Photogrammetry & Robotics Lab

Recursive Bayes Filter

Cyrill Stachniss

5 Minute Preparation for Today

https://www.ipb.uni-bonn.de/5min/

Recursive State Estimation

State Estimation

- Estimate the state x of a system given observations z and controls u
- **Goal:**

$$p(x \mid z, u)$$

State Estimation

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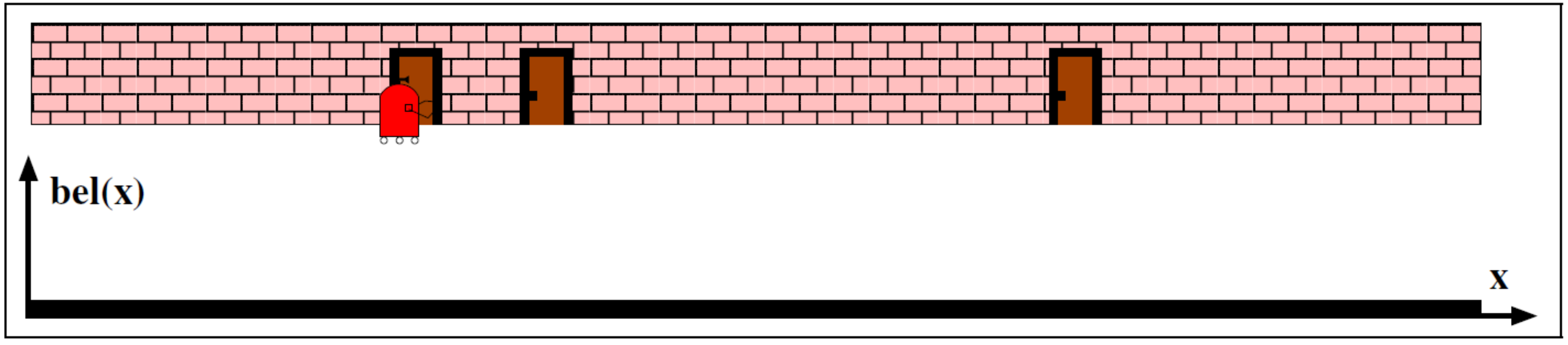
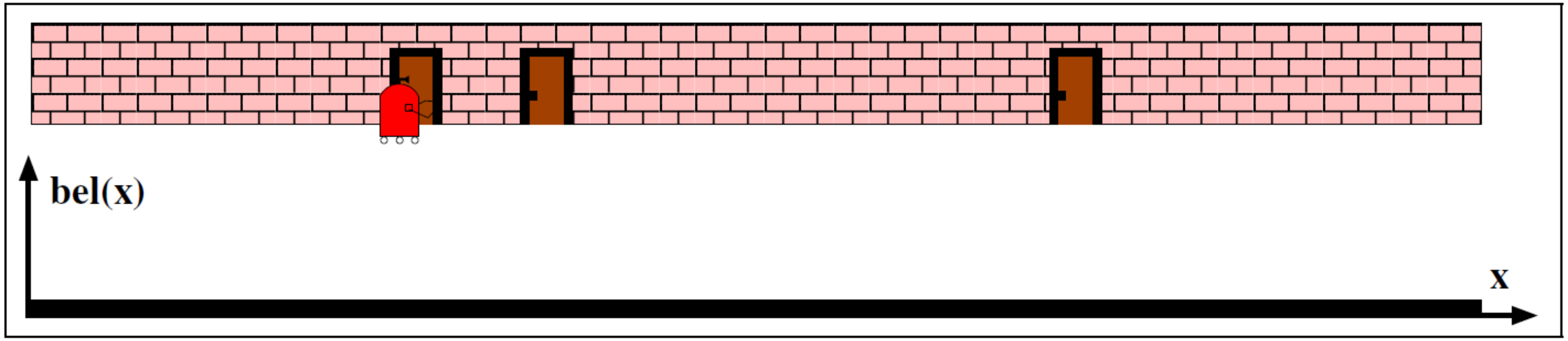
$$p(x_t | z_{1:t}, u_{1:t})$$

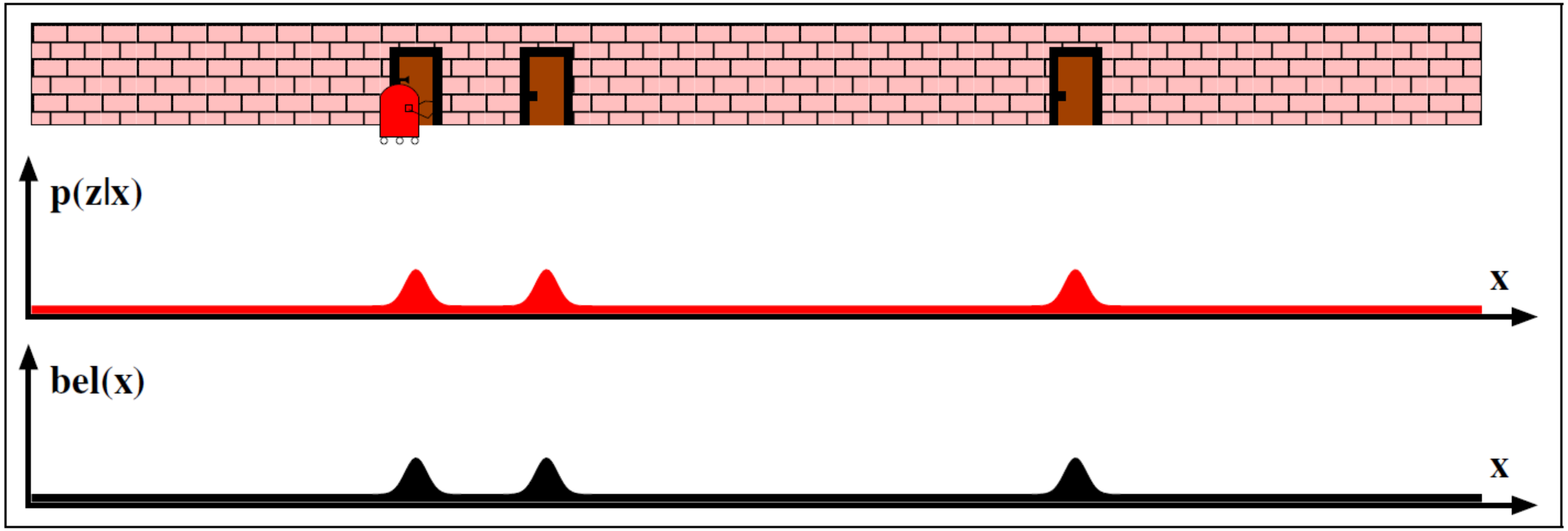
Recursive Bayes Filter



Courtesy: Thrun, Burgard and Fox

Recursive Bayes Filter





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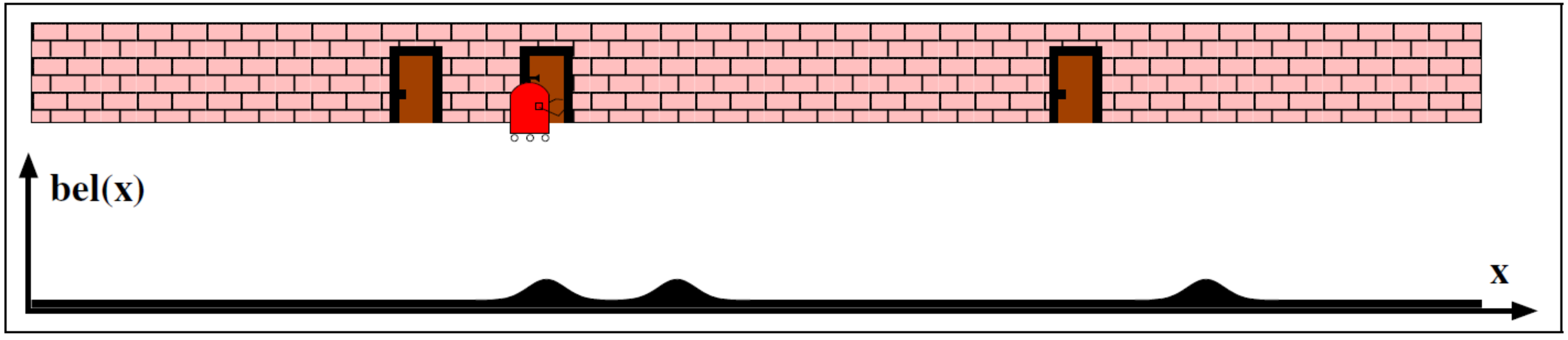
Recursive Bayes Filter



