Photogrammetry & Robotics Lab

EKF SLAM – Simultaneous Localization and Mapping

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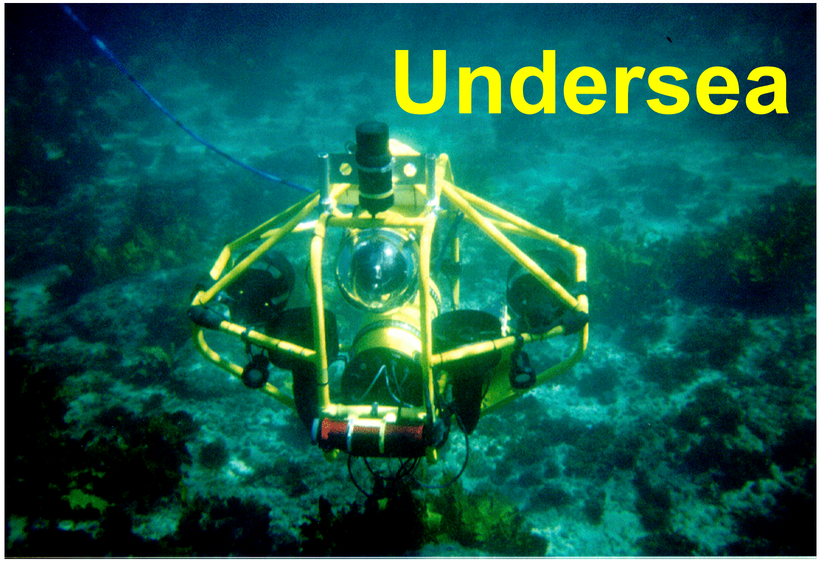
5 Minute Preparation for Today

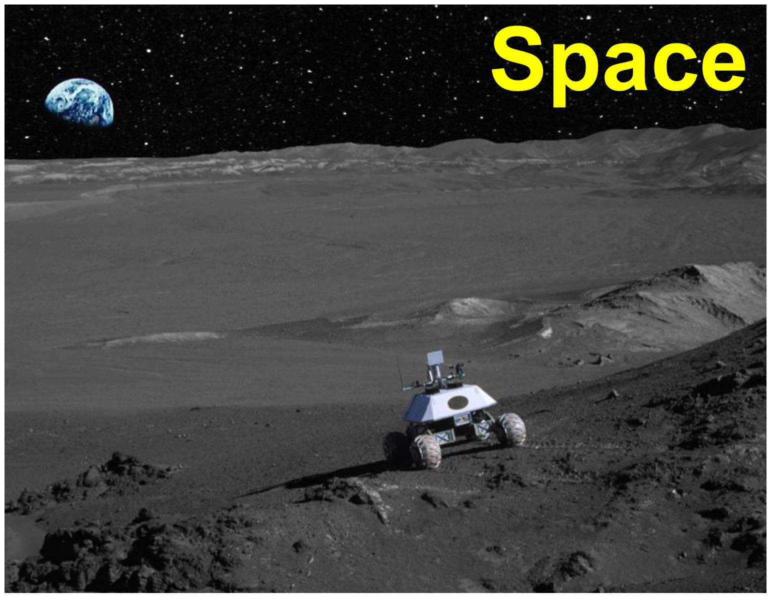
https://www.ipb.uni-bonn.de/5min/

SLAM: Simultaneous Localization and Mapping

- **Build a map** of the environment from a mobile sensor platform
- At the same time, **localize** a mobile sensor platform in the map build so far
- **Online** variant of the bundle adjustment problem

SLAM Applications





Courtesy: Evolution Robotics, H. Durrant-Whyte, NASA, S. Thrun

Definition of the SLAM Problem

Given

- The sent controls commands

$$u_{1:T} = \{u_1, u_2, u_3, \ldots, u_T\}$$

- Observations

$$z_{1:T} = \{z_1, z_2, z_3, \ldots, z_T\}$$

Wanted

- Map of the environment

$$m$$

- Path (or current pose) of the vehicle

$$x_{0:T} = \{x_0, x_1, x_2, \ldots, x_T\}$$

Bayes Filter

- Recursive filter with a **prediction step** and a **correction step**
- **Estimates:**

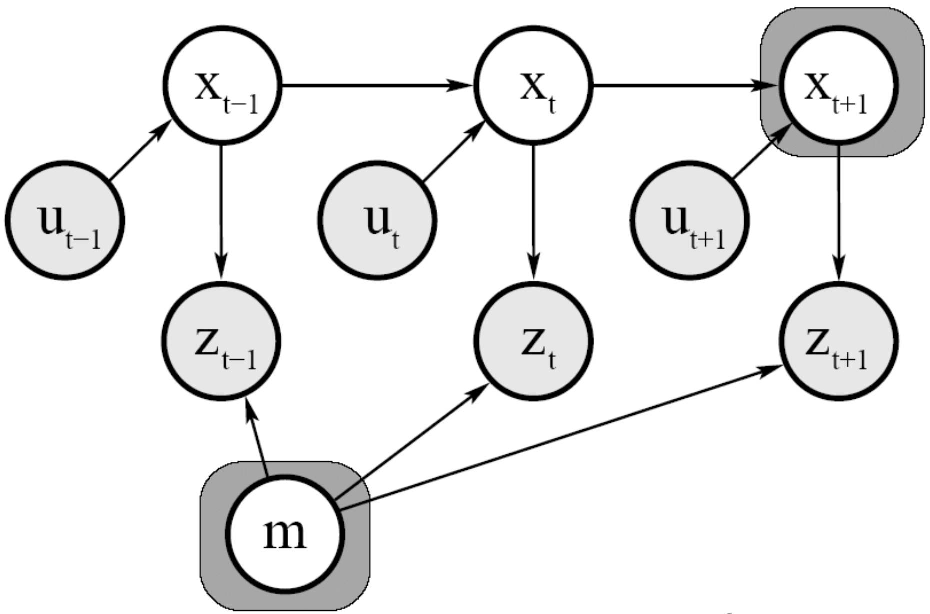
$$p(x_t, m \mid z_{1:t}, u_{1:t})$$

- Kalman Filter is a recursive Bayes Filter for the **linear Gaussian case**
- **EKF** for dealing with **non-linearities**

EKF for Online SLAM

We consider here the Kalman filter as a solution to the online SLAM problem

$$p(x_t, m \mid z_{1:t}, u_{1:t})$$



Courtesy: Thrun, Burgard, Fox

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$

3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

7: return μ_t, Σ_t

EKF SLAM

- Application of the EKF to SLAM
- Estimate robot's pose and locations of landmarks in the environment
- Assumption: known correspondences
- State space (for the 2D plane) is

$$x_t = (\underbrace{x, y, \theta}_{\text{robot's pose}}, \underbrace{m_{1,x}, m_{1,y}}_{\text{landmark 1}}, \ldots, \underbrace{m_{n,x}, m_{n,y}}_{\text{landmark n}})^T$$

EKF SLAM: State Representation

- Map with *n* landmarks: (3+2*n*)-dimensional Gaussian
- Belief is represented by

$$\underbrace{\begin{pmatrix} x_t \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_t x_t} & \Sigma_{x_t m_1} & \dots & \Sigma_{x_t m_n} \\ \Sigma_{m_1 x_t} & \Sigma_{m_1 m_1} & \dots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_t} & \Sigma_{m_n m_1} & \dots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: State Representation

- More compactly

$$\underbrace{\begin{pmatrix} x \\ m \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix}}_{\Sigma}$$

