Photogrammetry & Robotics Lab

Camera Calibration: Direct Linear Transform

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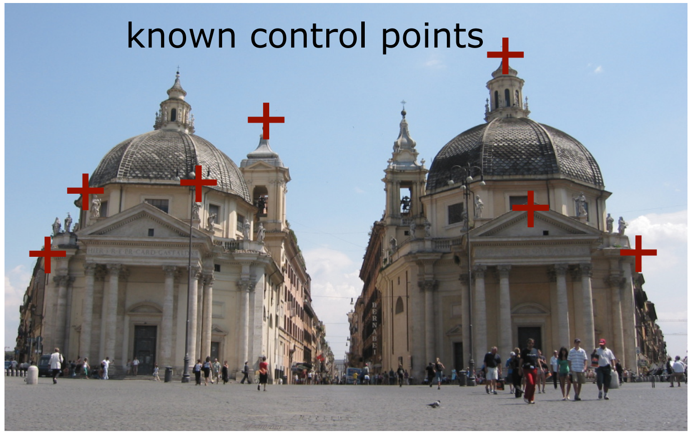
The slides have been created by Cyrill Stachniss.

3D Point to Pixel: Estimating the Parameters of P

$$\mathbf{x} = \mathsf{P}\mathbf{X}$$

pixel coordinate — trans-formation — world coordinate

Estimating Camera Parameters Given the Geometry

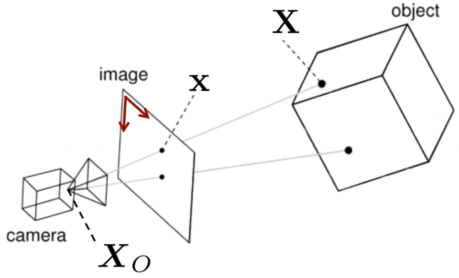


Estimate Ex- and Intrinsics

- **Wanted:** Extrinsic and intrinsic parameters of a camera
- **Given:** Coordinates of object points (control points)
- **Observed:** Coordinates (x, y) of those known 3D object points in the image

Mapping

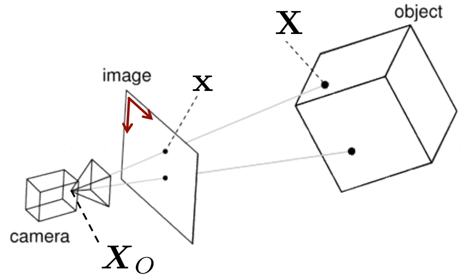
Direct linear transform (DLT) maps any object point $\mathbf{X}$ to the image point $\mathbf{x}$



Mapping

Direct linear transform (DLT) maps any object point $\mathbf{X}$ to the image point $\mathbf{x}$

$$\begin{aligned}\mathbf{x} &= \mathsf{K}R[I_3|-\boldsymbol{X}_O]\mathbf{X}\\ &= \mathsf{P}\ \mathbf{X}\end{aligned}$$



Mapping

Direct linear transform (DLT) maps any object point $\mathbf{X}$ to the image point $\mathbf{x}$

$$\begin{aligned}\underset{3\times1}{\mathbf{x}} &= \underset{3\times3}{\mathsf{K}}\ \underset{3\times3}{R}\ \underbrace{[\underset{3\times3}{I_3}|-\underset{3\times1}{\boldsymbol{X}_O}]}_{3\times4}\ \underset{4\times1}{\mathbf{X}}\\ &= \underset{3\times4}{\mathsf{P}}\ \underset{4\times1}{\mathbf{X}}\end{aligned}$$

Camera Parameters

$$\mathbf{x} = \mathsf{K}R[I_3|-\boldsymbol{X}_O]\mathbf{X} = \mathsf{P}\ \mathbf{X}$$

- **Intrinsics**
  - Camera-internal parameters
  - Given through K
- **Extrinsics**
  - Pose parameters of the camera
  - Given through $\boldsymbol{X}_O$ and R
- Projection matrix $\mathsf{P} = \mathsf{K}R[I_3|-\boldsymbol{X}_O]$ contains both, the in- and extrinsics

Direct Linear Transform (DLT)

Compute the **11 intrinsic and extrinsic parameters**

$$\mathbf{x} = \mathsf{K}R[I_3 | -\boldsymbol{X}_O]\mathbf{X}$$

observed image point

c, s, m, x_H, y_H

3 rotations

3 translations

control point coordinates (given)

How Many Points Are Needed?

$$\mathbf{x} = \mathsf{P}\ \mathbf{X}$$

Each point gives **???** observation equations

How Many Points Are Needed?