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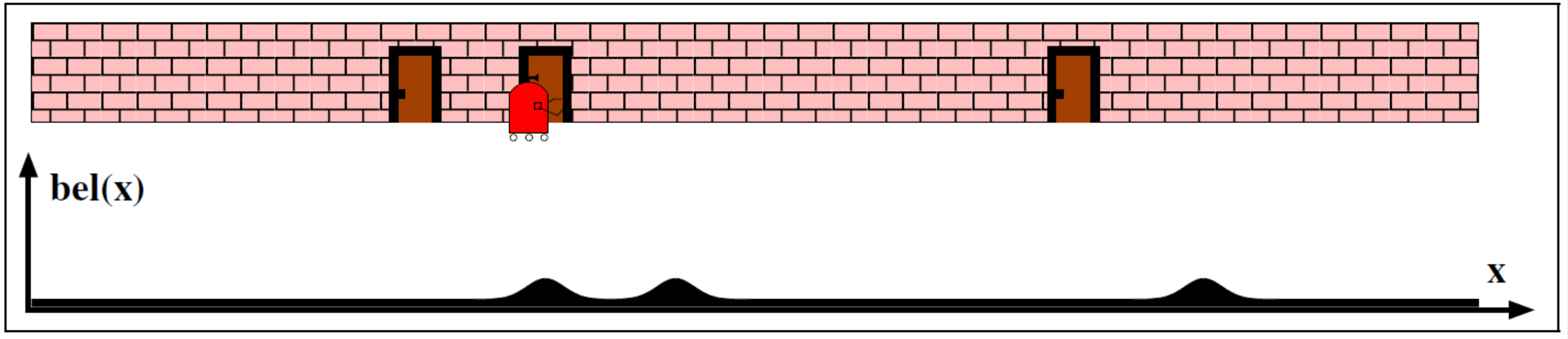
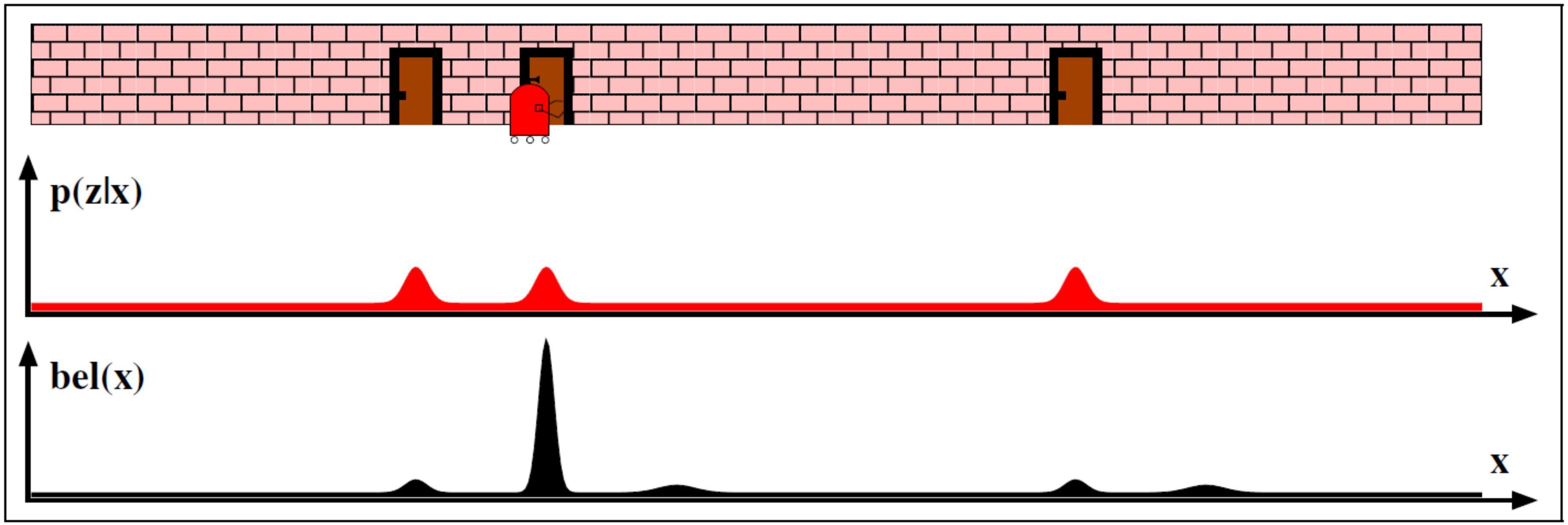
Recursive Bayes Filter





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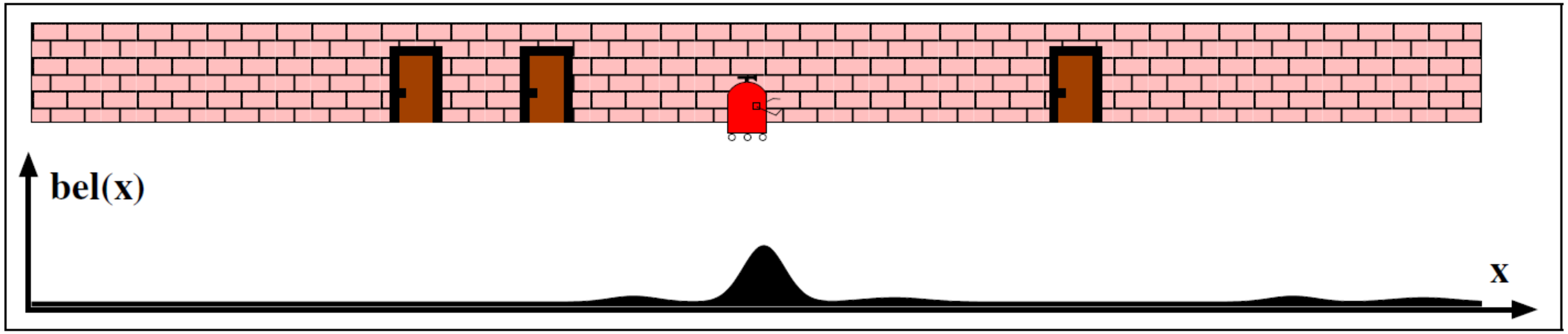
Recursive Bayes Filter



Courtesy: Thrun, Burgard and Fox

Tiny Reminder (Probability Theory)

(reminder)

Bayes' Rule

$$p(x, y) = p(x \mid y)\, p(y)$$

$$p(x, y) = p(y \mid x)\, p(x)$$

$$\Downarrow$$

$$\boxed{p(x \mid y) = \frac{p(y \mid x)\, p(x)}{p(y)} = \frac{\text{likelihood prior}}{\text{evidence}}}$$

(reminder)

Bayes' Rule with Background Knowledge z

$$p(x \mid y) = \frac{p(y \mid x)\, p(x)}{p(y)}$$

$$p(x \mid y, z) = \frac{p(y \mid x, z)\, p(x \mid z)}{p(y \mid z)}$$

(reminder)

Law of Total Probability and Marginalization

Law of Total Probability

$$p(x) = \sum_y p(x \mid y)\, p(y)$$

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

Marginalization

$$p(x) = \sum_y p(x, y)$$

$$p(x) = \int p(x, y)\, dy$$

Markov Property/Assumption

- "The future is independent from the past given the current state."
- Markov property = the conditional probability distribution of future states depends only upon the present state, not on the sequence of events that preceded it.
- Such a process has no memory

State Estimation

- Estimate the state x of a system given observations z and controls u
- **Goal:**

$$p(x_t | z_{1:t}, u_{1:t})$$

Recursive Bayes Filter 1

$$bel(x_t) = \underline{p(x_t \mid z_{1:t}, u_{1:t})}$$

definition of the belief

Recursive Bayes Filter 2

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \underline{\eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}
\end{aligned}
$$

Bayes' rule

Recursive Bayes Filter 3

$$bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$
$$= \quad \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})$$
$$= \quad \eta\, \underline{p(z_t \mid x_t)}\, p(x_t \mid z_{1:t-1}, u_{1:t})$$

Markov assumption

Recursive Bayes Filter 4

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \underline{\int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})} \\
&\qquad \underline{p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}}
\end{aligned}
$$

Law of total probability

Recursive Bayes Filter 5

$$
\begin{aligned}
bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int \underline{p(x_t \mid x_{t-1}, u_t)}\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}
\end{aligned}
$$

Markov assumption

Recursive Bayes Filter 6

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:\underline{t-1}})\, dx_{t-1}
\end{aligned}
$$

independence assumption

Recursive Bayes Filter 7

$$
\begin{aligned}
bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, \underline{bel(x_{t-1})}\, dx_{t-1}
\end{aligned}
$$

recursive term

Complete Derivation of the Recursive Bayes Filter

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, dx_{t-1}
\end{aligned}
$$

Prediction and Correction Step

- Bayes filter can be written as a two step process

$$bel(x_t) = \underline{\eta\, p(z_t \mid x_t)} \; \underline{\int p(x_t \mid u_t, x_{t-1})\; bel(x_{t-1})\, dx_{t-1}}$$

- **Prediction step**

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\; bel(x_{t-1})\, dx_{t-1}$$

- **Correction step**

$$bel(x_t) = \eta\, p(z_t \mid x_t)\; \overline{bel}(x_t)$$

Motion and Observation Model

- Prediction step

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1})}_{\textbf{motion model}} \, bel(x_{t-1}) \, dx_{t-1}$$

- Correction step

$$bel(x_t) = \eta \, \underbrace{p(z_t \mid x_t)}_{\substack{\textbf{observation model} \\ \textbf{(also: measurement or sensor model)}}} \, \overline{bel}(x_t)$$

Different Realizations

- The Bayes filter is a **framework** for recursive state estimation
- There are **different realizations**
- **Different properties**
  - Linear vs. non-linear models for motion and observation models
  - Gaussian distributions only?
  - Parametric vs. non-parametric filters
  - …