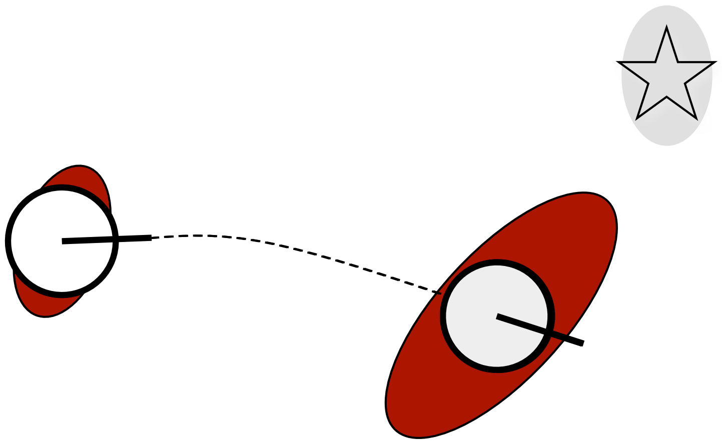
EKF SLAM: Filter Cycle

1. State prediction
2. Measurement prediction
3. Measurement
4. Data association
5. Update

EKF SLAM: Initial State

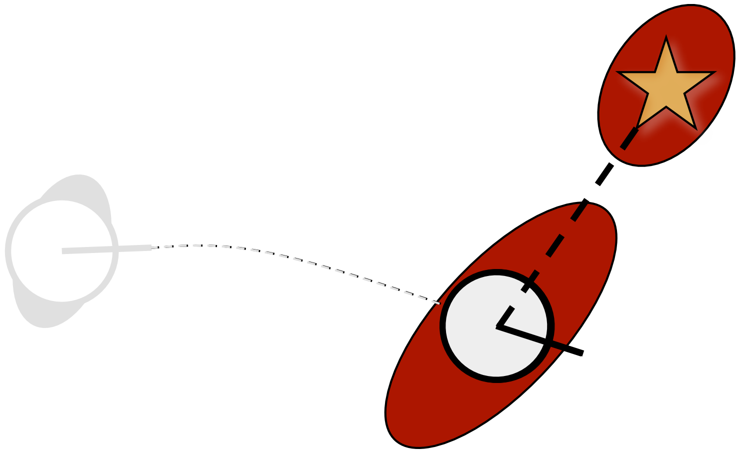
$$\underbrace{\begin{pmatrix} x_t \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_t x_t} & \Sigma_{x_t m_1} & \cdots & \Sigma_{x_t m_n} \\ \Sigma_{m_1 x_t} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_t} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Predicted Motion



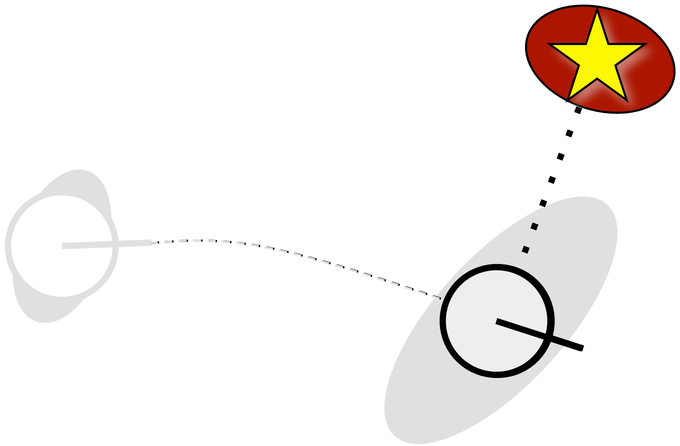
$$\underbrace{\begin{pmatrix} x_t \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_t x_t} & \Sigma_{x_t m_1} & \cdots & \Sigma_{x_t m_n} \\ \Sigma_{m_1 x_t} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_t} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Predicted Measurement



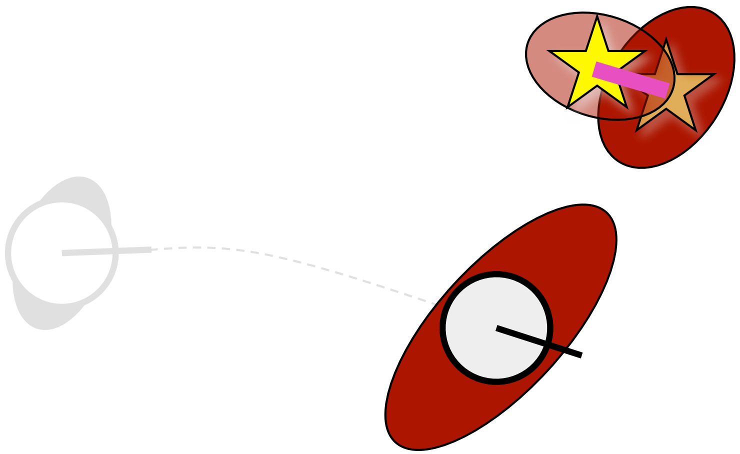
$$\underbrace{\begin{pmatrix} x_t \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_t x_t} & \Sigma_{x_t m_1} & \cdots & \Sigma_{x_t m_n} \\ \Sigma_{m_1 x_t} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_t} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Obtained Measurement



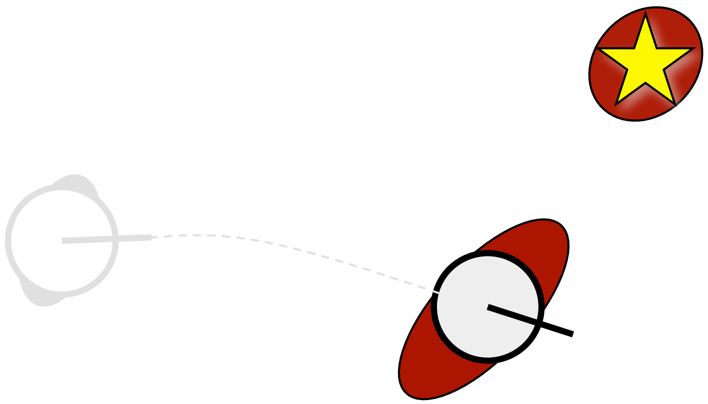
$$\underbrace{\begin{pmatrix} x_t \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_t x_t} & \Sigma_{x_t m_1} & \cdots & \Sigma_{x_t m_n} \\ \Sigma_{m_1 x_t} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_t} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Data Association and Difference Between h(x) and z



$$\underbrace{\begin{pmatrix} x_t \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_t x_t} & \Sigma_{x_t m_1} & \cdots & \Sigma_{x_t m_n} \\ \Sigma_{m_1 x_t} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_t} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Update Step



$$\underbrace{\begin{pmatrix} x_t \\ m_1 \\ \vdots \\ m_n \end{pmatrix}}_{\mu} \quad \underbrace{\begin{pmatrix} \Sigma_{x_t x_t} & \Sigma_{x_t m_1} & \cdots & \Sigma_{x_t m_n} \\ \Sigma_{m_1 x_t} & \Sigma_{m_1 m_1} & \cdots & \Sigma_{m_1 m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{m_n x_t} & \Sigma_{m_n m_1} & \cdots & \Sigma_{m_n m_n} \end{pmatrix}}_{\Sigma}$$

EKF SLAM: Concrete Example

Setup

- Platform moves in the 2D plane
- Velocity-based motion model
- Observation of point landmarks
- Range-bearing sensor
- Known data association
- Known number of landmarks

Initialization

- Platform starts in its own reference frame (all landmarks unknown)
- 2N+3 dimensions

$$\mu_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \end{pmatrix}^T$$

$$\Sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \infty & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \infty \end{pmatrix}$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$

$$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$$

$$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$$

$$\text{return } \mu_t, \Sigma_t$$

Prediction Step (Motion)

- Goal: Update state space based on the motion
- Motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t\Delta t) \\ \omega_t \Delta t \end{pmatrix}}_{g_{x,y,\theta}(u_t,(x,y,\theta)^T)}$$

- How to map that to the 2N+3 dim state space used in the EKF?

Update the State Space

- From the motion in the plane

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}$$

- to the 2N+3 dimensional space

$$\begin{pmatrix} x' \\ y' \\ \theta' \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \\ \vdots \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0\dots0 \\ 0 & 1 & 0 & 0\dots0 \\ 0 & 0 & 1 & \underbrace{0\dots0}_{2N cols} \end{pmatrix}^T}_{F_x^T} \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}}_{g(u_t,x_t)}$$

Extended Kalman Filter Algorithm

$$\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)\textbf{:}$$

~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ **DONE**

$$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$$

$$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$$

$$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$$

$$\text{return } \mu_t, \Sigma_t$$

Update Covariance

- The function g only affects the motion and **not** the landmarks

Jacobian of the motion (3x3)

$$G_t = \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix}$$

Identity (2N x 2N)

Jacobian of the Motion

$$G_t^x = \frac{\partial}{\partial(x, y, \theta)^T} \left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin\theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos\theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix} \right]$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}
\end{aligned}
$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t\,\Delta t \end{pmatrix} \\
&= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Jacobian of the Motion

$$
\begin{aligned}
G_t^x &= \frac{\partial}{\partial(x,y,\theta)^T}\left[\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t \Delta t \end{pmatrix}\right] \\
&= I + \frac{\partial}{\partial(x,y,\theta)^T}\begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ \omega_t \Delta t \end{pmatrix} \\
&= I + \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\frac{v_t}{\omega_t}\cos\theta + \frac{v_t}{\omega_t}\cos(\theta+\omega_t\Delta t) \\ 0 & 1 & -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta+\omega_t\Delta t) \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