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathsf{P} \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix}$$

Each point gives **???** observation equations

How Many Points Are Needed?

$$\begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} = \mathsf{P} \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix}$$

Each point gives **???** observation equations

How Many Points Are Needed?

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathsf{P} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Each point gives **two** observation equations, one for each image coordinate

$$x = \frac{p_{11}X + p_{12}Y + p_{13}Z + p_{14}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$
$$y = \frac{p_{21}X + p_{22}Y + p_{23}Z + p_{24}}{p_{31}X + p_{32}Y + p_{33}Z + p_{34}}$$

Spatial Resection vs. DLT

- **Calibrated camera**
  - 6 unknowns
  - We need **at least 3 points**
  - Problem solved by **spatial resection**
- **Uncalibrated camera**
  - 11 unknowns
  - We need **at least 6 points**
  - Assuming the model of an **affine camera**
  - Problem solved by **DLT**

DLT: Direct Linear Transform

Computing the Orientation of an Uncalibrated Camera Using ≥ 6 Known Points

DLT: Problem Specification

- Task: Estimate the 11 elements of P
- Given:
  - 3D coordinates $\mathbf{X}_i$ of $I \geq 6$ object points
  - Observed image coordinates $\mathbf{x}_i$ of an uncalibrated camera with the mapping

$$\mathbf{x}_i = \mathsf{P}\,\mathbf{X}_i \qquad i = 1, \ldots, I$$

  - Data association

Rearrange the DLT Equation

$$\mathbf{x}_i = \underset{3\times4}{\mathsf{P}}\,\mathbf{X}_i = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \mathbf{X}_i$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \underset{3\times4}{\mathsf{P}}\,\mathbf{X}_i = \begin{bmatrix} \boxed{p_{11} \quad p_{12} \quad p_{13} \quad p_{14}} \\ \boxed{p_{21} \quad p_{22} \quad p_{23} \quad p_{24}} \\ \boxed{p_{31} \quad p_{32} \quad p_{33} \quad p_{34}} \end{bmatrix} \mathbf{X}_i$$

Define the tree vectors A, B, C as follows

$$\boxed{\mathbf{A}} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \end{bmatrix} \qquad \boxed{\mathbf{B}} = \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \end{bmatrix} \qquad \boxed{\mathbf{C}} = \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix}$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \underset{3\times4}{\mathsf{P}}\,\mathbf{X}_i = \begin{bmatrix} \boxed{p_{11} \quad p_{12} \quad p_{13} \quad p_{14}} \\ \boxed{p_{21} \quad p_{22} \quad p_{23} \quad p_{24}} \\ \boxed{p_{31} \quad p_{32} \quad p_{33} \quad p_{34}} \end{bmatrix} \mathbf{X}_i$$

So what we can rewrite the equation as

$$\mathbf{x}_i = \mathsf{P}\mathbf{X}_i = \begin{bmatrix} \boxed{\mathbf{A}^\mathsf{T}} \\ \boxed{\mathbf{B}^\mathsf{T}} \\ \boxed{\mathbf{C}^\mathsf{T}} \end{bmatrix} \mathbf{X}_i$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \underset{3\times4}{\mathsf{P}}\,\mathbf{X}_i = \begin{bmatrix} \boxed{p_{11} \quad p_{12} \quad p_{13} \quad p_{14}} \\ \boxed{p_{21} \quad p_{22} \quad p_{23} \quad p_{24}} \\ \boxed{p_{31} \quad p_{32} \quad p_{33} \quad p_{34}} \end{bmatrix} \mathbf{X}_i$$