Popular Filters

- **Kalman filter & EKF**
  - Gaussians
  - Linear or linearized models

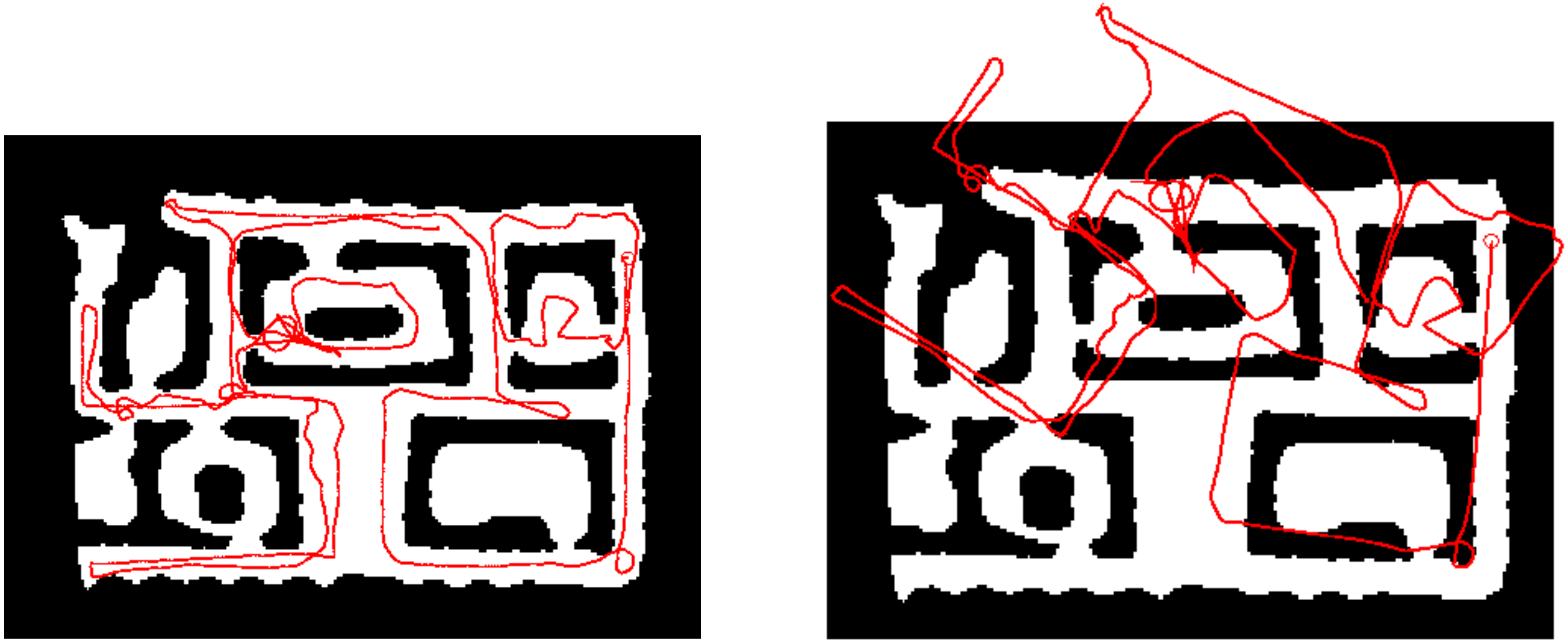
- **Particle filter**
  - Non-parametric
  - Arbitrary models (sampling required)

Motion Model

$$\overline{bel}(x_t) = \int \boxed{p(x_t \mid u_t, x_{t-1})}\, bel(x_{t-1})\, dx_{t-1}$$

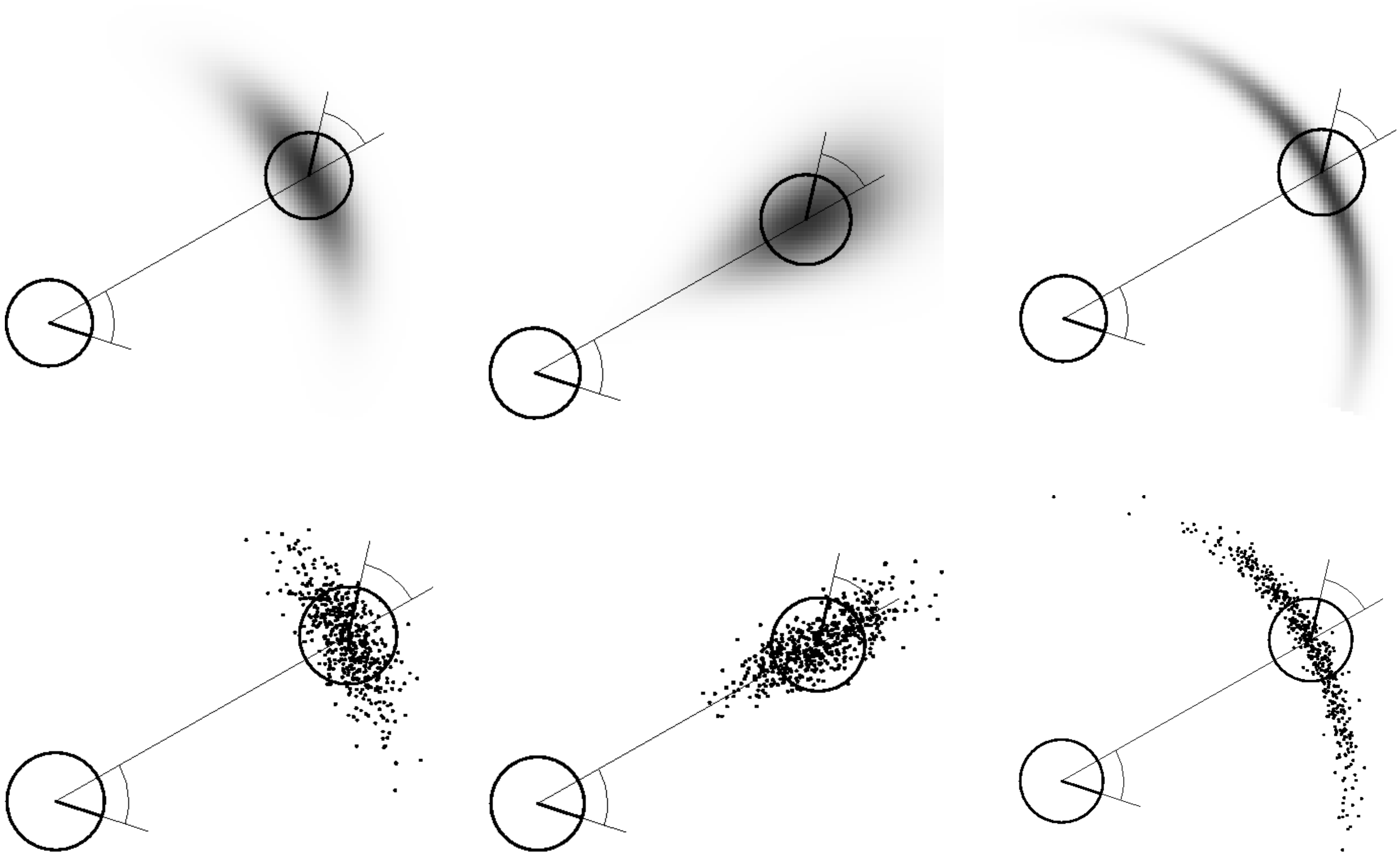
Basic Motion Models

- Motion is inherently uncertain
- How can we model this uncertainty?



Courtesy: Thrun, Burgard, Fox

Example: Odometry-Based Motion



Courtesy: Thrun, Burgard, Fox

Probabilistic Motion Models

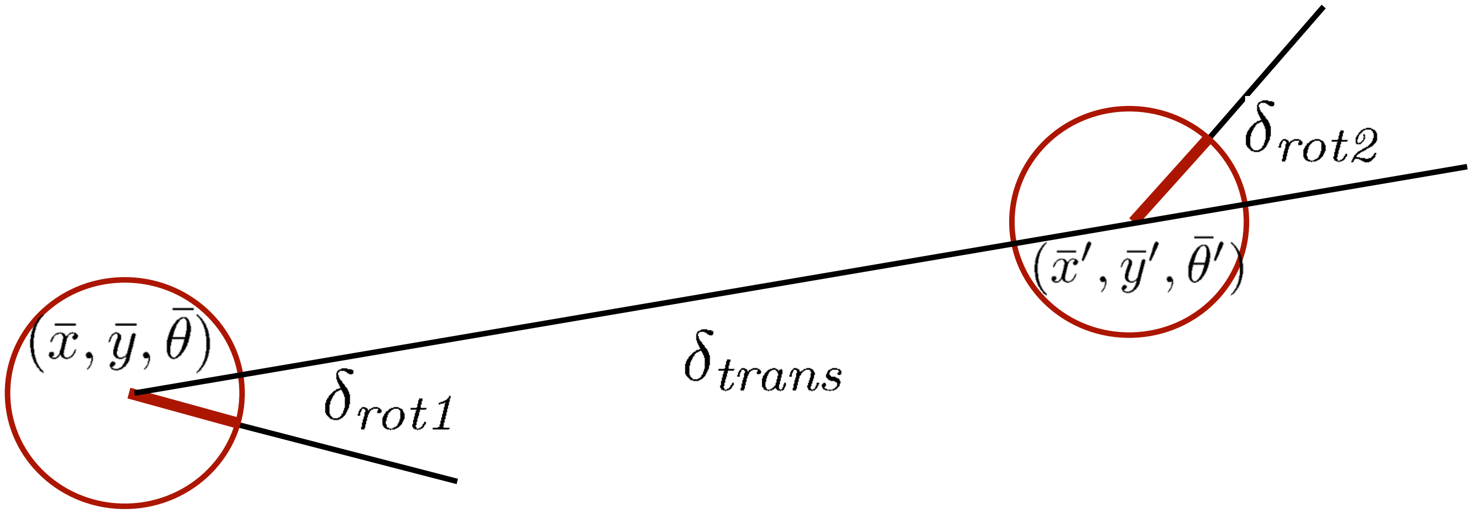
- Specifies a posterior probability that action *u* carries the robot from x_{t-1} to x_t