This Leads to the Update

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2: ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ **Apply & DONE**

3: $\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t^T + R_t$

$$
\begin{aligned}
\bar{\Sigma}_t &= G_t \Sigma_{t-1} G_t^T + R_t \\
&= \begin{pmatrix} G_t^x & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{xx} & \Sigma_{xm} \\ \Sigma_{mx} & \Sigma_{mm} \end{pmatrix} \begin{pmatrix} (G_t^x)^T & 0 \\ 0 & I \end{pmatrix} + R_t \\
&= \begin{pmatrix} G_t^x \Sigma_{xx} (G_t^x)^T & G_t^x \Sigma_{xm} \\ (G_t^x \Sigma_{xm})^T & \Sigma_{mm} \end{pmatrix} + R_t
\end{aligned}
$$

Extended Kalman Filter Algorithm

1: $\textbf{Extended\_Kalman\_filter}(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$ **DONE**

3: $\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t$ **DONE**

4: $K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$

7: $\text{return } \mu_t, \Sigma_t$

EKF SLAM:Prediction Step

EKF_SLAM_Prediction$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, R_t)$**:**

$$2: \quad F_x = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \end{pmatrix}$$

$$3: \quad \bar{\mu}_t = \mu_{t-1} + F_x^T \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$

$$4: \quad G_t = I + F_x^T \begin{pmatrix} 0 & 0 & -\frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & 0 \end{pmatrix} F_x$$

$$5: \quad \bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + \underbrace{F_x^T \, R_t^x \, F_x}_{R_t}$$

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: ~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ **DONE**

3: ~~$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$~~ **Apply & DONE**

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

7: $return \; \mu_t, \Sigma_t$

EKF SLAM: Correction Step

- Known data association
- $c_t^i = j$: i-th measurement at time t observes the landmark with index j
- Initialize landmark if unobserved
- Compute the expected observation
- Compute the Jacobian of h
- Proceed with computing the Kalman gain

Range-Bearing Observation

- Range-Bearing observation $z_t^i = (r_t^i, \phi_t^i)^T$
- If landmark has not been observed, we can initialize it with:

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

location of landmark j

estimated location of the platform

relative measurement

Expected Observation: h(x)

- Compute expected observation according to the current estimate

$$
\begin{aligned}
\delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\
q &= \delta^T \delta \\
\hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \\
&= h(\bar{\mu}_t)
\end{aligned}
$$

Jacobian for the Observation

- Based on

$$\begin{aligned} \delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \end{aligned}$$

- Compute the Jacobian

$$ {}^{\text{low}}H_t^i = \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} $$

↑

low-dim space $(x, y, \theta, m_{j,x}, m_{j,y})$

Jacobian for the Observation

- Based on

$$\begin{aligned} \delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \end{aligned}$$

- Compute the Jacobian

$$\begin{aligned} {}^{\text{low}}H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\ &= \begin{pmatrix} \frac{\partial \sqrt{q}}{\partial x} & \frac{\partial \sqrt{q}}{\partial y} & \ldots \\ \frac{\partial \operatorname{atan2}(\ldots)}{\partial x} & \frac{\partial \operatorname{atan2}(\ldots)}{\partial y} & \ldots \end{pmatrix} \end{aligned}$$

The First Component

- Based on

$$\begin{aligned} \delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \end{aligned}$$

- We obtain (by applying the chain rule)

$$\begin{aligned} \frac{\partial \sqrt{q}}{\partial x} &= \frac{1}{2} \frac{1}{\sqrt{q}} 2\, \delta_x \,(-1) \\ &= \frac{1}{q} (-\sqrt{q} \delta_x) \end{aligned}$$

Jacobian for the Observation

- Based on

$$\begin{aligned} \delta &= \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix} \\ q &= \delta^T \delta \\ \hat{z}_t^i &= \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix} \end{aligned}$$

- Compute the Jacobian

$$\begin{aligned} {}^{\text{low}}H_t^i &= \frac{\partial h(\bar{\mu}_t)}{\partial \bar{\mu}_t} \\ &= \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix} \end{aligned}$$

Jacobian for the Observation

- Use the computed Jacobian

$$^{\text{low}}H_t^i = \frac{1}{q}\begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & \delta_x \end{pmatrix}$$

- map it to the high dimensional space

$$H_t^i = {}^{\text{low}}H_t^i \, F_{x,j}$$

↓

$$F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 1 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 \\ 0 & 0 & 0 & 0\cdots0 & 1 & 0 & 0\cdots0 \\ 0 & 0 & 0 & \underbrace{0\cdots0}_{2j-2} & 0 & 1 & \underbrace{0\cdots0}_{2N-2j} \end{pmatrix}$$

Next Steps as Specified...

Extended_Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ **DONE**

~~$\bar{\Sigma}_t = G_t\, \Sigma_{t-1}\, G_t^T + R_t$~~ **DONE**

$K_t = \bar{\Sigma}_t\, H_t^T (H_t\, \bar{\Sigma}_t\, H_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$

$\Sigma_t = (I - K_t\, H_t)\, \bar{\Sigma}_t$

return μ_t, Σ_t

Extended Kalman Filter Algorithm

Extended_Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

~~$\bar{\mu}_t = g(u_t, \mu_{t-1})$~~ **DONE**

~~$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$~~ **DONE**

~~$K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$~~ **Apply & DONE**

~~$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$~~ **Apply & DONE**

~~$\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$~~ **Apply & DONE**

return μ_t, Σ_t

EKF SLAM – Correction (1/2)

EKF_SLAM_Correction

$$Q_t = \begin{pmatrix} {\sigma_r}^2 & 0 \\ 0 & {\sigma_\phi}^2 \end{pmatrix}$$

for all observed features $z_t^i = (r_t^i, \phi_t^i)^T$ *do*

$$j = c_t^i$$

if landmark j *never seen before*

$$\begin{pmatrix} \bar{\mu}_{j,x} \\ \bar{\mu}_{j,y} \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \end{pmatrix} + \begin{pmatrix} r_t^i \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ r_t^i \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \end{pmatrix}$$

endif

$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \bar{\mu}_{j,x} - \bar{\mu}_{t,x} \\ \bar{\mu}_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$

$$q = \delta^T \delta$$

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

EKF SLAM – Correction (2/2)

$$F_{x,j} = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \\ 0 & 0 & 0 & 0 \cdots 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 0 & \underbrace{0 \cdots 0}_{2j-2} & 0 & 1 & \underbrace{0 \cdots 0}_{2N-2j} \end{pmatrix}$$

$$H_t^i = \frac{1}{q} \begin{pmatrix} -\sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 & +\sqrt{q}\delta_x & \sqrt{q}\delta_y \\ \delta_y & -\delta_x & -q & -\delta_y & +\delta_x \end{pmatrix} F_{x,j}$$

$$K_t^i = \bar{\Sigma}_t \, H_t^{iT} (H_t^i \, \bar{\Sigma}_t \, H_t^{iT} + Q_t)^{-1}$$

