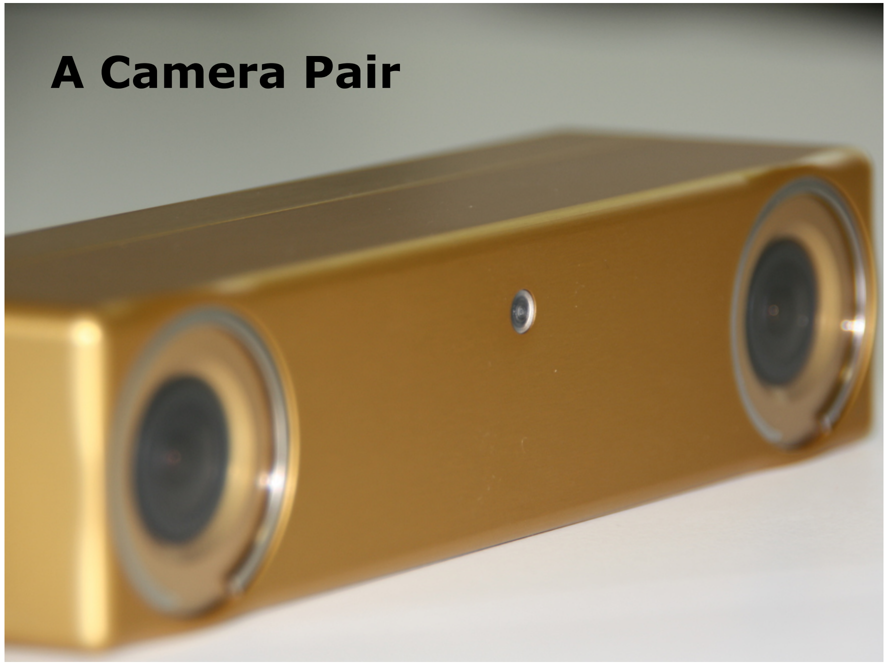
Photogrammetry & Robotics Lab

Relative Orientation, Fundamental and Essential Matrix

Cyrill Stachniss

The slides have been created by Cyrill Stachniss.



Camera Pair

- A stereo camera
- One camera that moves

Camera pair = two configurations from which images have been taken

Orientation Parameters for the Camera Pair and Relative Orientation

Orientation

- The orientation of the camera pair can be described using independent orientations for each camera

How many parameters are needed?

- Calibrated cameras: **?** parameters (angle preserving mapping)
- Uncalibrated cameras: **?** parameters (straight-line preserving mapping)

Orientation

- The orientation of the camera pair can be described using independent orientations for each camera

How many parameters are needed?

- Calibrated cameras: **12** parameters (angle preserving mapping)
- Uncalibrated cameras: **22** parameters (straight-line preserving mapping)

Can We Estimate the Camera Motion without Knowing the Scene?

Which Parameters Can We Obtain and Which Not?

Cameras Measure Directions

- We cannot obtain the (global) **translation** and **rotation** (if the cameras maintain their relative transformation) as well as the **scale**

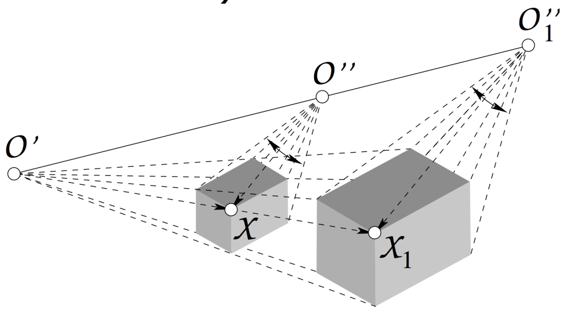


Image courtesy: Förstner & Wrobel

What We Can Compute

- The **rotation** R of the second camera w.r.t. the first one (3 parameters)
- The **direction** B of the line connecting the to centers of projection (2 params)
- We do **not know** their **distance** (the length of B)