$$p(x_t \mid u_t, x_{t-1})$$

Odometry Model

- Motion from $(\bar{x}, \bar{y}, \bar{\theta})$ to $(\bar{x}', \bar{y}', \bar{\theta}')$
- Odometry information $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$

$$
\begin{aligned}
\delta_{trans} &= \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2} \\
\delta_{rot1} &= \operatorname{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta} \\
\delta_{rot2} &= \bar{\theta}' - \bar{\theta} - \delta_{rot1}
\end{aligned}
$$



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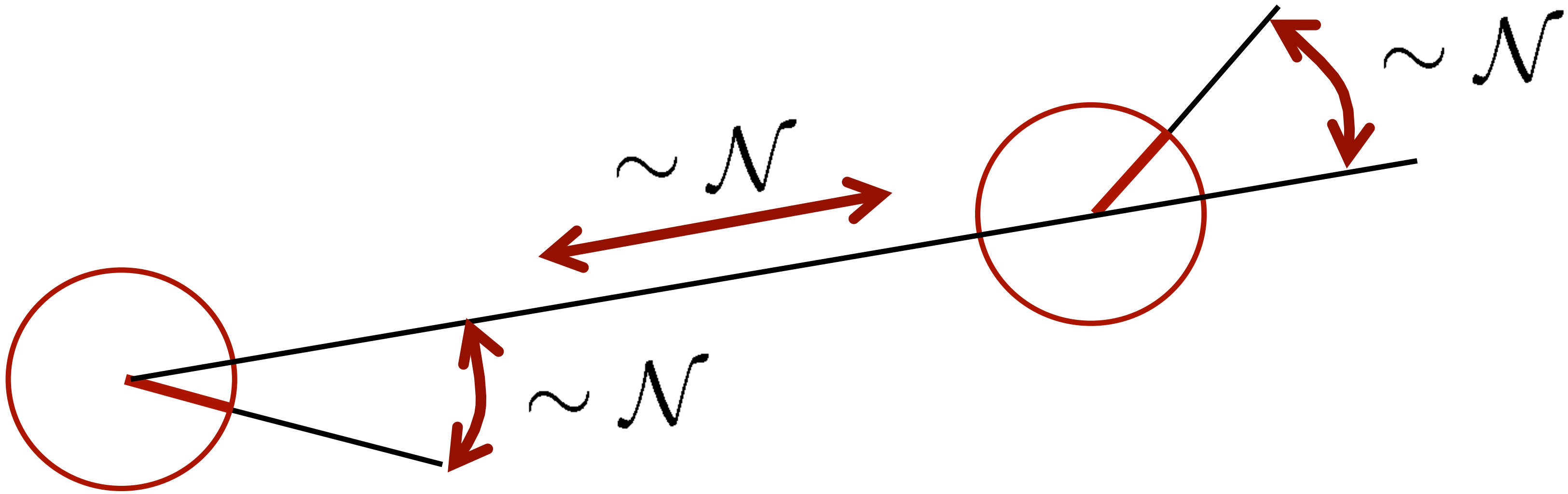
Probability Distribution

- Noise in odometry $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$
- Example: Gaussian noise

$$u \sim \mathcal{N}(0, \Sigma)$$



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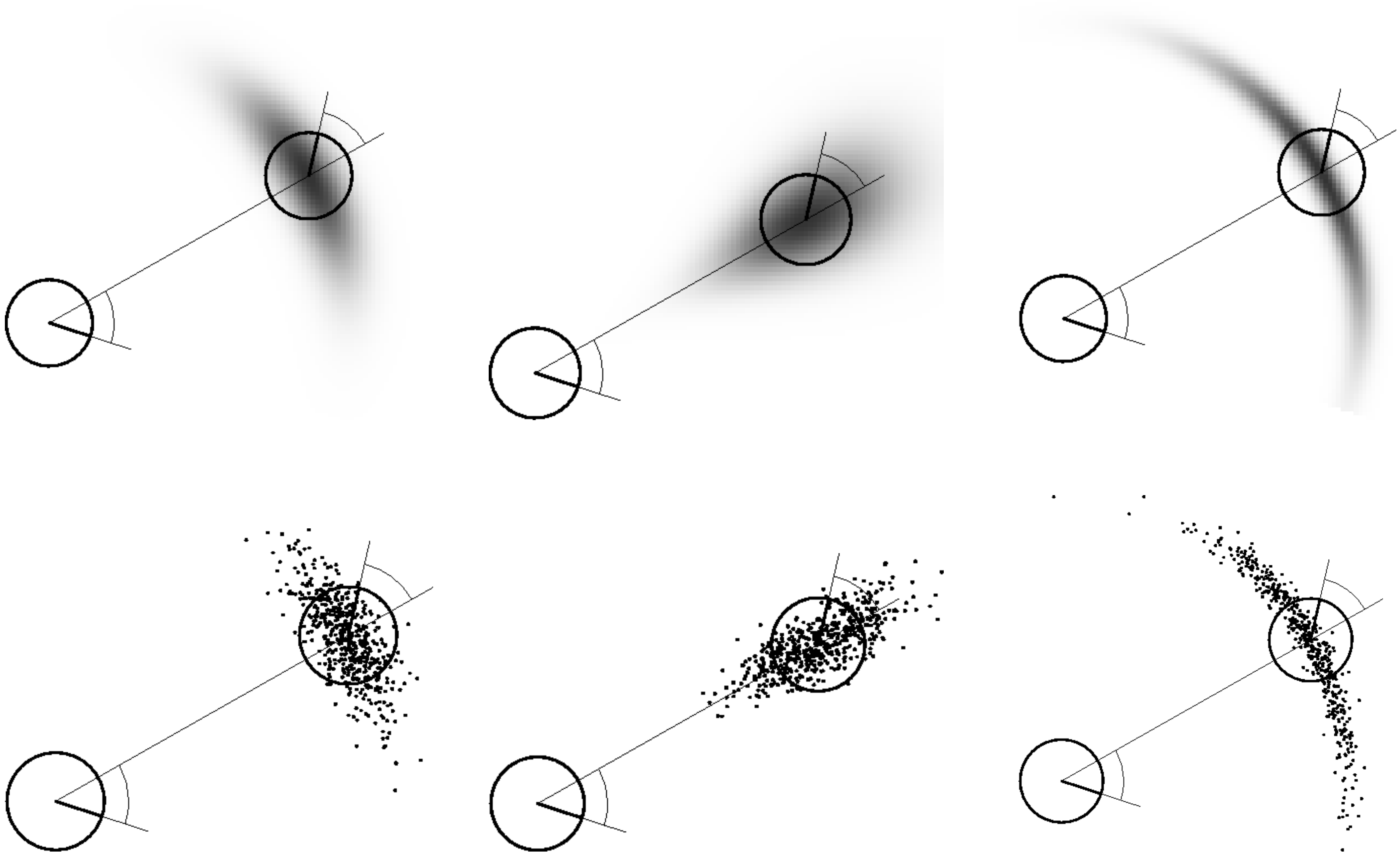
Example: Odometry-Based Motion



Courtesy: Thrun, Burgard, Fox

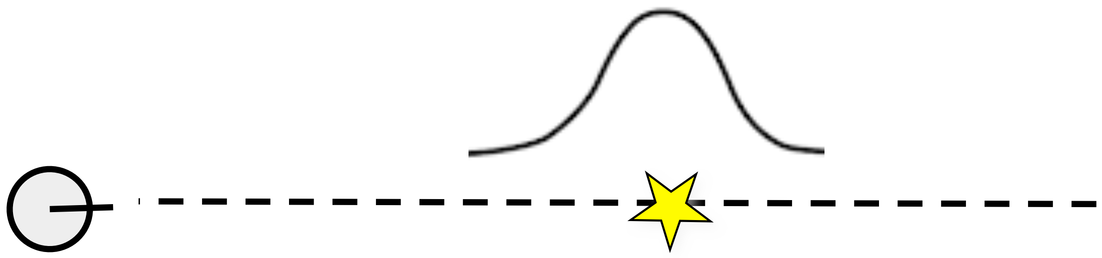
Observation Model

$$bel(x_t) = \eta \boxed{p(z_t \mid x_t)} \overline{bel}(x_{t-1})$$

Range Sensors

Example: Simple Observation Model with Gaussian Noise

- Range sensor estimating the distance to the closest obstacle
- Gaussian noise in the range reading



(optional)

Model for Laser Scanners

- Scan z consists of K measurements.