So what we can rewrite the equation as

$$\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \mathbf{x}_i = \mathsf{P}\mathbf{X}_i = \begin{bmatrix} \boxed{\mathbf{A}^\mathsf{T}} \\ \boxed{\mathbf{B}^\mathsf{T}} \\ \boxed{\mathbf{C}^\mathsf{T}} \end{bmatrix} \mathbf{X}_i = \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{X}_i \\ \mathbf{B}^\mathsf{T}\mathbf{X}_i \\ \mathbf{C}^\mathsf{T}\mathbf{X}_i \end{bmatrix}$$

Rearrange the DLT Equation

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{X}_i \\ \mathbf{B}^\mathsf{T}\mathbf{X}_i \\ \mathbf{C}^\mathsf{T}\mathbf{X}_i \end{bmatrix}$$

$$\Rightarrow$$

$$x_i = \frac{u_i}{w_i} = \frac{\mathbf{A}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i} \qquad y_i = \frac{v_i}{w_i} = \frac{\mathbf{B}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i}$$

Rearrange the DLT Equation

$$x_i = \frac{\mathbf{A}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i} \quad \Rightarrow \quad x_i\,\mathbf{C}^\mathsf{T}\mathbf{X}_i - \mathbf{A}^\mathsf{T}\mathbf{X}_i = 0$$

$$y_i = \frac{\mathbf{B}^\mathsf{T}\mathbf{X}_i}{\mathbf{C}^\mathsf{T}\mathbf{X}_i} \quad \Rightarrow \quad y_i\,\mathbf{C}^\mathsf{T}\mathbf{X}_i - \mathbf{B}^\mathsf{T}\mathbf{X}_i = 0$$

Leads to an system of equation, which is **linear in the parameters $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$**

$$\begin{array}{rrr} -\mathbf{X}_i^\mathsf{T}\mathbf{A} & & +x_i\,\mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \\ & -\mathbf{X}_i^\mathsf{T}\mathbf{B} & +y_i\,\mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \end{array}$$

Estimating the Elements of P

- Collect the elements of P within a parameter vector $\boldsymbol{p}$

$$\boldsymbol{p} = (p_k) = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix} = \mathrm{vec}(\mathsf{P}^\mathsf{T})$$

rows of P as column-vectors, one below the other (12x1)

Estimating the Elements of P

- Rewrite $\begin{array}{rrr} -\mathbf{X}_i^\mathsf{T}\mathbf{A} & & +x_i\,\mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \\ & -\mathbf{X}_i^\mathsf{T}\mathbf{B} & +y_i\,\mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \end{array}$
- as $\begin{array}{l} \boldsymbol{a}_{x_i}^\mathsf{T}\boldsymbol{p} = 0 \\ \boldsymbol{a}_{y_i}^\mathsf{T}\boldsymbol{p} = 0 \end{array}$

Estimating the Elements of P

- Rewrite $\begin{aligned} -\mathbf{X}_i^\mathsf{T}\mathbf{A} \qquad\qquad & +x_i\, \mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \\ -\mathbf{X}_i^\mathsf{T}\mathbf{B} & +y_i\, \mathbf{X}_i^\mathsf{T}\mathbf{C} = 0 \end{aligned}$
- as $\begin{aligned} \boldsymbol{a}_{x_i}^\mathsf{T}\boldsymbol{p} = 0 \\ \boldsymbol{a}_{y_i}^\mathsf{T}\boldsymbol{p} = 0 \end{aligned}$
- with

$$\begin{aligned}
\boldsymbol{p} &= (p_k) = \mathrm{vec}(\mathsf{P}^\mathsf{T}) \\
\boldsymbol{a}_{x_i}^\mathsf{T} &= (-\mathbf{X}_i^\mathsf{T}, \mathbf{0}^\mathsf{T}, x_i\mathbf{X}_i^\mathsf{T}) \\
&= (-X_i,\, -Y_i,\, -Z_i,\, -1, 0,\, 0,\, 0,\, 0, x_iX_i,\, x_iY_i,\, x_iZ_i,\, x_i) \\
\boldsymbol{a}_{y_i}^\mathsf{T} &= (\mathbf{0}^\mathsf{T}, -\mathbf{X}_i^\mathsf{T},\, y_i\mathbf{X}_i^\mathsf{T}) \\
&= (0,\, 0,\, 0,\, 0, -X_i,\, -Y_i,\, -Z_i,\, -1, y_iX_i,\, y_iY_i,\, y_iZ_i,\, y_i)
\end{aligned}$$