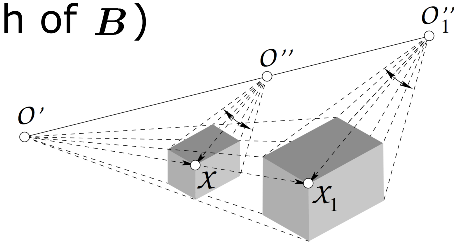


Image courtesy: Förstner & Wrobel

For Calibrated Cameras

- We need 2x6=12 parameters for two calibrated cameras for the orientation
- With a calibrated camera, we obtain an angle-preserving model of the object
- Without additional information, we can **only obtain** 12-7 = **5 parameters** (not 7=translation, rotation, scale)

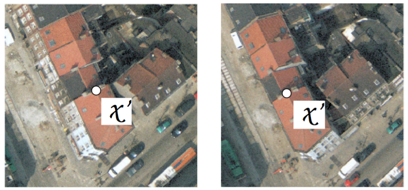
first camera

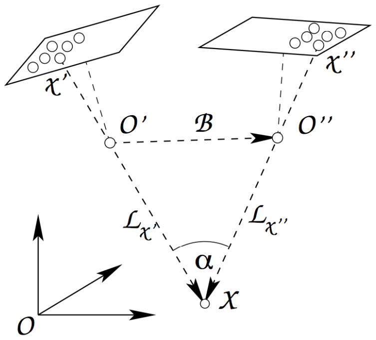
distance between cameras

Photogrammetric Model

- Given two cameras images, we can reconstruct the object only **up to a similarity transform**
- Called a **photogrammetric model**
- The **orientation of the photogrammetric model** is called the **absolute orientation**
- For obtaining the absolute orientation, we need at least **3 points** in 3D (to estimate the 7 parameters)

What Is Needed for Computing an **3D Model of a Scene?**





For Uncalibrated Cameras

- **Straight-line preserving** but **not angle preserving**
- Object can be reconstructed up to a straight-line preserving mapping
- Projective transform (15 parameters)
- Thus, for uncalibrated cameras, we can only **obtain 22-15=7** parameters given two images
- We need at **least 5 points** in 3D (15 coordinates) for the absolute o.

Relative Orientation Summary

Cameras	#params /img	#params /img pair	#params for RO	#params for AO	min #P
calibrated	**6**	**12**	**5**	**7**	**3**
not calibrated	**11**	**22**	**7**	**15**	**5**

RO = relative orientation

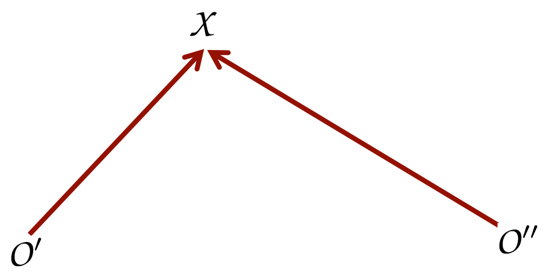
AO = absolute orientation

min #P = min. number of control points

Coplanarity Constraint for Straight-Line Preserving (Uncalibrated) Cameras to Obtain the Fundamental Matrix

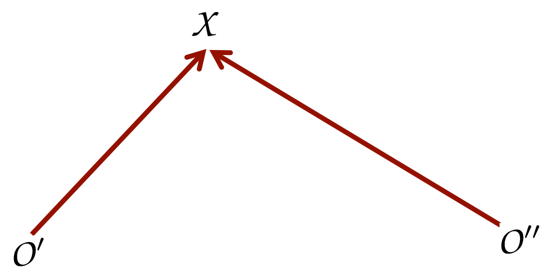
Which Parameters Can We Compute **Without Additional** Information About the Scene?

We start with a perfect orientation and the intersection of two corresponding rays



Coplanarity Constraint

- Consider perfect orientation and the intersection of two corresponding rays
- The rays lie within one plane in 3D

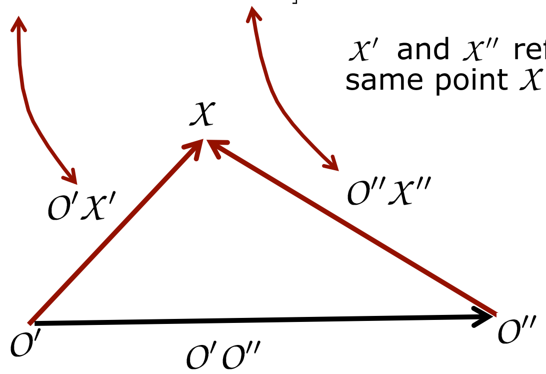


Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$

X' and X'' refer to the same point X in space



Scalar Triple Product

- The operator $[\cdot, \cdot, \cdot]$ is the triple product
- Dot product of one of the vectors with the cross product of the other two

$$[A, B, C] = (A \times B) \cdot C$$

- It is the volume of the parallelepiped of three vectors
- $[A, B, C] = 0$ means that the vectors lie in one plane