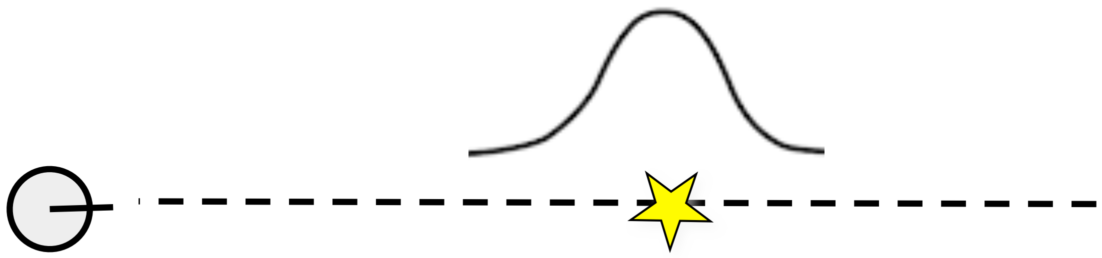
$$z_t = \{z_t^1, \ldots, z_t^k\}$$

- Individual measurements are independent given the sensor position

$$p(z_t \mid x_t, m) = \prod_{i=1}^{k} p(z_t^i \mid x_t, m)$$

Simplest Ray-Cast Model

- Ray-cast models consider the first obstacle along the line of sight
- Gaussian noise in the distance



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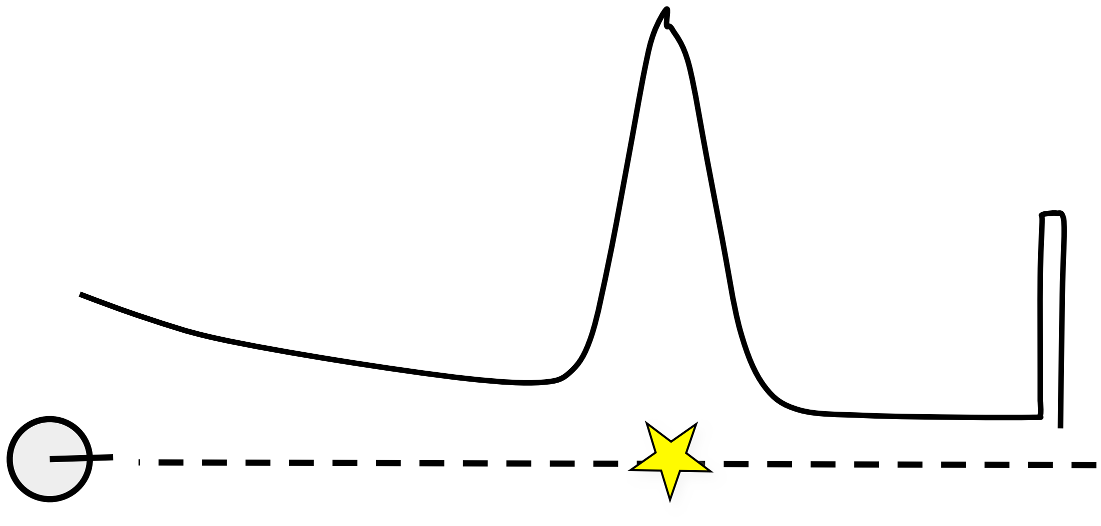
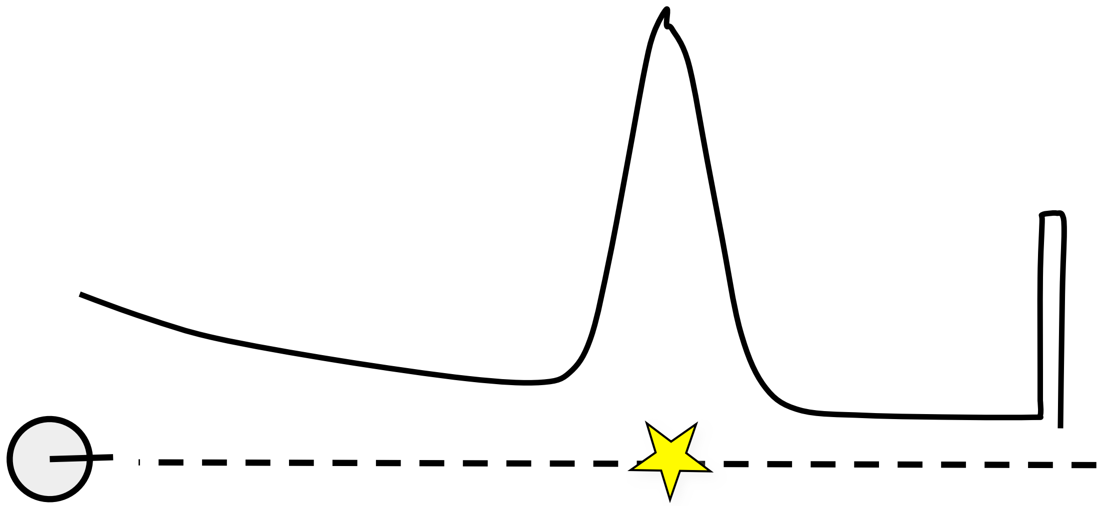
More Advanced Ray-Cast Model

- Ray-cast models consider the first obstacle along the line of sight
- A more advanced model may look like that. Why?



More Advanced Ray-Cast Model

- Ray-cast models consider the first obstacle along the line of sight
- Mixture of four models: considers different effects (dynamic objects, random, max-range, noise)



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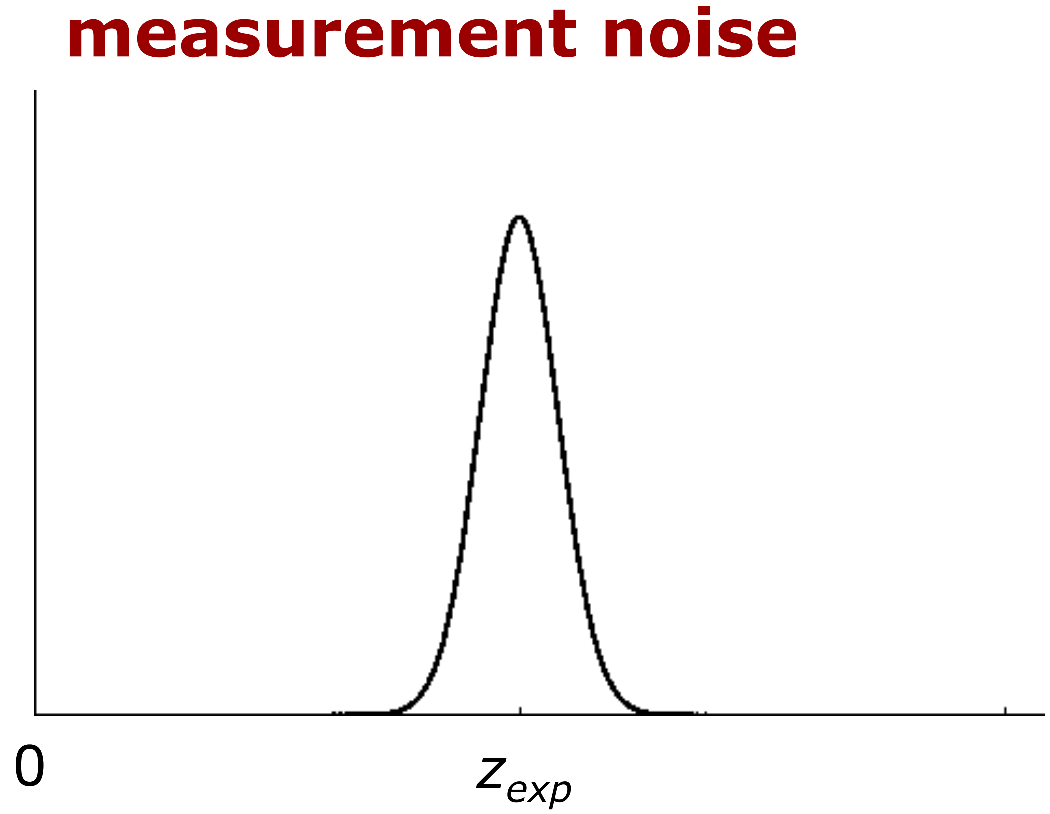
Beam-Based Proximity Model