Verifying Correctness

$$\boldsymbol{a}_{x_i}^\mathsf{T}\boldsymbol{p} \;=\; (-X_i,\, -Y_i,\, -Z_i,\, -1, 0,\, 0,\, 0,\, 0, x_iX_i,\, x_iY_i,\, x_iZ_i,\, x_i) \begin{bmatrix} p_{11}\\ p_{12}\\ p_{13}\\ p_{14}\\ p_{21}\\ p_{22}\\ p_{23}\\ p_{24}\\ p_{31}\\ p_{32}\\ p_{33}\\ p_{34} \end{bmatrix}$$

Verifying Correctness

$$\boldsymbol{a}_{x_i}^\mathsf{T}\boldsymbol{p} \;=\; (-X_i,\, -Y_i,\, -Z_i,\, -1, 0,\, 0,\, 0,\, 0, x_iX_i,\, x_iY_i,\, x_iZ_i,\, x_i) \begin{bmatrix} \boxed{\begin{matrix} p_{11}\\ p_{12}\\ p_{13}\\ p_{14}\end{matrix}} \\ \boxed{\begin{matrix} p_{21}\\ p_{22}\\ p_{23}\\ p_{24}\end{matrix}} \\ \boxed{\begin{matrix} p_{31}\\ p_{32}\\ p_{33}\\ p_{34} \end{matrix}} \end{bmatrix} \begin{matrix} \boldsymbol{A} \\ \\ \boldsymbol{B} \\ \\ \boldsymbol{C} \end{matrix}$$

$$\begin{bmatrix} \boldsymbol{A}\\ \boldsymbol{B}\\ \boldsymbol{C} \end{bmatrix}$$

Verifying Correctness

$$\begin{aligned}
\boldsymbol{a}_{x_i}^\mathsf{T}\boldsymbol{p} &= (\underbrace{\boxed{-X_i,\, -Y_i,\, -Z_i,\, -1}}_{-\mathbf{X}_i^\mathsf{T},}\underbrace{\boxed{0,\, 0,\, 0,\, 0}}_{\mathbf{0},}\underbrace{\boxed{x_iX_i,\, x_iY_i,\, x_iZ_i,\, x_i}}_{x_i\mathbf{X}_i^\mathsf{T}}) \begin{bmatrix} p_{11}\\ p_{12}\\ p_{13}\\ p_{14}\\ p_{21}\\ p_{22}\\ p_{23}\\ p_{24}\\ p_{31}\\ p_{32}\\ p_{33}\\ p_{34} \end{bmatrix} \begin{matrix} \boldsymbol{A} \\ \\ \boldsymbol{B} \\ \\ \boldsymbol{C} \end{matrix} \\
&= (\quad -\mathbf{X}_i^\mathsf{T}, \quad \mathbf{0}, \quad x_i\mathbf{X}_i^\mathsf{T} \quad) \begin{bmatrix} \boldsymbol{A}\\ \boldsymbol{B}\\ \boldsymbol{C} \end{bmatrix}
\end{aligned}$$

Verifying Correctness

$$
\begin{aligned}
\boldsymbol{a}_{x_i}^\top \boldsymbol{p} &= (-X_i,\, -Y_i,\, -Z_i,\, -1, 0,\, 0,\, 0,\, 0, x_i X_i,\, x_i Y_i,\, x_i Z_i,\, x_i) \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} \\
&= \left(-\mathbf{X}_i^\top, \quad \mathbf{0}, \quad x_i \mathbf{X}_i^\top \right) \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix} \\
&= -\mathbf{X}_i^\top \mathbf{A} \quad + x_i\, \mathbf{X}_i^\top \mathbf{C}
\end{aligned}
$$