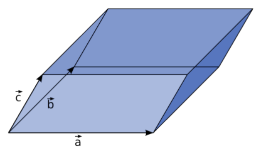
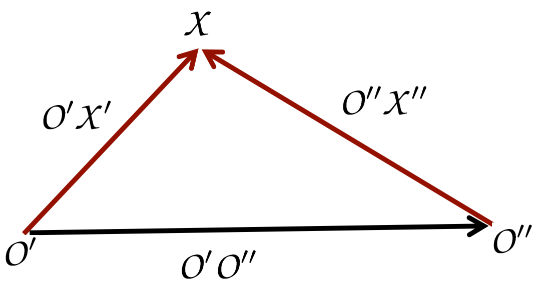


Image courtesy: Wikipedia (Niabot)

Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[\mathcal{O}'\mathcal{X}' \quad \mathcal{O}'\mathcal{O}'' \quad \mathcal{O}''\mathcal{X}''] = 0$$



Coplanarity Constraint for Uncalibrated Cameras

- The directions of the vectors $\mathcal{O}'\mathcal{X}'$ and $\mathcal{O}''\mathcal{X}''$ can be derived from the image coordinates $\mathbf{x}', \mathbf{x}''$

$$\mathbf{x}' = \mathsf{P}'\mathbf{X} \qquad \mathbf{x}'' = \mathsf{P}''\mathbf{X}$$

- with the projection matrices

$$\mathsf{P}' = \mathsf{K}'R'[I_3| - \boldsymbol{X}_{O'}] \qquad \mathsf{P}'' = \mathsf{K}''R''[I_3| - \boldsymbol{X}_{O''}]$$

Reminder: $[I_3| - \boldsymbol{X}_{O''}] = \begin{bmatrix} 1 & 0 & 0 & -X_{O''} \\ 0 & 1 & 0 & -Y_{O''} \\ 0 & 0 & 1 & -Z_{O''} \end{bmatrix}$

Directions to a Point

- The normalized directions of the vectors $\mathcal{O}''\mathcal{X}''$ and $\mathcal{O}'\mathcal{X}'$ are

$$\boxed{{}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'} \longleftarrow \text{image coord.}$$

- as the normalized projection

$${}^n\mathbf{x}' = [I_3| - \boldsymbol{X}_{O'}]\mathbf{X} \longleftarrow \text{world coord.}$$

- provides the direction to from the center of projection to the point in 3D
- Analogous:

$${}^n\mathbf{x}'' = (R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}''$$

Base Vector

- The base vector $\mathcal{O}'\mathcal{O}''$ directly results from the coordinates of the projection centers

$$\boxed{\mathrm{b} = \boldsymbol{B} = \boldsymbol{X}_{O''} - \boldsymbol{X}_{O'}}$$

Coplanarity Constraint

- Using the previous relations, the coplanarity constraint

$$[\mathcal{O}'\mathcal{X}' \quad \mathcal{O}'\mathcal{O}'' \quad \mathcal{O}''\mathcal{X}''] = 0$$

- can be rewritten as

$$\begin{aligned} [{}^n\mathbf{x}' \quad \mathbf{b} \quad {}^n\mathbf{x}''] &= 0 \\ {}^n\mathbf{x}' \cdot (\mathbf{b} \times {}^n\mathbf{x}'') &= 0 \\ {}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0 \end{aligned}$$

skew-symmetric matrix

Derivation

- Why is this correct?

$$\begin{aligned} {}^n\mathbf{x}' \cdot (\mathbf{b} \times {}^n\mathbf{x}'') &= 0 \\ {}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0 \end{aligned}$$

Derivation

- Why is this correct?

$$\begin{aligned} {}^n\mathbf{x}' \cdot (\mathbf{b} \times {}^n\mathbf{x}'') &= 0 \\ {}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0 \end{aligned}$$

