing t and k is given by

$$\Delta\widehat{\beta} = (A^{\mathsf{T}}\Sigma_{yy}^{-1}A)^{-1}A^{\mathsf{T}}\Sigma_{yy}^{-1}H\Delta s \tag{7}$$

This expression can be used to evaluate the influence of each parameter s_k on each unknown parameter β_j. For example we can evaluate the effect of changes in the focal distance in the camera to the 3D–reconstruction.

The effect of Uncertainty in Calibration Parameters. The uncertainty in the calibration parameters can be expressed by the second order moments $\underline{s} \sim M(s, \Sigma_{ss}(s))$. The effect on the coordinates can be evaluated the same way as the random observational errors under (1) by using the *actual covariance matrix of the observations*, which takes the random observational and the random systematic errors into account: From $\Delta\underline{y} = \mathsf{E}(\Delta y) + H\underline{\Delta s} + \underline{e}$ (eq. (6)) one obtains $\Sigma_{yy}^{\text{true}} = \Sigma_{yy} + H\Sigma_{ss}H^{\mathsf{T}}$ by error propagation. This leads to the actual covariance matrix of the unknown parameters

$$\Sigma_{\widehat{\beta\beta}}^{\text{true}} = \begin{pmatrix} \Sigma_{\hat{t}\hat{t}} & \Sigma_{\hat{t}\hat{k}} \\ \Sigma_{\hat{k}\hat{t}} & \Sigma_{\hat{k}\hat{k}} \end{pmatrix}^{\text{true}} = N^{-1}A^{\mathsf{T}}\Sigma_{yy}^{-1} \cdot \Sigma_{yy}^{\text{true}} \cdot \Sigma_{yy}^{-1}AN^{-1} \tag{8}$$

$$= N^{-1} + N^{-1}A^{\mathsf{T}}\Sigma_{yy}^{-1}H\Sigma_{ss}H^{\mathsf{T}}\Sigma_{yy}^{-1}AN^{-1} \doteq \Sigma_{\widehat{\beta\beta}}(e) + \Sigma_{\widehat{\beta\beta}}(s) \tag{9}$$

which only in case the calibration parameters are estimated with infinite accuracy ($\Sigma_{\widehat{\beta\beta}}(s) = 0$) equals N^{-1}. We can recognize the two sources of uncertainty in 3D–reconstruction, one from the feature measurement in image processing and one from the calibration step.

3 The Examples

In this section we demonstrate our approach in two simple examples. The results are compared with an algebraic solution.

3.1 The setup

We investigate lateral and forward motion with a single camera. The structure and the motion are estimated simultaneously from two images. We consider a 'cube' of data–points in a distance of 10 meters and a camera motion about 1 meter. The coordinate system is fixed between the two projection centers. In the adjustment process there is a rank deficiency of 7. We fix the two projection centers and the rotation around the basis (figure 1).

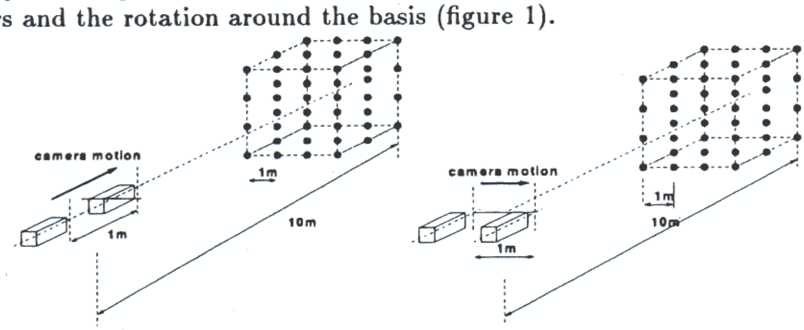

Fig. 1. Experimental setup for forward motion and lateral motion

A pinhole camera–model is used in the investigation:

$$x_{ij} = c\frac{r_1(\Omega_j)(p_i - p_{0j})}{r_3(\Omega_j)(p_i - p_{0j})} + \Delta x_{ij} \qquad y_{ij} = c\frac{r_2(\Omega_j)(p_i - p_{0j})}{r_3(\Omega_j)(p_i - p_{0j})} + \Delta y_{ij} \tag{10}$$

where x_{ij} and y_{ij} are the coordinates of the image of the i-th point in camera in position j. $\boldsymbol{R}(\boldsymbol{\Omega}_j)$ is the rotation matrix for image j depending on the rotation parameters $\boldsymbol{\Omega}_j$, $\boldsymbol{p}_i$ is the 3-vector of the coordinates of the object point i, $\boldsymbol{p}_{0j}$ is the 3-vector of the coordinates of the projection center and $\Delta x_{ij}(\boldsymbol{s})$ and $\Delta y_{ij}(\boldsymbol{s})$ are correction terms. In our example we consider a physical motivated model. It includes 4 Parameters to describe the interior orientation of the camera:

$$\Delta x_{ij} = x_H \;+\; x_{ij} A_1 (r^2 - r_0^2) \qquad \Delta y_{ij} = y_H \;+\; y_{ij} A_1 (r^2 - r_0^2) \tag{11}$$