Verifying Correctness

$$
\begin{aligned}
\boldsymbol{a}_{y_i}^\top \boldsymbol{p} &= (0,\, 0,\, 0,\, 0,\, -X_i,\, -Y_i,\, -Z_i,\, -1,\, y_i X_i,\, y_i Y_i,\, y_i Z_i,\, y_i) \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{14} \\ p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \\ p_{31} \\ p_{32} \\ p_{33} \\ p_{34} \end{bmatrix} \\
&= \left(\mathbf{0}, \quad -\mathbf{X}_i^\top, \quad y_i \mathbf{X}_i^\top \right) \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix} \\
&= -\mathbf{X}_i^\top \mathbf{B} \quad + y_i\, \mathbf{X}_i^\top \mathbf{C}
\end{aligned}
$$

Estimating the Elements of P

- For each point, we have

$$\boldsymbol{a}_{x_i}^\top \boldsymbol{p} = 0$$

$$\boldsymbol{a}_{y_i}^\top \boldsymbol{p} = 0$$

- Stacking everything together

$$
\begin{bmatrix} \boldsymbol{a}_{x_1}^\top \\ \boldsymbol{a}_{y_1}^\top \\ \cdots \\ \boldsymbol{a}_{x_i}^\top \\ \boldsymbol{a}_{y_i}^\top \\ \cdots \\ \boldsymbol{a}_{x_I}^\top \\ \boldsymbol{a}_{y_I}^\top \end{bmatrix} \boldsymbol{p} = \underset{2I \times 12}{\mathsf{M}} \; \underset{12 \times 1}{\boldsymbol{p}} \stackrel{!}{=} 0
$$

Solving the Linear System (Homogeneous System => SVD)

- Solving a system of linear equations of the form $A\,\boldsymbol{x} = \mathbf{0}$ is equivalent to finding the null space of A
- Thus, we can apply the SVD to solve M $\boldsymbol{p} \stackrel{!}{=} \mathbf{0}$
- Choose $\boldsymbol{p}$ as the singular vector belonging to the singular value of 0

Redundant Observations

- In case of redundant observations, we will have contradictions $(\mathsf{M}\boldsymbol{p} \neq \mathbf{0})$:

$$\mathsf{M}\ \boldsymbol{p} = \boldsymbol{w}$$

- Find $\boldsymbol{p}$ such that it minimizes

$$\begin{aligned} \Omega &= \boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} \\ \Rightarrow \quad \widehat{\boldsymbol{p}} &= \arg\min_{\boldsymbol{p}}\ \boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} \\ &= \arg\min_{\boldsymbol{p}}\ \boldsymbol{p}^{\mathsf{T}}\mathsf{M}^{\mathsf{T}}\mathsf{M}\boldsymbol{p} \end{aligned}$$

with $||P||_2 = \sum_{ij} p_{ij}^2 = \|\boldsymbol{p}\| = 1$

Redundant Observations

- Singular value decomposition (SVD)

$$\underset{2I\times 12}{\mathsf{M}} = \underset{2I\times 12}{U}\ \underset{12\times 12}{\mathsf{S}}\ \underset{12\times 12}{V^{\mathsf{T}}} = \sum_{i=1}^{12} s_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathsf{T}}$$

- Choosing $\boldsymbol{p} = \boldsymbol{v}_{12}$ (the singular vector belonging to the smallest singular value s_{12}) minimizes Ω

Obtaining the Projection Matrix

- Choosing $\boldsymbol{p} = \boldsymbol{v}_{12}$ minimizes Ω and thus is our estimate of P:

$$\boldsymbol{p} = \begin{bmatrix} p_{11} \\ \vdots \\ p_{34} \end{bmatrix} \Rightarrow \mathsf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}$$

Does this always work?