- Results from the cross product as

$$\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -b_3x_2 + b_2x_3 \\ b_3x_1 - b_1x_3 \\ -b_2x_1 + b_1x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}}_{S_b} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$$

- with S_b being a skew-symmetric matrix

Coplanarity Constraint

- By combining ${}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' = 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}}(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}S_b(R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}'' = 0$$

Coplanarity Constraint

- By combining ${}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' \quad = \quad 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' = 0$$

$$\begin{aligned} \mathsf{F} &= (\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1} \\ &= (\mathsf{K}')^{-\mathsf{T}}R'\mathsf{S}_b R''^{\mathsf{T}}(\mathsf{K}'')^{-1} \end{aligned}$$

Fundamental Matrix

- The matrix F is the **fundamental matrix** (for uncalibrated cameras):

$$\boxed{\mathsf{F} \quad = \quad (\mathsf{K}')^{-\mathsf{T}}R'\mathsf{S}_b R''^{\mathsf{T}}(\mathsf{K}'')^{-1}}$$

- It allow for expressing the **coplanarity constraint** by

$$\boxed{\mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' \quad = \quad 0}$$

Fundamental Matrix

- The **fundamental matrix** is the matrix that fulfills the equation

$$\mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' \quad = \quad 0$$

for corresponding points

- The fundamental matrix contains the all the available **information about the relative orientation** of **two images** from uncalibrated cameras

Fundamental Matrix From the Camera Projection Matrices

- If the projection matrices are given, we can derive the fundamental matrix?

$$\mathsf{P}', \mathsf{P}'' \rightarrow \mathsf{F}$$

- Let the projection matrices be partitioned into a left 3×3 matrix and a 3-vector as $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}']$.

Fundamental Matrix From the Camera Projection Matrices

- We have $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}'] = [\underbrace{\mathsf{K}'\mathsf{R}'}_{\mathsf{A}'}|\underbrace{-\mathsf{K}'\mathsf{R}'\boldsymbol{X}_{O'}}_{\mathbf{a}'}]$
- and can recover the projection center

$$\mathsf{A}'^{-1}\mathbf{a}' = (\mathsf{K}'\mathsf{R}')^{-1}\mathbf{a}' = -\mathsf{R}'^{\top}\mathsf{K}'^{-1}\mathsf{K}'\mathsf{R}'\boldsymbol{X}_{O'} = -\boldsymbol{X}_{O'}$$

$$\boldsymbol{X}_{O'} = -\mathsf{A}'^{-1}\mathbf{a}'$$

- so that the base line is given by

$$\mathbf{b}'_{12} = \mathsf{A}''^{-1}\mathbf{a}'' - \mathsf{A}'^{-1}\mathbf{a}'$$

Fundamental Matrix From the Camera Projection Matrices

- We have $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}'] = [\underbrace{\mathsf{K}'\mathsf{R}'}_{\mathsf{A}'}|\underbrace{-\mathsf{K}'\mathsf{R}'\boldsymbol{X}_{O'}}_{\mathbf{a}'}]$
- and $\mathbf{b}'_{12} = \mathsf{A}''^{-1}\mathbf{a}'' - \mathsf{A}'^{-1}\mathbf{a}'$
- and thus can compute the F

$$\mathsf{F} \;=\; (\mathsf{K}')^{-\mathsf{T}}R'\mathsf{S}_b R''^{\mathsf{T}}(\mathsf{K}'')^{-1} \;=\; \mathsf{A}'^{-\top}\mathsf{S}_{b'_{12}}\mathsf{A}''^{-1}$$

- with $S_b = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$

Alternative Definition

- In the context of many images, we will call F_{ij} that fundamental matrix which yields the constraint $\mathbf{x}'^{\mathsf{T}}_i\mathsf{F}_{ij}\mathbf{x}''_j = 0$
- Thus in our case, we have $\mathsf{F} = \mathsf{F}_{12}$
- Our definition of F is not the same as in the book by Hartley and Zisserman
- The definition in Hartley and Zisserman is based on $\mathbf{x}''^{\top}_i\mathsf{F}_{ij}\mathbf{x}'_j = 0$, i.e. $\mathsf{F} = \mathsf{F}_{21} = \mathsf{F}^{\mathsf{T}}_{12}$
- The transposition needs to be taken into account when comparing expressions