(c–principal distance, x_H, y_H–principal point, A_1–radial distortion, $r^2 = x_{ij}^2 + y_{ij}^2$, r_0-constant, depending on the size of the image [WE80]). We took the numerical values from calibration of a real camera–system ([AF97], $c = 1144pel$, $x_H = 3.5pel$, $y_H = 10.8pel$, $A_1 = -6.1e-8$).

3.2 Algebraic Prediction

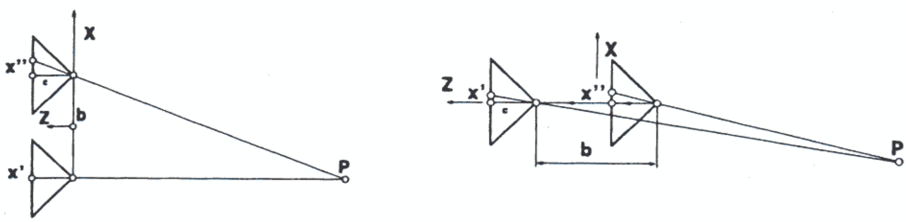

Fig. 2. The geometry for lateral (left) and for forward motion (right)

First we are making some predictions about the effect of errors in the principal distance c on the 3D–reconstruction. By fixation of the motion parameters the 3D–coordinate of one point can easily be calculated by triangulation (cf. fig. 2): for forward motion:

$$X \;=\; \frac{b}{c}\frac{x'x''}{(x'-x'')}; \qquad Z \;=\; \frac{b}{2}\frac{x'+x''}{(x'-x'')} \tag{12}$$

and for lateral motion:

$$X \;=\; \frac{b}{2}\frac{(x''+x')}{(x''-x')}; \qquad Z \;=\; \frac{bc}{x''-x'}. \tag{13}$$

Observe that the Z coordinate in forward motion and the X and Y coordinate in lateral motion are independent from the principal distance c. Changes and errors in the calibration of this parameter do not effect the estimation in these coordinates. On the other hand, c directly effects the X, Y coordinates (forward motion)

$$X \;+\; \Delta X \;=\; \frac{b}{c+\Delta c}\frac{x'x''}{(x'-x'')}; \qquad \Delta X \;=\; \frac{\Delta c}{c+\Delta c} X \tag{14}$$

and the Z–coordinates in (lateral motion):

$$Z \;+\; \Delta Z \;=\; \frac{b(c+\Delta c)}{x''-x'}; \qquad \Delta Z \;=\; \frac{\Delta c}{c} Z \tag{15}$$

The size of the error depends also on the position of the points in space. Points with a greater distance from the motion trajectory are more effected. We can do this analysis for all interior orientation parameters. But an analytic solution without fixation of the motion parameters is more complicated, so at this point we explore the numerical method.

3.3 Effect of calibration errors on 3D–Coordinates

Bias. The elements $(\partial\beta_j/\partial s_k)$ in $\boldsymbol{N}^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\Sigma}_{yy}\boldsymbol{H}\boldsymbol{\Delta s}$ (eq. (7)) directly give the influence on individual estimates. Table 1 collects the **maximum** influence numbers for the interior orientation parameters (according (11)) onto the 3D–coordinates. The coefficient A_1 needs a comment for interpretation: A value of $2.2\ 10^{-8}$ leads to a maximum distortion of 1 [pel] in the image plane. From all

		∂c	∂x_H	∂y_H	∂A_1
Forward motion	∂x	0.0013	-6.5 10^{-9}	-2.6 10^{-8}	-2.9 10^{5}
	∂y	0.0013	-1.3 10^{-8}	-3.2 10^{-8}	-2.7 10^{5}
	∂z	0.0	-0.0012	-0.0013	-1.4 10^{6}
Lateral motion	∂x	0.0	-0.0082	0.0	4.6 10^{6}
	∂y	0.0	0.0	-0.0087	4.6 10^{6}
	∂z	-0.0087	0.0	0.0	3.1 10^{7}

Table 1. Maximum influence of calibration parameters $c[pel]$, $x_H[pel]$, $y_H[pel]$, $A_1[1/pel^2]$ on the coordinates of the 3D–reconstruction $[m]$

influence numbers (not only the maximum numbers) we get some interesting results:

- Errors in the principle distance c do not influence the Z–coordinate in forward motion and the X and Y coordinate in lateral motion. The size of the influence number depends on the X and Y point coordinate in forward motion and the Z coordinate in lateral motion. The result is analogical to the algebraic solution
- Errors in the principal point coordinates causes errors in the Z–coordinate in forward and X and Y–coordinates in lateral motion. Points far from the motion trajectory are more effected.
- Calibration errors in A_1 have an appr. 10 times larger influence on the coordinates in the case of lateral motion than for forward motion.