Critical Surfaces

- M is of rank 11, if
  - Number of points ≥6
  - Assumption: no gross errors
- **No solution**, if all points $\mathbf{X}_i$ are located on a **plane**

$$\mathsf{M} = \begin{bmatrix} \dots \\ \boldsymbol{a}_{x_i}^{\mathsf{T}} \\ \boldsymbol{a}_{y_i}^{\mathsf{T}} \\ \dots \end{bmatrix}$$

e.g., assume all Z_i=0

$$= \begin{bmatrix} & & & & & \dots & & & & & & \\ -X_i & -Y_i & -Z_i & -1 & 0 & 0 & 0 & 0 & x_iX_i & x_iY_i & x_iZ_i & x_i \\ 0 & 0 & 0 & 0 & -X_i & -Y_i & -Z_i & -1 & y_iX_i & y_iY_i & y_iZ_i & y_i \\ & & & & & \dots & & & & & & \end{bmatrix}$$

Critical Surfaces

- M is of rank 11, if
  - Number of points ≥6
  - Assumption: no gross errors
- **No solution**, if all points $\mathbf{X}_i$ are located on a **plane**

$$\mathsf{M} = \begin{bmatrix} \dots \\ \boldsymbol{a}_{x_i}^{\mathsf{T}} \\ \boldsymbol{a}_{y_i}^{\mathsf{T}} \\ \dots \end{bmatrix}$$

rank deficiency

$$= \begin{bmatrix} & & & & & \dots & & & & & & \\ -X_i & -Y_i & 0 & -1 & 0 & 0 & 0 & 0 & x_iX_i & x_iY_i & 0 & x_i \\ 0 & 0 & 0 & 0 & -X_i & -Y_i & 0 & -1 & y_iX_i & y_iY_i & 0 & y_i \\ & & & & & \dots & & & & & & \end{bmatrix}$$

Critical Surfaces

- M is of rank 11, if
  - Number of points ≥6
  - Assumption: no gross errors
- **No solution**, if
  - All points $\mathbf{X}_i$ are located on a **plane**
  - (All points $\mathbf{X}_i$ and the projection center $\boldsymbol{X}_o$ are located on a twisted cubic curve)

From P to $\mathsf{K}, R, \boldsymbol{X}_O$

Decomposition of P

- We have P , how to obtain $\mathsf{K}, R, \boldsymbol{X}_O$?
- Structure of the projection matrix

$$\mathsf{P} = [\mathsf{K}R | -\mathsf{K}R\boldsymbol{X}_O] = [\mathsf{H}|\mathbf{h}]$$

- with

$$\mathsf{H} = \mathsf{K}R \qquad \mathbf{h} = -\mathsf{K}R\mathbf{X}_O$$

Decomposition of P

$$\mathsf{H} = \mathsf{K}R \qquad \mathbf{h} = -\mathsf{K}R\mathbf{X}_O$$

- We get the **projection center** through

$$\boldsymbol{X}_O = -\mathsf{H}^{-1}\,\mathbf{h}$$

Decomposition of P

$$\mathsf{H} = \mathsf{K}R \qquad \mathbf{h} = -\mathsf{K}R\mathbf{X}_O$$

- We get the projection center through

$$\boldsymbol{X}_O = -\mathsf{H}^{-1}\,\mathbf{h}$$

Rotation matrix:

- Let's look to the structure $\mathsf{H} = \mathsf{K}R$
- What do we know about the matrices?

Decomposition of P

$$\mathsf{H} = \mathsf{K}R \qquad \mathbf{h} = -\mathsf{K}R\mathbf{X}_O$$

Rotation matrix:

- Let's look to the structure $\mathsf{H} = \mathsf{K}R$
- K is a triangular matrix
- R is a rotation matrix

Is there a matrix decomposition into
a rotation matrix and a triangular on?

Decomposition of P

$$\mathsf{H} = \mathsf{K}R \qquad \mathbf{h} = -\mathsf{K}R\mathbf{X}_O$$

Rotation matrix:

- Let's look to the structure $\mathsf{H} = \mathsf{K}R$
- K is a triangular matrix
- R is a rotation matrix

Is there a matrix decomposition into
a rotation matrix and a triangular on?