Essential Matrix (for Calibrated Cameras)

Using Calibrated Cameras

- Most photogrammetric systems rely on calibrated cameras
- Calibrated cameras simplify the orientation problem
- Often, we assume that both cameras have the same calibration matrix
- Assumption here: no distortions or other imaging errors

Coplanarity Constraint

- For calibrated cameras the coplanarity constraint can be simplified
- Based on the calibration matrices, we obtain the **directions** as

$$\underbrace{{}^k\mathbf{x}'}_{\text{direction in camera frame}} = \mathsf{K}'^{-1}\underbrace{\mathbf{x}'}_{\text{coordinates in the image}} \qquad {}^k\mathbf{x}'' = \mathsf{K}''^{-1}\mathbf{x}''$$

- This relation results from

$$\mathbf{x}' = \mathsf{P}'\mathbf{X}' = \mathsf{K}'R'[I_3 | -\boldsymbol{X}'_O]\mathbf{X}' = \mathsf{K}'\,{}^k\mathbf{x}'$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' &= 0 \\ \mathbf{x}'^{\mathsf{T}}\underbrace{(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}}_{\mathsf{F}}\mathbf{x}'' &= 0 \end{aligned}$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' &= 0 \\ \mathbf{x}'^{\mathsf{T}}\underbrace{(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}}_{\mathsf{F}}\mathbf{x}'' &= 0 \\ \underbrace{\mathbf{x}'^{\mathsf{T}}(\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}}R'\mathsf{S}_bR''^{\mathsf{T}}\underbrace{(\mathsf{K}'')^{-1}\mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \end{aligned}$$

Coplanarity Constraint

- Exploiting the fundamental matrix

$$\begin{aligned}
\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\
\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\
\underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \\
{}^k\mathbf{x}'^{\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}}\ {}^k\mathbf{x}'' &= 0
\end{aligned}$$

same form as the fundamental matrix but for calibrated cameras

Essential Matrix

- From F to the essential matrix E

$$\begin{aligned}
\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' &= 0 \\
\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}} (R')^{-\mathsf{T}} \mathsf{S}_b (R'')^{-1} (\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' &= 0 \\
\underbrace{\mathbf{x}'^{\mathsf{T}} (\mathsf{K}')^{-\mathsf{T}}}_{{}^k\mathbf{x}'^{\mathsf{T}}} R' \mathsf{S}_b R''^{\mathsf{T}} \underbrace{(\mathsf{K}'')^{-1} \mathbf{x}''}_{{}^k\mathbf{x}''} &= 0 \\
{}^k\mathbf{x}'^{\mathsf{T}} \underbrace{R' \mathsf{S}_b R''^{\mathsf{T}}}_{\mathsf{E}}\ {}^k\mathbf{x}'' &= 0 \\
{}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\ {}^k\mathbf{x}'' &= 0
\end{aligned}$$

essential matrix
(DE: Essentielle Matrix)

Essential Matrix

- We derived a **specialization of the fundamental matrix**
- For the calibrated cameras, it is called the **essential matrix**

$$\boxed{\mathsf{E} = R' \mathsf{S}_b R''^{\mathsf{T}}}$$

- We can write the coplanarity constraint for calibrated cameras as

$$\boxed{{}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\ {}^k\mathbf{x}'' = 0}$$

Essential Matrix

- The essential matrix as **five** degrees of freedom
- There are **five** parameters that determine the relative orientation of the image pair for calibrated cameras
- There are 4=9-5 constraints to its 9 elements (3 by 3 matrix)
- The essential matrix is homogenous and singular

$${}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\ {}^k\mathbf{x}'' = 0$$

Popular Parameterizations for the Relative Orientation

Five Parameters – How?

- Five parameters that determine the relative orientation of the image pair

How to parameterize the essential matrix?