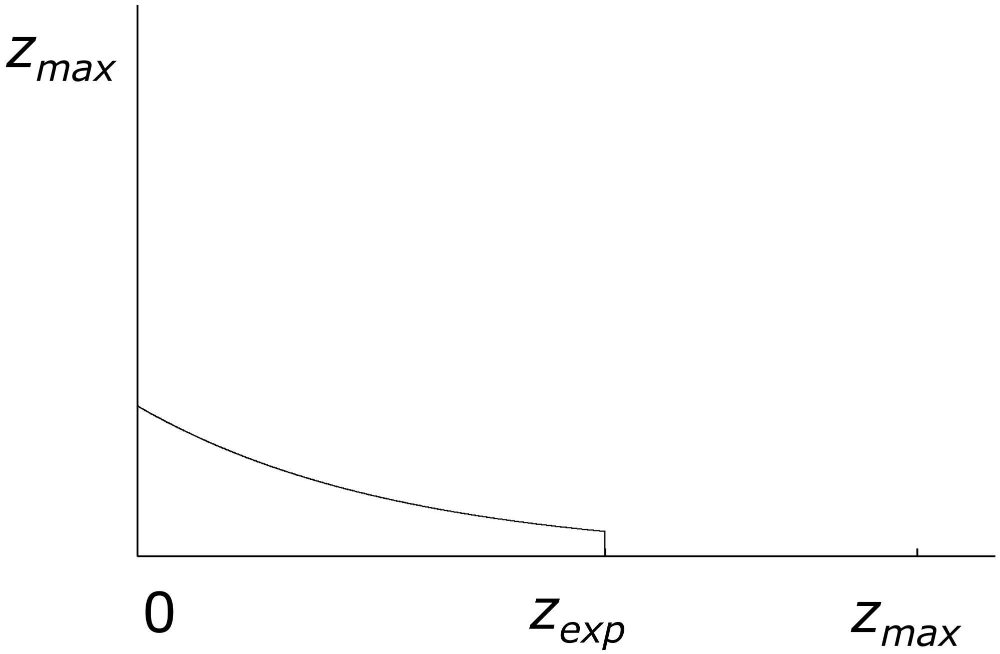


$$P_{hit}(z \mid x, m) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{(z - z_{exp})^2}{\sigma^2}\right)$$

Image courtesy: Burgard

Beam-Based Proximity Model

unexpected obstacles

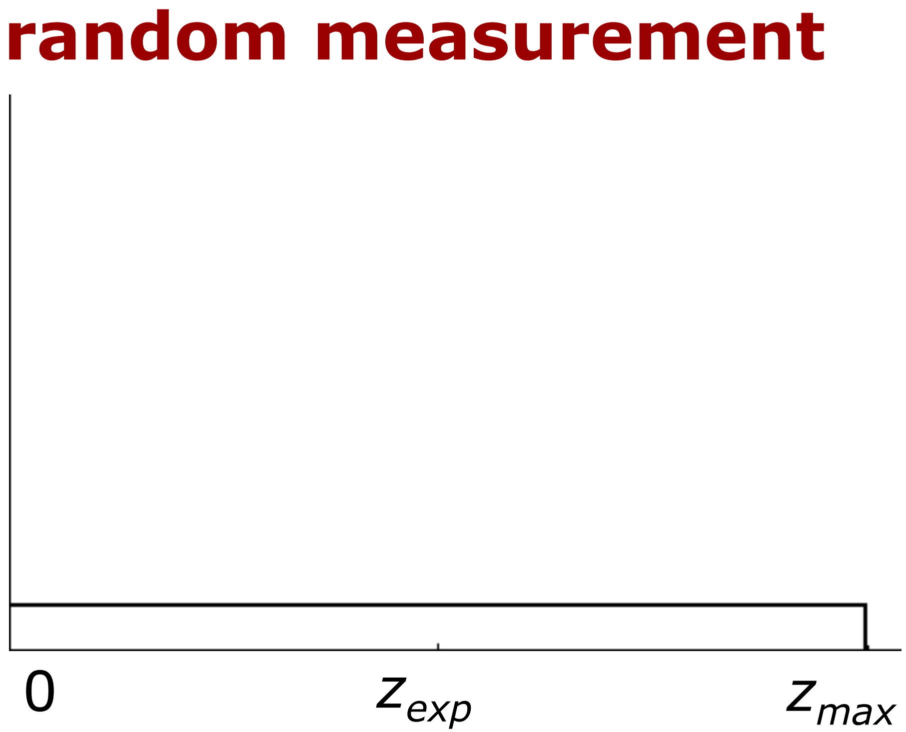


$$P_{unexp}(z \mid x, m) = \begin{cases} \lambda \exp(-\lambda z) & z < z_{exp} \\ 0 & \text{otherwise} \end{cases}$$

Image courtesy: Burgard

(optional)

Beam-Based Proximity Model

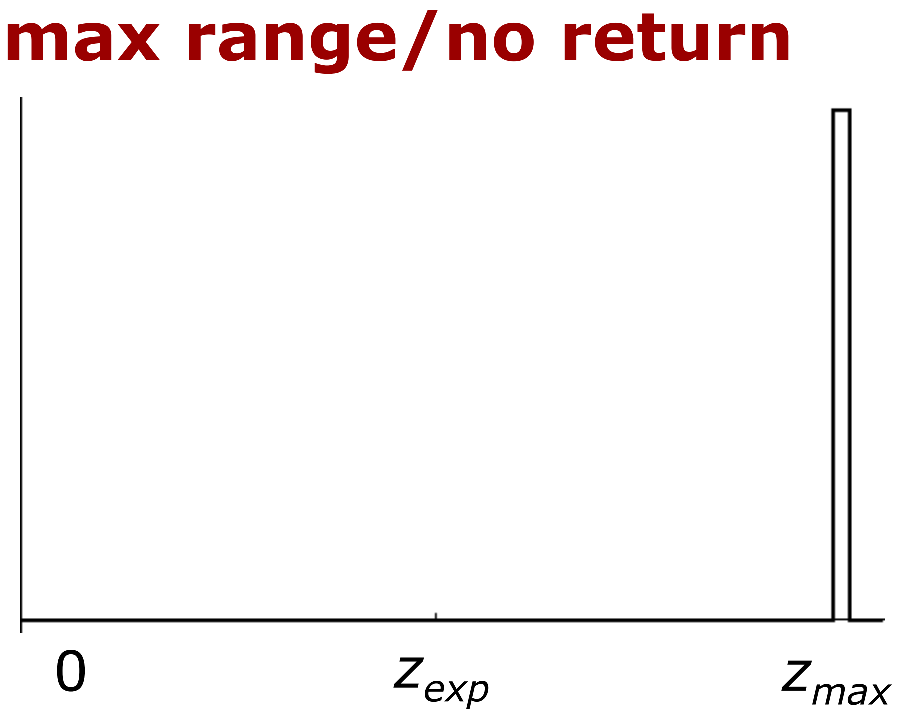


$$P_{rand}(z \mid x, m) = \text{const.}$$

Image courtesy: Burgard

(optional)

Beam-Based Proximity Model

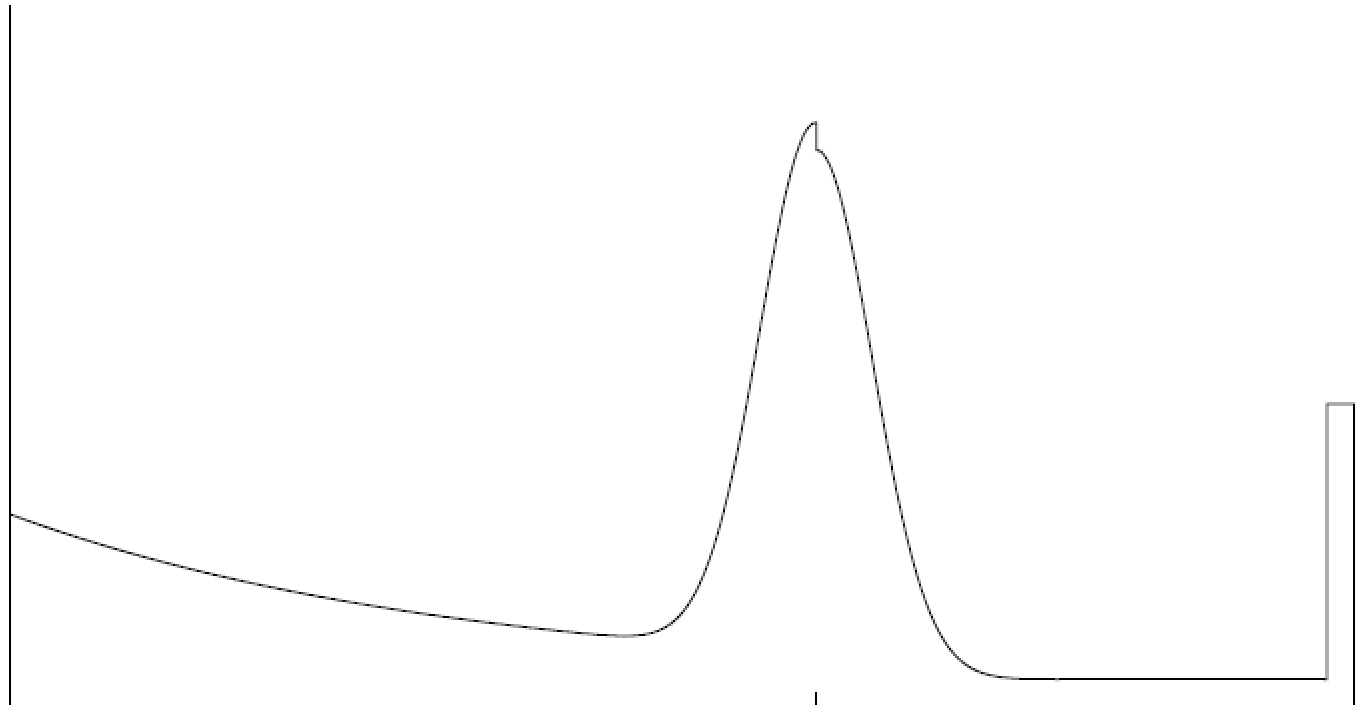


$$P_{max}(z \mid x, m) = \begin{cases} \text{const.} & z = z_{max} \\ 0 & \text{otherwise} \end{cases}$$

Image courtesy: Burgard

(optional)

Resulting Mixture Density



How can we determine the parameters?

