1: Homogeneous coordinates and transformations in 2D

Learning objective: This set of exercises should enable you to represent 2D points and apply basic 2D transformations in homogeneous form.

A. Homogeneous coordinates

1. **(15)** The following task is motivated by providing exercises with coordinates not being real values, but only rational, in order to easily check results.

   Given are the following five points $\mathcal{x}_i$ with their homogeneous coordinates:

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -11 \\ -60 \\ 61 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 220 \\ -21 \\ -221 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 165 \\ -52 \\ 173 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} -36 \\ -77 \\ -85 \end{bmatrix}. \tag{1}$$

   (a) (**5**) Give the non-homogeneous coordinates of the five points.

   (b) (**5**) Plot the five points.

   (c) (**2**) On which curve $\mathcal{C}$ do the five points lie? Why?

   (d) (**3**) Give a 6-th point $\mathbf{x}_6 = [u_6, v_6, w_6]^\mathsf{T}$, different from the other five, with the following properties: (1) It should lie on $\mathcal{C}$, (2) it should lie in the second quadrant, (3) all coordinates should be integers with at least 4 digits, the last not zero, and (4) the third coordinate w_6 should be negative. Hint: Have a look at the concept of *Pythagorean triples*.

2. **(20)** The following exercise is meant to illustrate the usefulness of using homogeneous coordinates for describing curves through given points. Points are generated according to the following scheme:

$$\boldsymbol{x}_i = \boldsymbol{a} + i\boldsymbol{d} \quad \text{with} \quad i = 1, \ldots . \tag{2}$$

   (a) (**2**) On which curve $\mathcal{C}$ do these points lie?

   (b) (**6**) Write the generating rule in homogeneous coordinates using the corresponding entities $\mathbf{x}_i$, $\mathbf{a}$, and $\mathbf{d}$. Express them as a function of the non-homogeneous entities $\boldsymbol{x}_i$, $\boldsymbol{a}$, and $\boldsymbol{d}$. Explain, why the differences in the derivation of the homogeneous from the non-homogeneous entities. Hint: have a look at slide 20 in lecture 1.

   (c) (**12**) Given are two points $\mathbf{x} = [2, 3, 1]^\mathsf{T}$ and $\mathbf{y} = [8, 10, 2]^\mathsf{T}$. Plot the points

$$\mathbf{z}(\alpha) = \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \quad \text{with} \quad \alpha = -5, -4, ..., 4, 5 \tag{3}$$

      i. What curve does it represent? Be as specific as possible.

      ii. What happens for $\alpha = 0, 1, 2$? What points are in the direct neighbourhood of $\alpha = 2$?

      iii. Can you generalize this result for arbitrary pairs of distinct points? How? Under which constraints?

      iv. Three distinct points with homogeneous coordinates $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are colinear, if the 3×3 matrix $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ is singular. Why? For the proof assume the points are elements of $\mathbb{R}^2$. *Hint:* Start with showing, that if the determinant is zero, then – in the general case – the directions from $\mathcal{x}_1$ to the other two points are identical.

B. Translations, rotations, and dilations

1. **(6)** Give the homogeneous matrices for the following transformations of 2D points.

   (a) (**2**) Translation $\mathcal{T}_t$ with translation vector $\boldsymbol{t} = [-4\,, -1]^\mathsf{T}$

   (b) (**2**) Rotation $\mathcal{R}_\phi$ with $\phi = -90°$ around the origin

(c) (**2**) Mirroring at the origin and scaling $\mathcal{D}_\lambda$ with a factor 0.5

2. **(4)** Given is the point $p([2\,,1]^\mathsf{T})$. Provide the *non-homogeneous coordinates* of the transformed points p', p'', and p''' after applying the following transformations using the matrices from task 1:
   (a) (**1**) translation $p' = \mathcal{T}_t(p)$,
   (b) (**2**) then rotation $p'' = \mathcal{R}_\phi(p')$,
   (c) (**1**) then signed scaling $p''' = \mathcal{D}_\lambda(p'')$.