$$\bar{\mu}_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$$

$$\bar{\Sigma}_t = (I - K_t^i \, H_t^i) \, \bar{\Sigma}_t$$

endfor

$$\mu_t = \bar{\mu}_t$$

$$\Sigma_t = \bar{\Sigma}_t$$

return μ_t, Σ_t

Implementation Notes

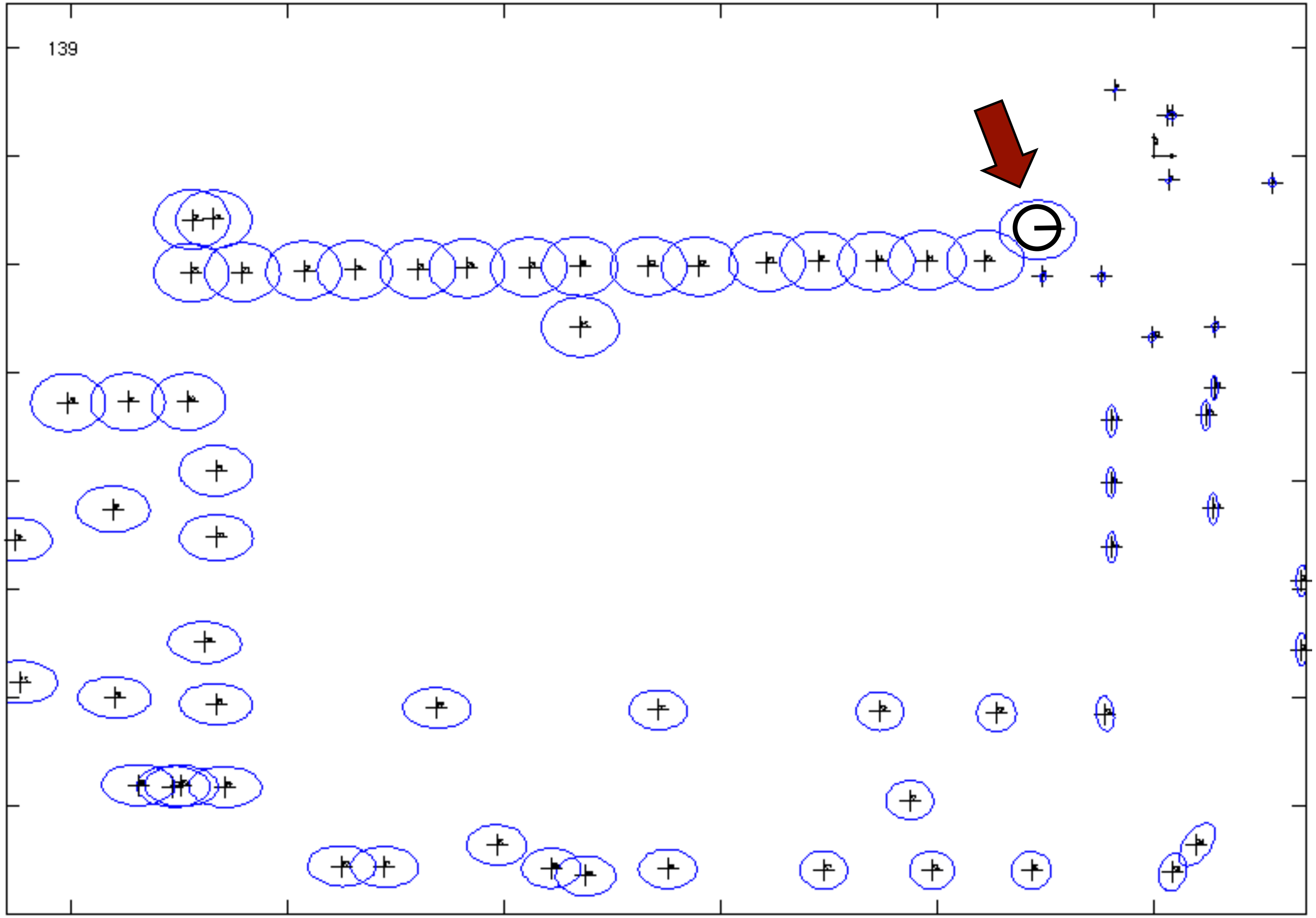
- Measurement update in a single step requires only one full belief update
- Always normalize the angular components
- You may not need to create the F matrices explicitly (e.g., in Matlab)

Done!

Loop Closing

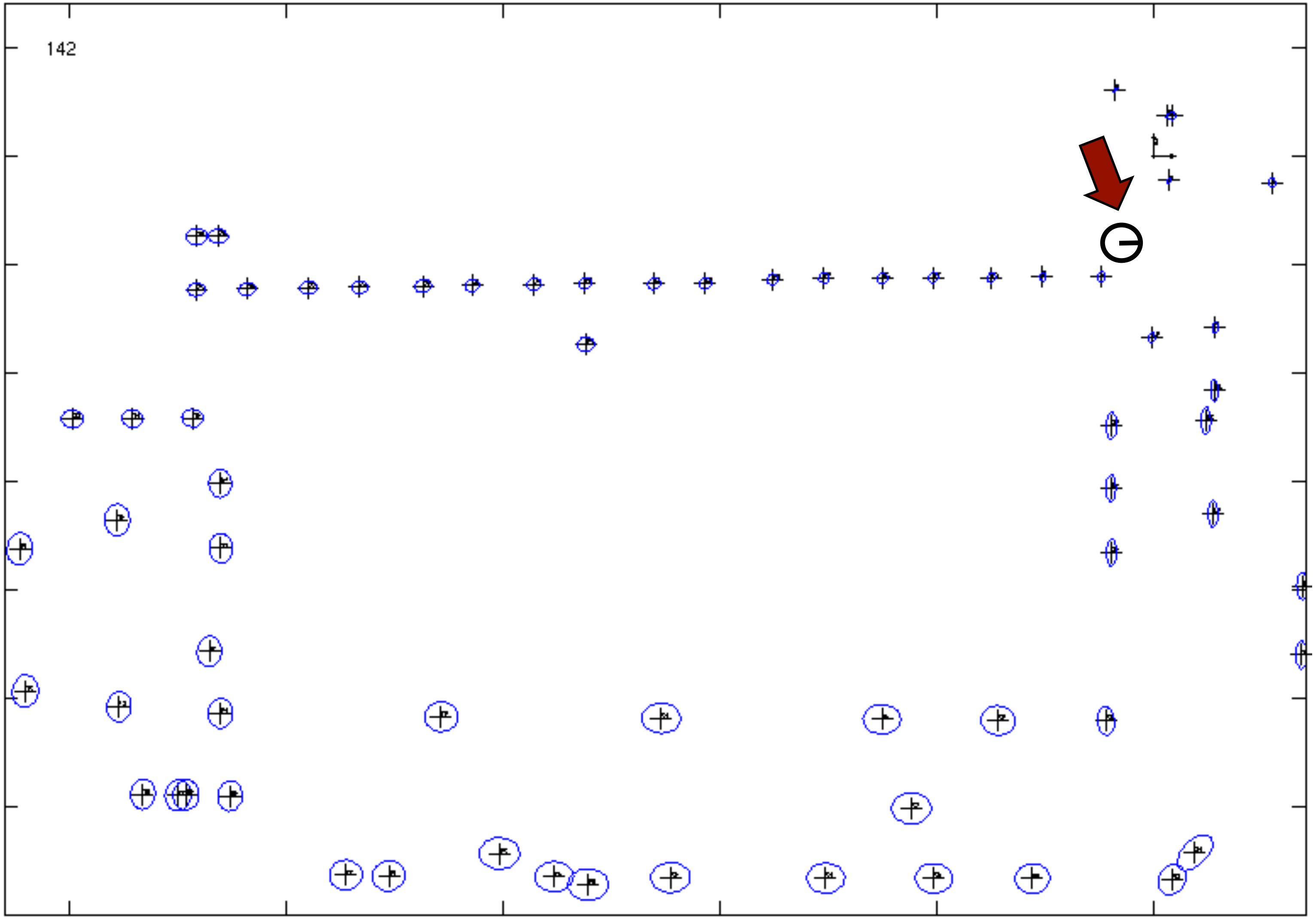
- Loop closing means revisiting (and recognizing) an already mapped area
- Data association under
  - high ambiguity
  - possible environment symmetries
- Uncertainties **collapse** after a loop closure (whether the closure was correct or not)