Image courtesy: Burgard

(optional)

Raw Sensor Data

Measured distances for expected distance of 3m.

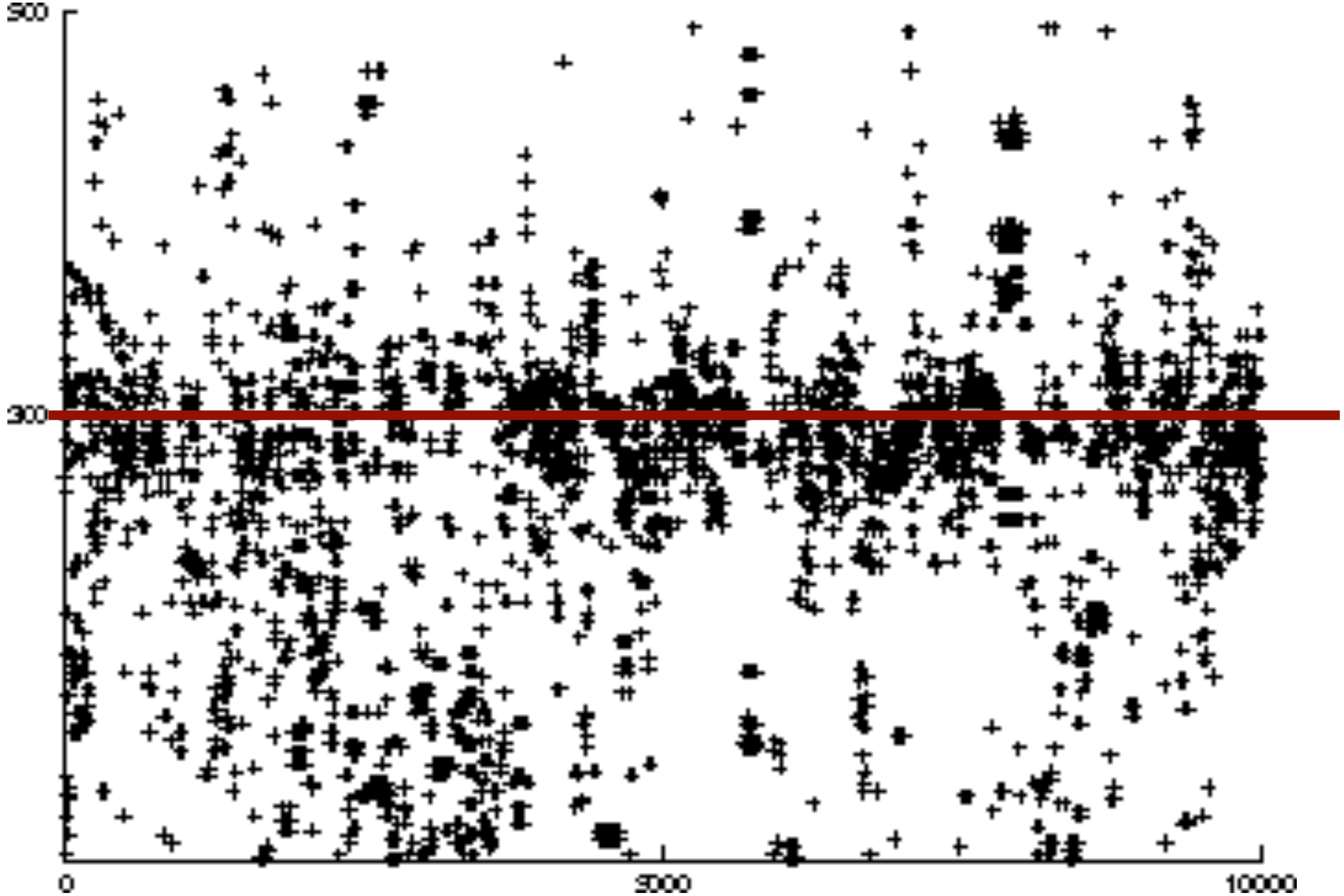
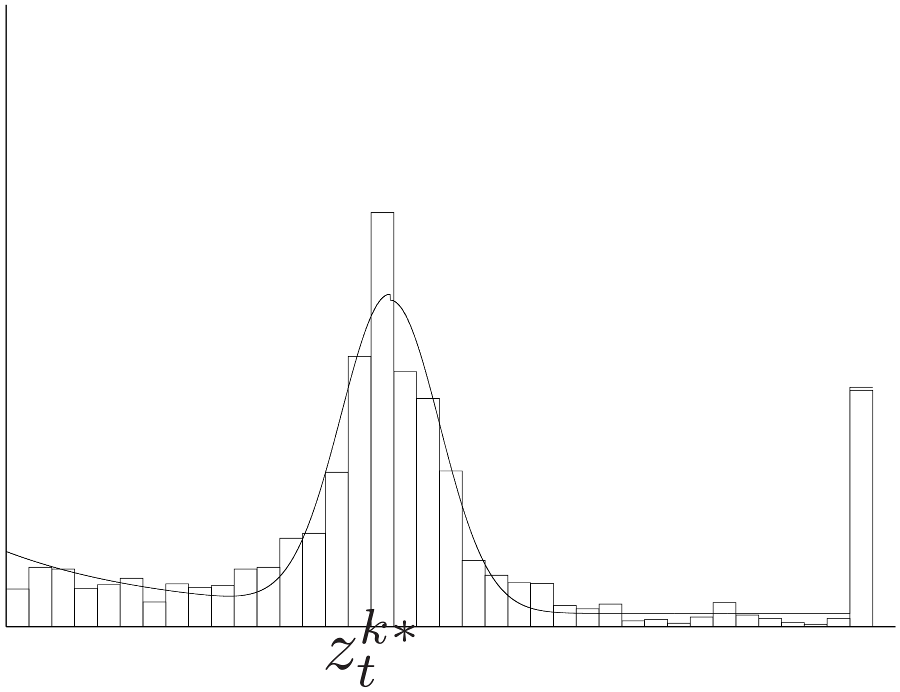


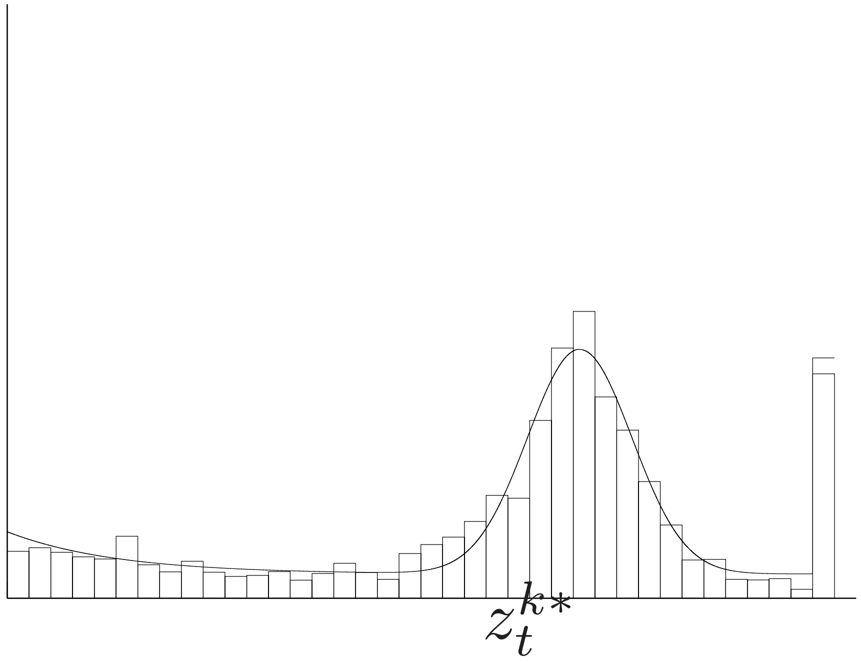
Image courtesy: Burgard

(optional)

Results



300cm



400cm

Image courtesy: Burgard

(optional)