3. **(6)** Using your favourite drawing tool, and plot the starting point and the transformed points together with the coordinate system. Make all entities explicit and name them. Check the result of the previous task visually. (Useful functions in MATLAB: `figure, hold on, grid on, axis equal, plot, text, axis, xlabel, ylabel`)

4. **(5)** How many different points p''' could you obtain by exchanging the sequence of the three transformations? Give reasons for your answer.

total: 56

2: Analysing and concatenating 2D transformations

Learning objective: This set of exercises should enable you to analyse and concatenate 2D transformations in various contexts.

A. Special transformations

1. **(20)** Translated basic motions.

   Perform the following trnaformations using homogeneous matrices and vectors.

   (a) (**3**) Rotation $\mathcal{R}_{p\phi}$ by $\phi = 45°$ around the point $p([0, -1]^{\mathsf{T}})$.

   (b) (**2**) Scaling $\mathcal{D}_{q\lambda}$ by $\lambda = 2$ and the scaling centre $q([-1, 0]^{\mathsf{T}}$.

   (c) (**2**) Reflection/mirroring $\mathcal{M}_x$ at the point $x([2, 1]^{\mathsf{T}})$

   (d) (**8**) Reflection/mirroring $\mathcal{M}_l$ at the straight line $l(3x + 4y - 5 = 0)$.

   For the given point $a([5, -3]^{\mathsf{T}})$ give the transformed coordinates

   (e) (**3**) $\boldsymbol{b}_1 = \mathcal{R}_{p\phi}(a)$

   (f) (**3**) $\boldsymbol{b}_2 = \mathcal{D}_{q\lambda}(a)$

   (g) (**3**) $\boldsymbol{b}_3 = \mathcal{S}_x(a)$

   (h) (**3**) $\boldsymbol{b}_4 = \mathcal{R}_l(a)$

   and graphically check the results using two arbitrary additional points.

2. **(4)** Rotation around a given point.

   Prove the following: A rotation with angle ϕ around the point $\boldsymbol{d}$ leads to the homogeneous transformation matrix:

$$\mathsf{R}_p(\phi, \boldsymbol{d}) = \begin{bmatrix} R(\phi) & (I_2 - R(\phi))\boldsymbol{d} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \tag{4}$$

3. **(8)** Mirroring at a straight line.

   Prove the following: Mirroring a point at the straight line $x\cos\phi + y\sin\phi - d = 0$ leads to the homogeneous transformation matrix

$$\mathsf{Y}(\phi, d) = \begin{bmatrix} -\cos(2\phi) & -\sin(2\phi) & 2d\cos\phi \\ -\sin(2\phi) & \cos(2\phi) & 2d\sin\phi \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

B. General mappings of homogeneous coordinates

4. **(20)** The following exercise is meant to demonstrate that general mappings of homogeneous coordinates may map a square to a figure which is not connected in the plane $\mathbb{R}^2$.

   Given are the four points on the unit square

$$\boldsymbol{x}_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{6}$$

   (a) (**4**) Show, that the following homogeneous transformation

$$\mathbf{x}' = \mathsf{H}\mathbf{x} \quad \text{with} \quad \mathsf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{7}$$

   exchanges the first two points and leaves the other two unchanged.

(b) (**10**) Transform the points χ_5 and χ_6 sitting in the middle between χ_1 and χ_2 and in the middle between χ_3 and χ_4. Plot the six points $\{\mathbf{x}_i\}$ with $i = (1,5,2,3,6,4)$ in one figure and the points $\{\mathbf{x}'_i\}$ in a second figure. Discuss the result.

(c) (**6**) Generate an odd number of equally spaced on the four sides of the square, e.g., $\chi_7([-1,a]^\mathsf{T}), a \in [-1,1]$between χ_1 and χ_4. Plot the transformed points, without connecting them by line segments and, best, give each side of the square individual colors. Discuss the result. *Hint:* Use a plot for $\Delta a = 2/(2k+1)$, for some $k \in \mathbb{N}$.

5. **(35)** The following exercise discusses homogeneous coordinates in 1D and, what is called, the Cayley transformation, which we will use for representing rotations in 3D.