Before the Loop Closure



Courtesy: K. Arras

After the Loop Closure



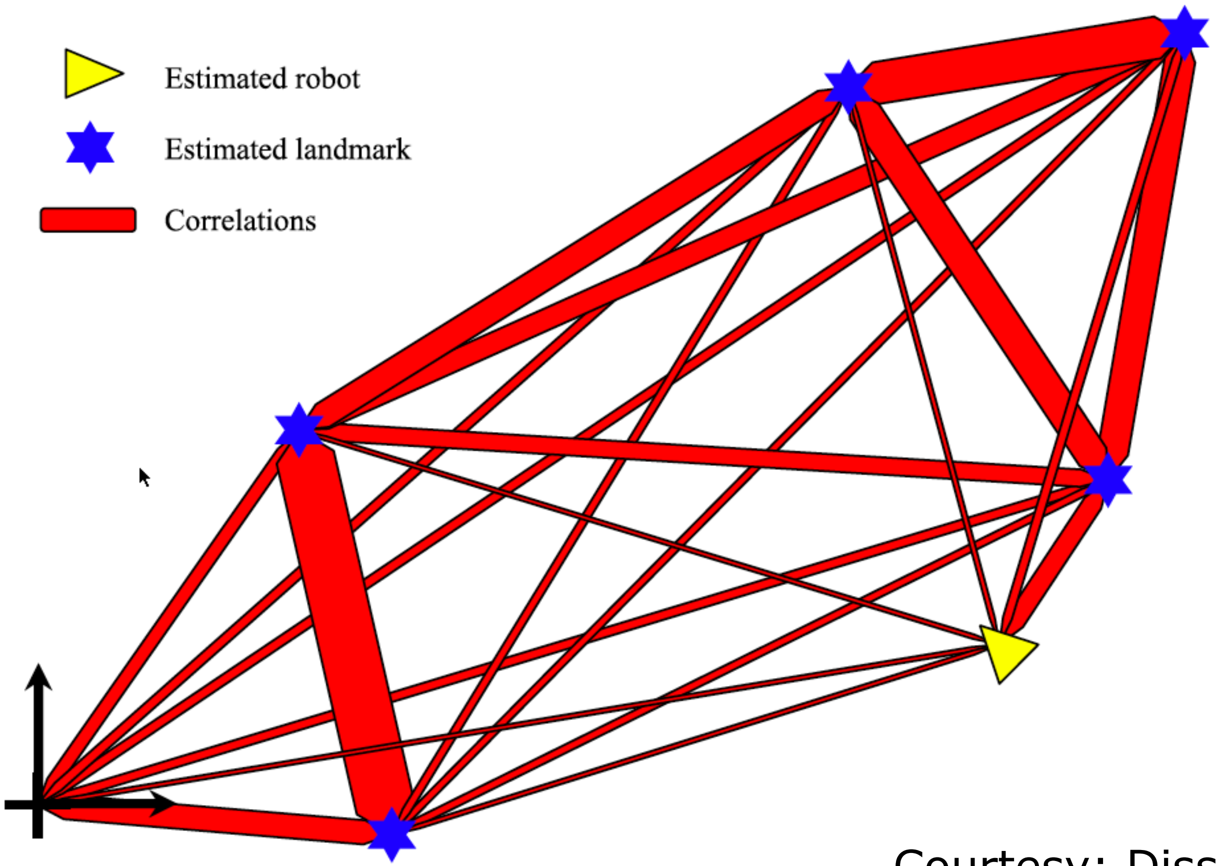
Courtesy: K. Arras

Loop Closures in SLAM

- Loop closing **reduces** the uncertainty in robot and landmark estimates
- This can be exploited when exploring an environment for the sake of better (e.g. more accurate) maps
- **Wrong loop closures lead to filter divergence**