Beam-Endpoint Model

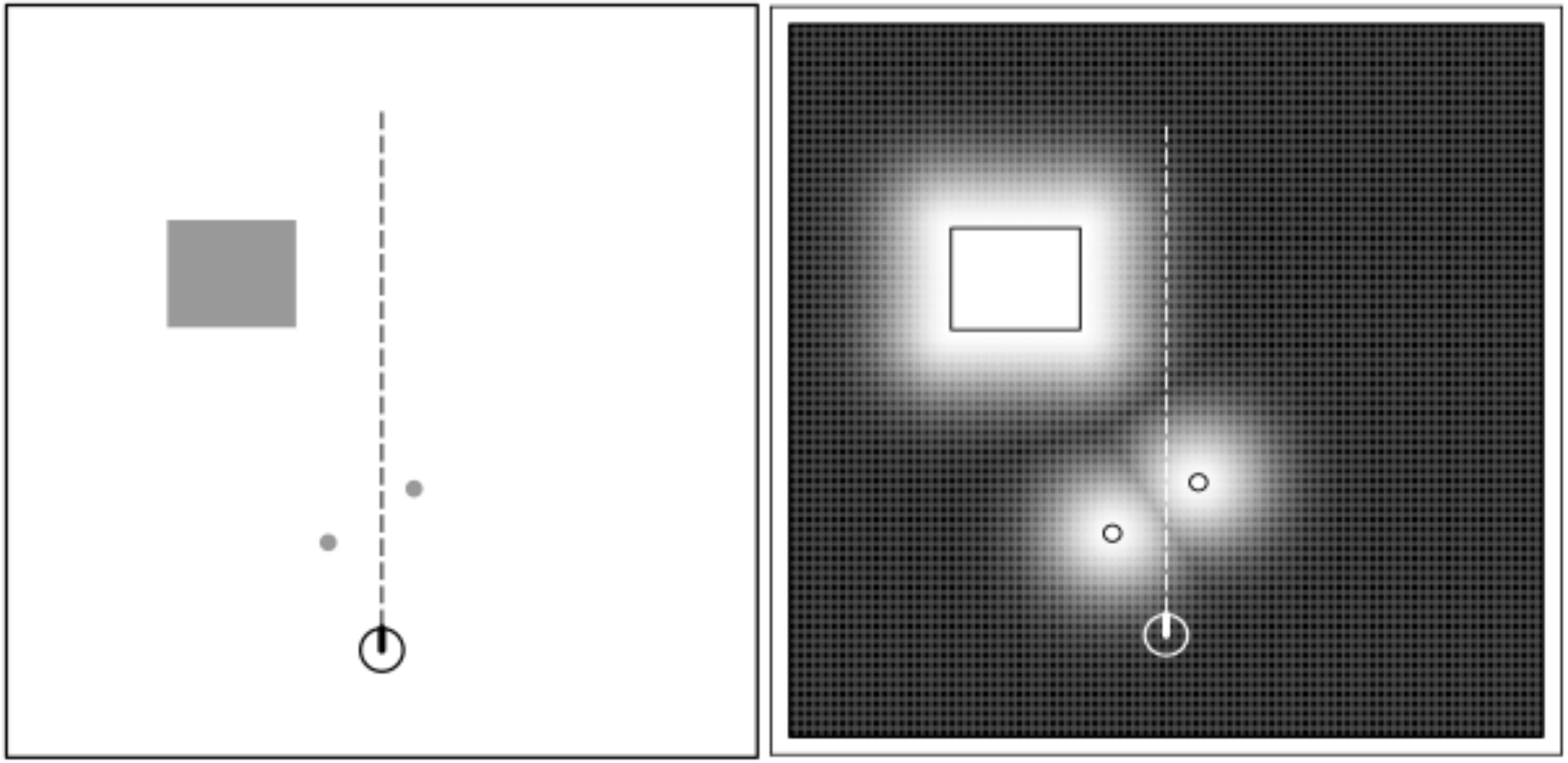
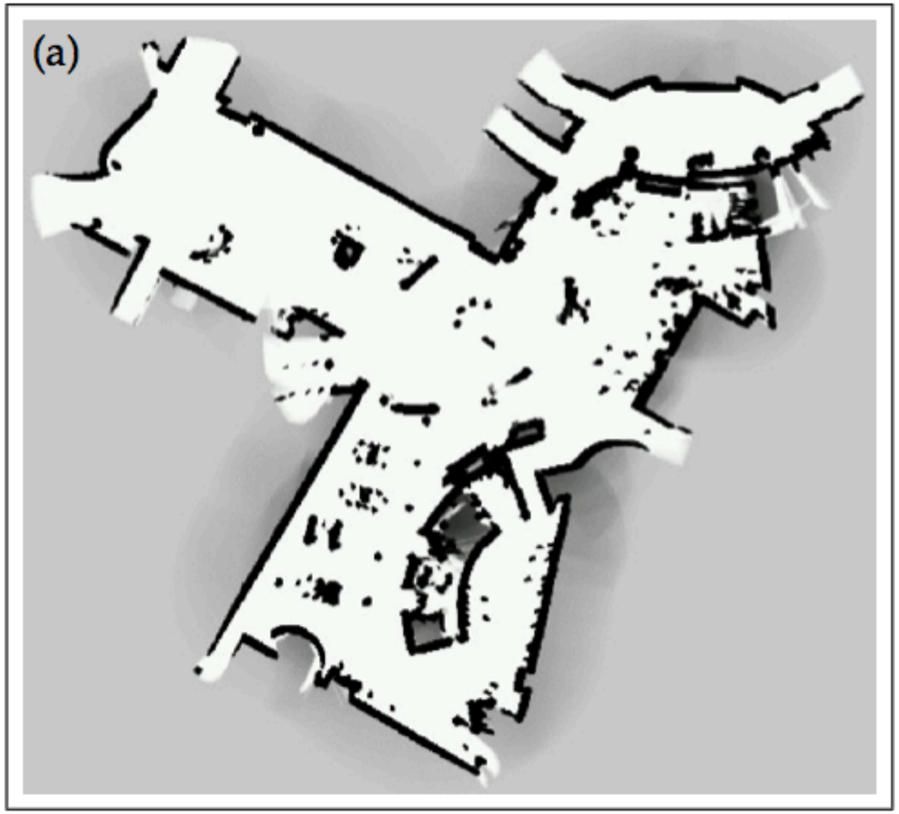
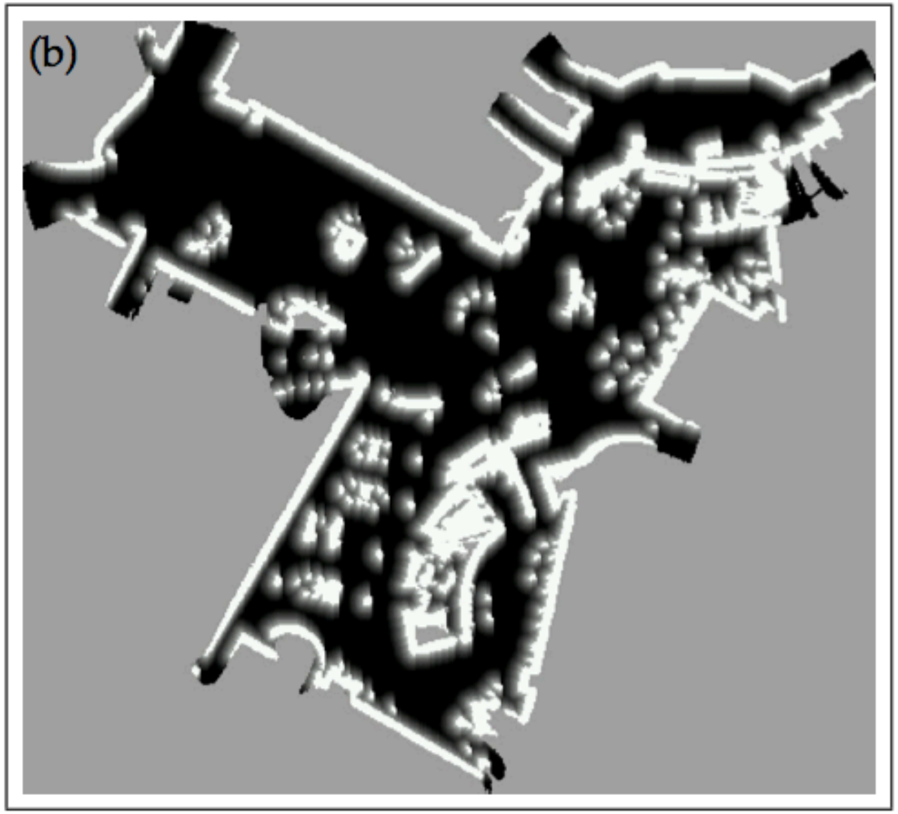


Image courtesy: Roy / Thrun, Burgard, Fox

Beam-Endpoint Model



map



likelihood field

Courtesy: N. Roy

Model for Perceiving Landmarks with Range-Bearing Sensors

- Range-bearing $z_t^i = (r_t^i, \phi_t^i)^T$
- Pose $(x, y, \theta)^T$
- Observation of feature j at location $(m_{j,x}, m_{j,y})^T$

$$\begin{pmatrix} r_t^i \\ \phi_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \end{pmatrix} + Q_t$$

What if monocular cameras are used?

Summary

- Probabilities occur is most of the problems addressed here
- Bayes filter is a framework for state estimation
- There are different realizations of the Bayes filter that we will study in this course (e.g., EKF, particle filter)
- Motion and observation model are central models in the Bayes filter to be specified

Literature

Probability Primer

- Thrun et al. "Probabilistic Robotics", Chapter 2.1 & 2.2

On the Bayes filter

- Thrun et al. "Probabilistic Robotics", Chapter 2.3