QR-decomposition

Decomposition of P

$$\mathsf{H} = \mathsf{K}R \qquad \mathbf{h} = -\mathsf{K}R\mathbf{X}_O$$

- We perform this for H^{-1} given the order of rotation and triangular matrix
- QR decomposition of H^{-1} yields rotation and calibration matrix

$$\mathsf{H}^{-1} = (\mathsf{K}\,R)^{-1} = R^{-1}\,\mathsf{K}^{-1} = \underset{\text{Q}}{R^{\mathsf{T}}}\,\underset{\text{R}}{\mathsf{K}^{-1}}$$

Decomposition of P

- The matrix $\mathsf{H} = \mathsf{K}R$ is homogenous
- Thus, is the calibration matrix
- Due to homogeneity normalize

$$K \leftarrow \frac{1}{\mathsf{K}_{33}}\mathsf{K}$$

Decomposition of P

- Decomposition $\mathsf{H}^{-1} = R^{\mathsf{T}}\,\mathsf{K}^{-1}$ results in K with **positive** diagonal elements
- To get **negative camera constant**, choose

$$\mathsf{K} \leftarrow \mathsf{K}R(z,\pi) \qquad R \leftarrow R(z,\pi)R$$

using

$$R(z,\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

decomposition still holds $\mathsf{H} = \mathsf{K}R(z,\pi)\;R(z,\pi)R = \mathsf{K}R$

DLT in a Nutshell

1. Build the M for the linear system

$$\mathsf{M} = \begin{bmatrix} \boldsymbol{a}_{x_1}^{\mathsf{T}} \\ \boldsymbol{a}_{y_1}^{\mathsf{T}} \\ \cdots \\ \boldsymbol{a}_{x_I}^{\mathsf{T}} \\ \boldsymbol{a}_{y_I}^{\mathsf{T}} \end{bmatrix} \qquad \underset{\text{2Ix12}}{\mathsf{M}}\;\underset{\text{12x1}}{\boldsymbol{p}} \stackrel{!}{=} 0$$

with

$$\begin{aligned} \boldsymbol{a}_{x_i}^{\mathsf{T}} &= (-X_i,\,-Y_i,\,-Z_i,\,-1, 0,\,0,\,0,\,0, x_iX_i,\,x_iY_i,\,x_iZ_i,\,x_i) \\ \boldsymbol{a}_{y_i}^{\mathsf{T}} &= (0,\,0,\,0,\,0, -X_i,\,-Y_i,\,-Z_i,\,-1, y_iX_i,\,y_iY_i,\,y_iZ_i,\,y_i) \end{aligned}$$

DLT in a Nutshell

2. Solve by SVD $\mathsf{M} = U\,S\,V^{\mathsf{T}}$
   Solution is last column of V

$$\boldsymbol{p} = \boldsymbol{v}_{12} \;\Rightarrow\; \mathsf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

DLT in a Nutshell

3. If individual parameters are needed

$$\mathsf{P} = \mathsf{K}R\,[I_3 | -\boldsymbol{X}_O] = [\mathsf{H}|\mathbf{h}]$$

$$\boldsymbol{X}_O = -\mathsf{H}^{-1}\,\mathbf{h}$$

$$\mathrm{QR}(\mathsf{H}^{-1}) = R^{\mathsf{T}}\,\mathsf{K}^{-1}$$

$$R \leftarrow R(z,\pi)R$$

$$K \leftarrow \frac{1}{\mathsf{K}_{33}}\mathsf{K}R(z,\pi)$$

Discussion DLT

- We realize $\mathsf{P} \leftrightarrow (\mathsf{K}, R, \boldsymbol{X}_O)$ both ways
- We are free to choose sign of c
- Solution is instable if the control points lie approximately on a plane
- Solution is statistically not optimal (no uncertainties of point coordinates)

Summary

- Direct linear transforms estimates the intrinsic and extrinsic of a camera
- Computes the parameters for the mapping of the uncalibrated camera
- Requires at least 6 known control points in 3D
- Direct solution (no initial guess)

Literature

- Förstner & Wrobel, Photogrammetric Computer Vision, Chapter 11.2
- Förstner, Scriptum Photogrammetrie I, Chapter 13.3

Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- **I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.**
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

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