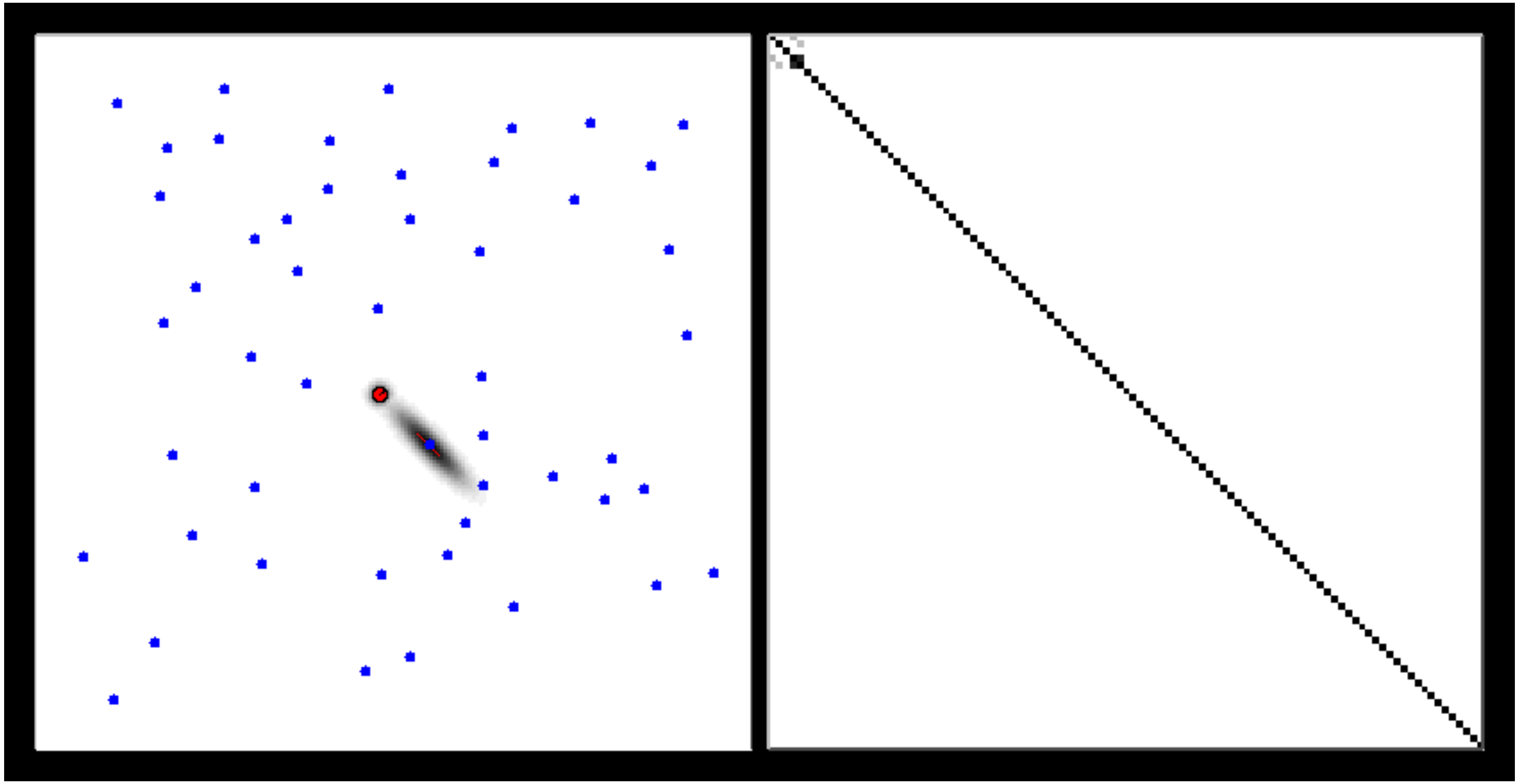
EKF SLAM Correlations

- In the limit, the landmark estimates become **fully correlated**



Courtesy: Dissanayake

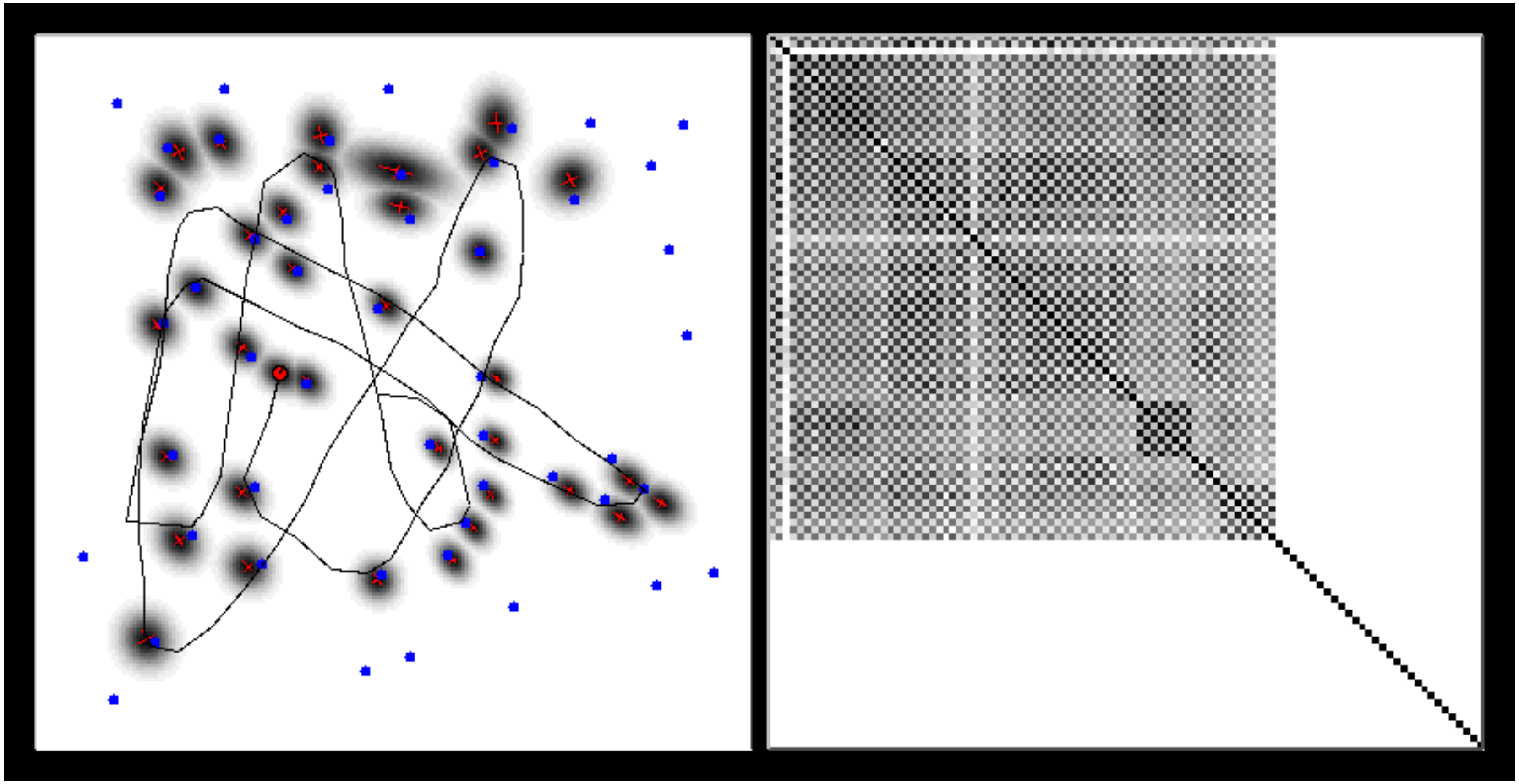
EKF SLAM Correlations



Map Correlation matrix

Courtesy: M. Montemerlo

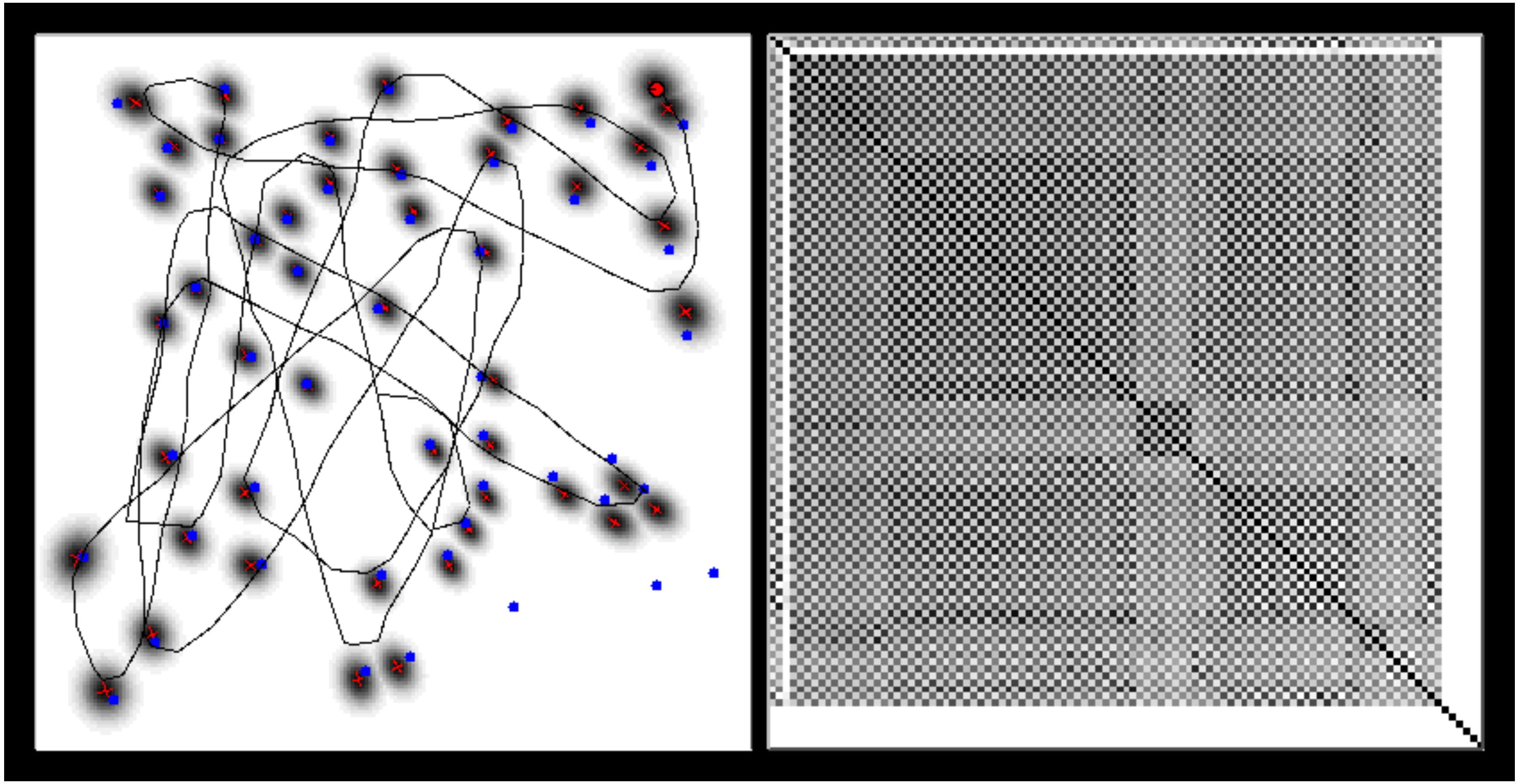
EKF SLAM Correlations



Map Correlation matrix

Courtesy: M. Montemerlo

EKF SLAM Correlations



Map Correlation matrix

Courtesy: M. Montemerlo

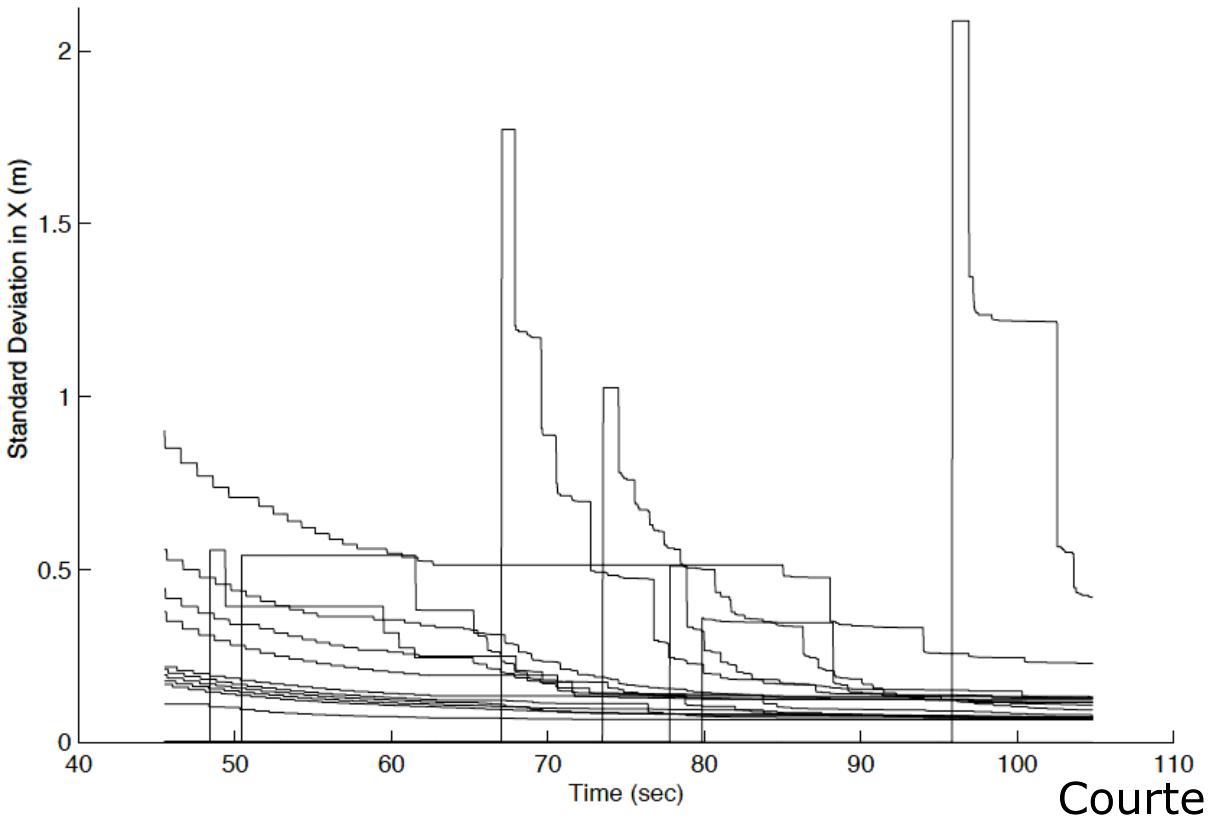
EKF SLAM Correlations

- The correlation between the robot's pose and the landmarks **cannot** be ignored
- Assuming independence generates too optimistic estimates of the uncertainty