The maximal effect of calibration errors could be observed on the 'corner' of our 'point–cube'. This can be intuitively expected.

Uncertainty. According to eq. (8) we can propagate the uncertainty from the calibration to the 3D–reconstruction. We consider a example point $(1.5, 1.5, 10)[m]$ on the corner of the cube and assume a precision in point measurement of about $0.5pel$ in the image plane, which is a reasonable assumption for non–signalized points. The camera is calibrated with a medium precision (see standard deviations in table 2). The first column reports the estimated precision of the point coordinates without taking the calibration errors into account (table 2). The

		(*)	(**)				(***)
			c $\sigma c = 55pel(5\%c)$	x_H $\sigma_{xH} = 25pel$	y_H $\sigma_{yH} = 25pel$	A_1 $\sigma_{A1} = 1e-7$	$\sum$
Forward	σ_x	0.117	0.072	0.0	0.0	0.023	0.193
	σ_y	0.119	0.072	0.0	0.0	0.025	0.195
	σ_z	0.313	0.0	0.031	0.033	0.108	0.431
Lateral	σ_x	0.105	0.0	0.206	0.0	0.463	0.507
	σ_y	0.088	0.0	0.0	0.218	0.455	0.505
	σ_z	0.572	0.481	0.0	0.0	3.084	3.122

Table 2. Uncertainty in camera calibration and effect on 3D–reconstruction (point $(1.5, 1.5, 10)[m]$), precision of 3D–reconstruction (standard–deviation $[m]$), (*)–error free calibration, (**)–effect of uncertainty in one single parameter, (***)–all error–sources integrated

influence of the uncertainty from the single calibration parameters are shown in the next columns. The last column describes the estimated precision including both influences, point measurement and calibration.

Observe that the critical parameters in our example are the calibration of the principal distance c and the radial distortion coefficient A_1. For lateral motion the effect of the camera–calibration is more critical than for forward motion.

3.4 Errors by linearization ?

One could argue, that the simple linearization step would falsify our error approximation. This can faced by adjusting the structure and the motion parameters with modified interior orientation within a adjustment by nonlinear optimization. Then the reconstruction error can be compared with the linear approximation. We did this adjustment with larger modifications of the interior orientation parameters. The effect of the modifications on our example point for forward motion are reported in table 3. The results for lateral motion are similar. The approximation from the linear algorithm are also shown. Comparing the results it can be observed, that the differences between linear approximation and the nonlinear adjustment can be neglected in practise.

	$\Delta c = -100pel$	$\Delta x_H = 25pel$	$\Delta y_H = 25pel$	$\Delta A_1 = 1e-7$
$\Delta x_{nonlinear}$	-0.144	0.010	0.006	-0.024
$\Delta x_{Approx.}$	-0.130	0.000	0.000	-0.024
$\Delta y_{nonlinear}$	-0.144	0.005	0.011	-0.025
$\Delta y_{Approx.}$	-0.130	0.000	0.000	-0.025
$\Delta z_{nonlinear}$	0.000	-0.123	-0.130	-0.109
$\Delta z_{Approx.}$	0.000	-0.120	-0.130	-0.110

Table 3. Nonlinear adjustment in comparison to linear approximation, Modifications in 3D–reconstruction caused by changes of the interior orientation, forward motion, example point $(1.5, 1.5, 10)[m]$

4 Conclusions

The paper presented methods for sensitivity analysis and demonstrated its applicability for evaluating the effect of systematic and stochastic calibration errors in structure from motion estimation. The proposed methodology is useful during planning, for online self diagnosis and for final evaluation and is transferable to all types of parameter estimation problems in Computer Vision. To find analytic solutions is often hard. The numerical analysis is very flexible and different functional models can be investigated in a short time.

We also got some interesting results for structure from motion. The effect of errors and uncertainties in the camera parameters are very different. Errors in the principle distance modify directly the reconstruction. The sensitivity for calibration errors for forward and lateral motion is different. For example for forward motion the effect of errors in the principal point coordinates are very small. Without calibrating this parameters (assumption image center) a 3D–reconstruction would be reasonable depending on the requirements in the application. This could be checked in a planning stage by the proposed approach.

The next step in our work is to apply these tools in real applications. There are interesting questions to answer, for example: which precision of camera calibration is needed to fulfill the requirements in an application or can the desired measurement precision be reached by the camera setup.

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