The Popular Parameterizations

- Five parameters that determine the relative orientation of the image pair
- Three popular parameterizations

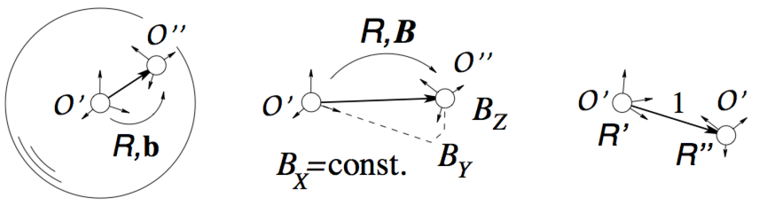


Image courtesy: Förstner

The Popular Parameterizations

1. General parameterization of dependent images
2. Photogrammetric parameterization of dependent images
3. Parameterization with independent images

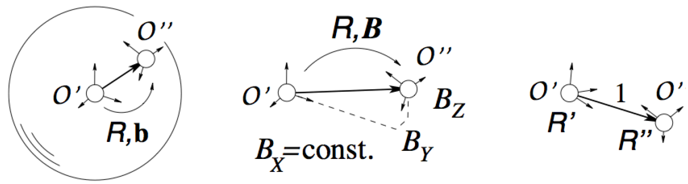


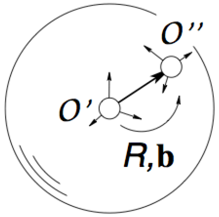
Image courtesy: Förstner

General Parameterization of Dependent Images

The general parameterization of dependent images uses a

- **normalized direction vector** b
- **rotation matrix** R

DE: Allgemeine Parametrisierung des Folgebildanschluss

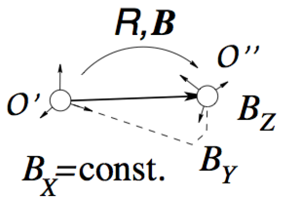


Photogrammetric parametrizat. of Dependent Images

Photogrammetric parameterization of dependent images uses

- **two components B_Y and B_Z of the base direction (B_X=const)**
- a **rotation matrix** R

DE: Klassich-photogrammetrische Parametrisierung des Folgebildanschluss

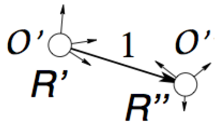


Parameterization with Independent Images

The parameterization with independent images uses

- a **rotation matrix** $R'(\omega', \phi', \kappa')$
- a **rotation matrix** $R''(\omega'', \phi'', \kappa'')$
- a fixed basis of constant length

DE: Parametrisierung mit Bilddrehungen

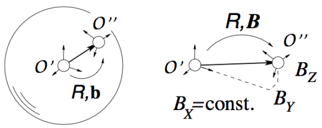


Parameterization of Dependent Images

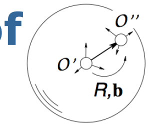
(DE: Parametrisierung des Folgebildanschluss)

For the Parameterizations of Dependent Images

- The reference frame is the frame of the first camera
- Describe the second camera relative to the first one
- Rotation mat. of the first cam is $R' = I_3$
- The rotation of the R.O. is then $R = R''$



General Parameterization of Dependent Images

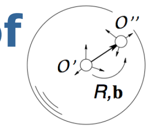


- The orientation of the second camera is $R = R''$ and we obtain from the coplanarity constraint

$$ {}^k\mathbf{x}'^{\mathsf{T}} \mathsf{S}_b R^{\mathsf{T}}\, {}^k\mathbf{x}'' = 0 \qquad \text{with} \qquad |\mathbf{b}| = 1 $$

- 6 parameters + 1 constraint $|\mathbf{b}| = 1$

General Parameterization of Dependent Images



- The resulting 5 degree of freedom are

$$ (\underbrace{B_X, B_Y, B_Z}_{\mathbf{b}}, \underbrace{\omega, \phi, \kappa}_{R}) \quad \text{with} \quad B_X^2 + B_Y^2 + B_Z^2 = 1 $$

Summary

- Parameters of image pairs
- Relative orientation
- Fundamental matrix F
- Coplanarity constraint $\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0$
- Essential matrix E
  (F for the calibrated camera pair)
- Coplanarity constraint ${}^k\mathbf{x}'^{\mathsf{T}} \mathsf{E}\, {}^k\mathbf{x}'' = 0$
- Parameterization of the relative orientation

Literature

- Förstner, Wrobel: Photogrammetric Computer Vision, Ch. 12.2

Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

Cyrill Stachniss, cyrill.stachniss@igg.uni-bonn.de, 2014