Courtesy: J.M. Castellanos

EKF SLAM Uncertainties

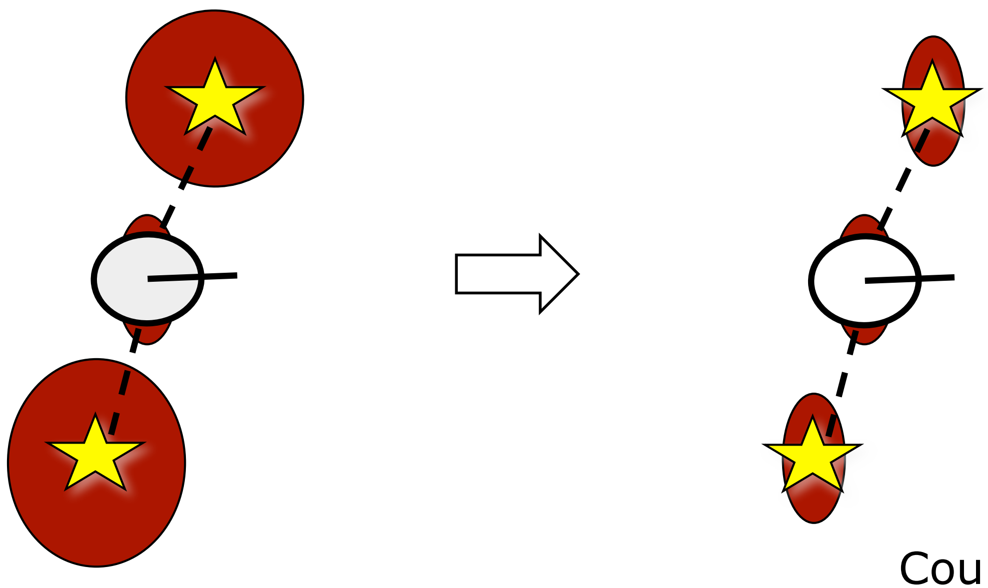
- The **determinant** of any sub-matrix of the map covariance matrix **decreases monotonically**
- New landmarks are initialized with **maximum uncertainty**



Courtesy: Dissanayake

EKF SLAM in the Limit

In the limit, the covariance associated with any single landmark location estimate is determined only by the initial covariance in the vehicle location estimate.

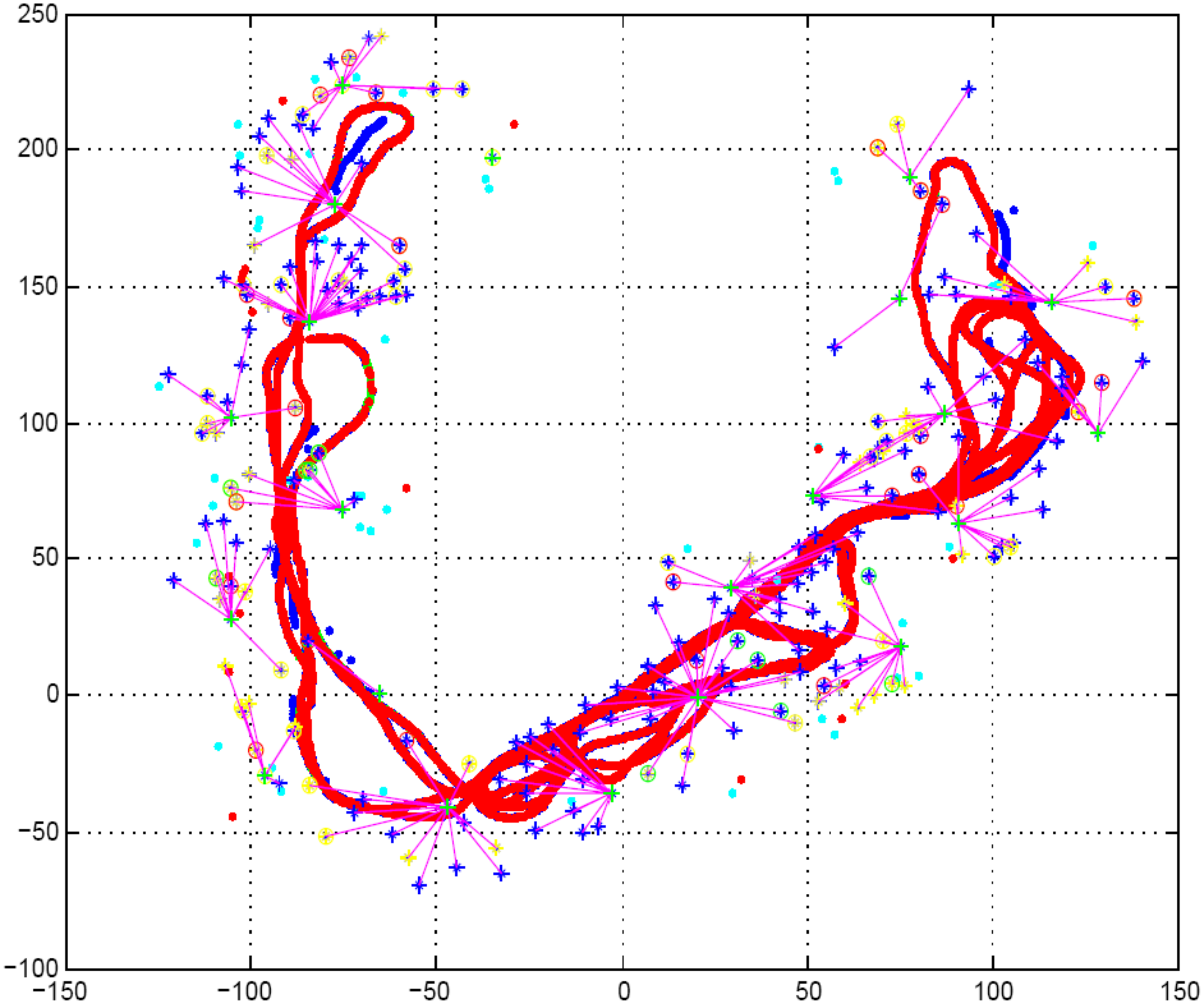


Courtesy: Dissanayake

Example: Victoria Park Dataset

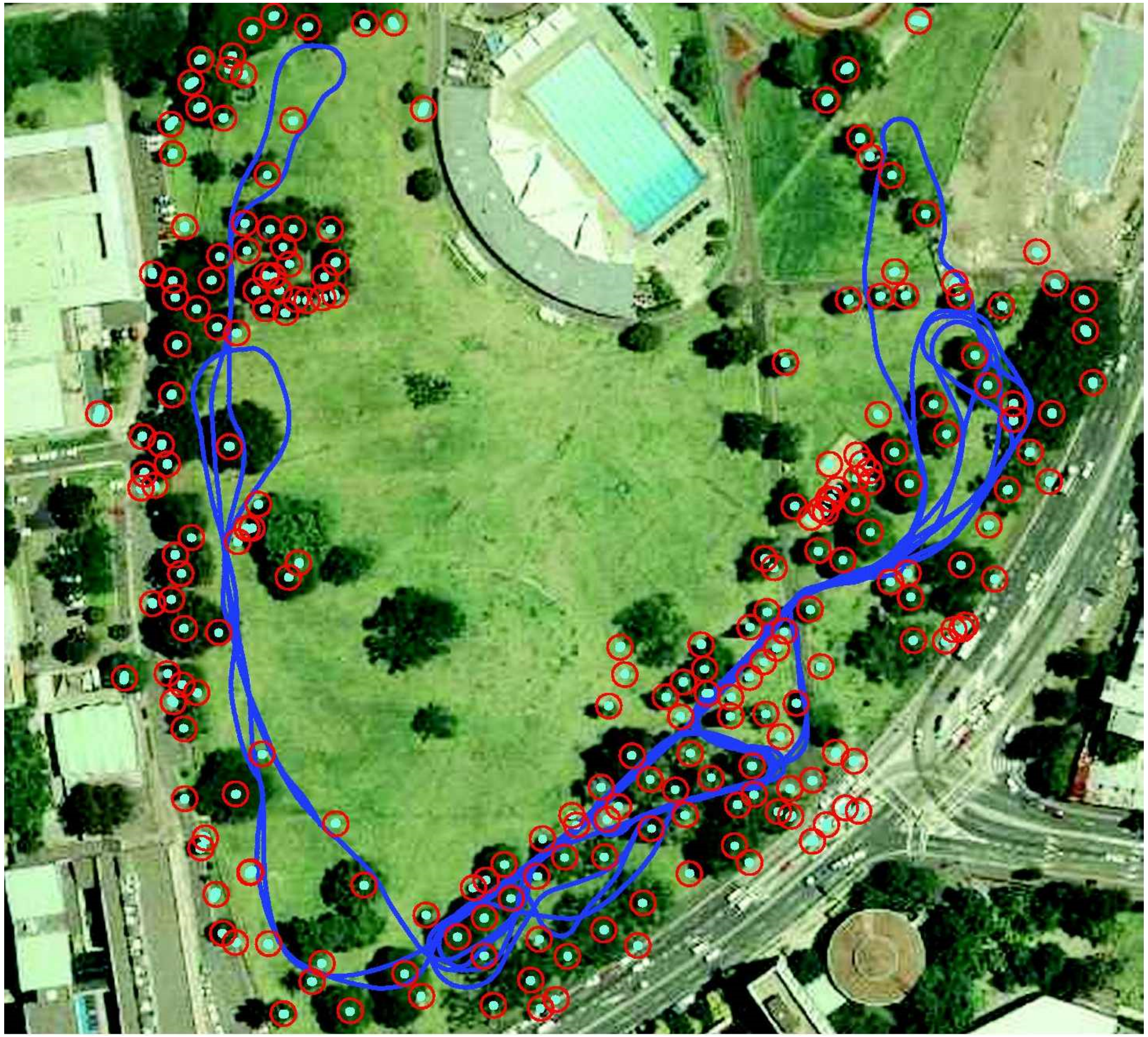
Courtesy: E. Nebot

Victoria Park: EKF Estimate



Courtesy: E. Nebot

Victoria Park: Landmarks

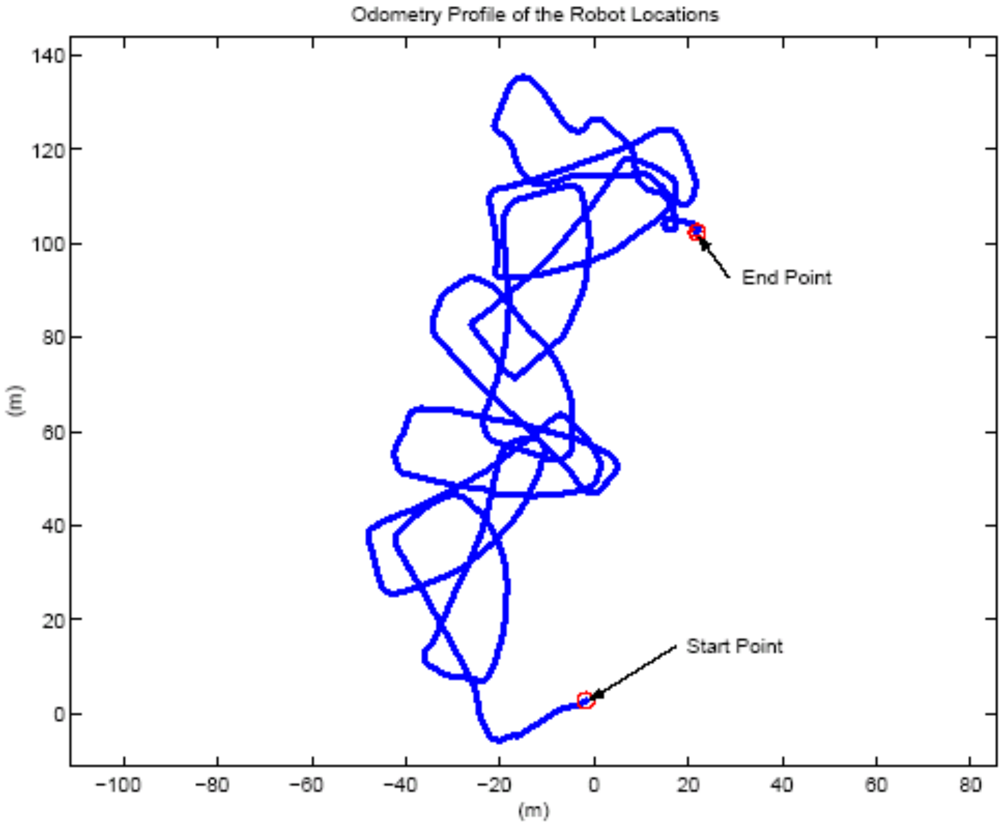


Courtesy: E. Nebot

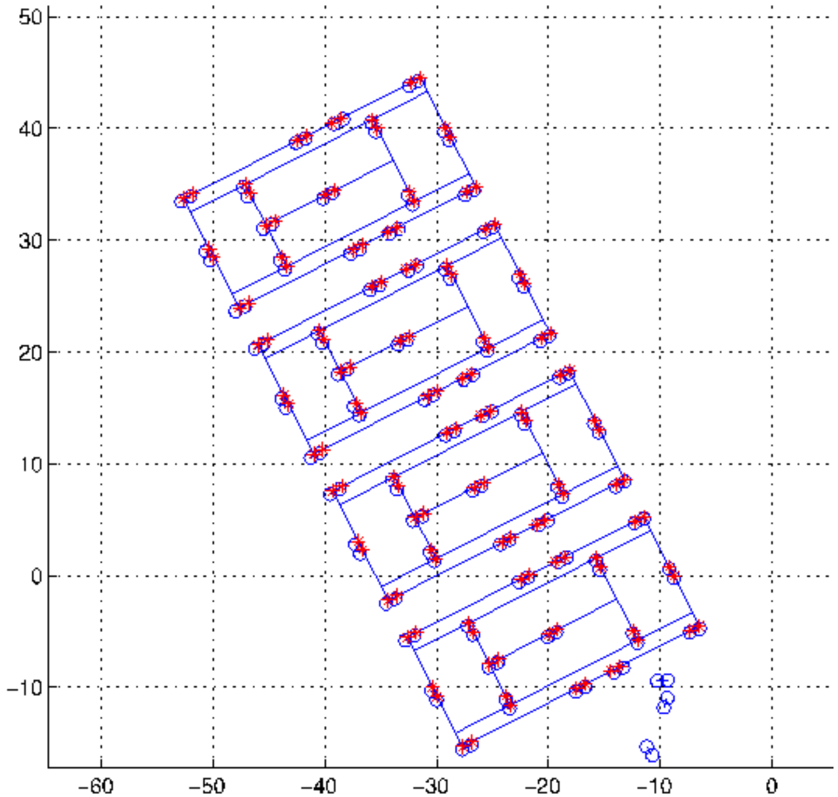
Example: Tennis Court Dataset

Courtesy: J. Leonard and M. Walter

EKF SLAM on a Tennis Court



odometry



estimated trajectory

Courtesy: J. Leonard and M. Walter

EKF SLAM Complexity

- Cubic complexity w.r.t. the measurement dimensionality
- Cost per step: dominated by the number of landmarks: $O(n^2)$
- Memory consumption: $O(n^2)$
- The EKF becomes computationally intractable for large maps!

EKF SLAM Summary

- Using the EKF to estimate pose and map in an SLAM fashion
- The first probabilistic SLAM approaches used the EKF
- Successful in medium-scale scenes
- Approximations exists to reduce the computational complexity
- Todays mainly used for short-term estimates (VO)

EKF SLAM Summary

- Unimodal (Gaussian) estimates only
- Convergence proof for the linear Gaussian case and then equivalent to least squares
- The smaller the noise the better the estimate in the non-linear case
- Can diverge if non-linearities are large

Literature

EKF SLAM

- Thrun et al.: “Probabilistic Robotics”, Chapter 10