Any 2-vector

$$\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{with} \quad \mathbf{x} \neq 0 \tag{8}$$

represents a point on the real axis if $v \neq 0$, namely

$$x = \frac{u}{v} \quad \text{if} \quad v \neq 0 \quad \text{and} \quad \mathbf{x} = v \begin{bmatrix} x \\ 1 \end{bmatrix}. \tag{9}$$

Two homogeneous vectors $\mathbf{x}$ and $\mathbf{y}$ represent the same 1D point, if $\mathbf{x} = \lambda\mathbf{y}$ for some factor $\lambda \neq 0$.

Discuss the following motions on $\mathbb{R}$ described by

$$\mathbf{x}'_k = \mathsf{H}_k\mathbf{x} \quad \text{with} \quad k = 1,2,3,4,5\,. \tag{10}$$

(a) (**5**) What motion on the real line do the following matrices describe:

$$\mathsf{H}_1 = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}, \quad \mathsf{H}_2 = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathsf{H}_3 = \begin{bmatrix} s & d \\ 0 & 1 \end{bmatrix}, \quad \mathsf{H}_4 = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}, \quad \mathsf{H}_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}? \tag{11}$$

(b) (**5**) For which points $\chi(x)$ and parameters are these motions defined?

(c) (**10**) Give all fixed points, if any in $\mathbb{R}$, for the five transformations assuming arbitrary parameters i.e. for which $\chi' = \chi$?

(d) (**2**) The transformation $y = f_c(x)$ represented by $\mathbf{y} = \mathsf{H}_C\mathbf{x}$ with

$$\mathsf{H}_C = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{12}$$

is the Cayley transformation of x. Give $y = f_C(x)$ explicitely. For which x is the transformation well defined in $\mathbb{R}$?

(e) (**3**) Show that the inverse function is $x = f_C^{-1}(y) = f_C(y)$ hence $x = f_C(f_C(x))$. Use homogeneous coordinates.

(f) (**10**) Which of the five transformations H_k in item 3a do have the property that the mapping and its inverse are identical, possibly restricting the mappings by specifying some parameters? Interpret the results geometrically.

C. Concatenated motions

6. **(22)** Moving on a slot car racing track.

On a planar slot car racing track a model-car is positioned at $\mathcal{A}$, s. Fig. 1). The curves are semicircles. The width of the car is half to the trackwidth such that it drives on one of the two parallel trails. The car moves in two steps: First it drives to $\mathcal{B}$, then it takes a left turn to $\mathcal{C}$. The global coordinate system is in the middle of the racing track, its x-axis is parallel to the straight parts. The car coordinate system is in the middle of the car, pointing forward.

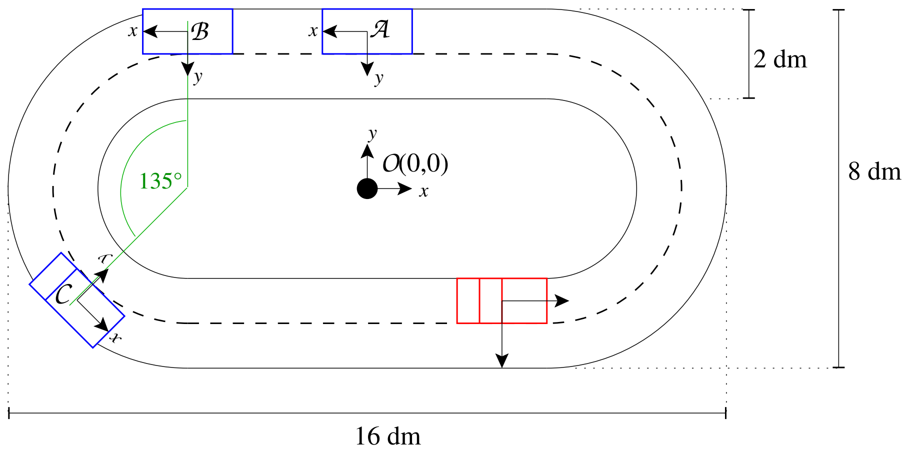

Figure 1: Moving on a slot car racing track. Dimensions in dm

(a) (**14**) Give the frames M_A, M_B, and M_C of the car at the positions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in the global coordinate system. Plot the scene using the function `plotScene` which takes the three frames as input, see Fig. 2.

(b) (**8**) Give the relative motions ${}_B\mathcal{M}^A$ (from $\mathcal{A}$ to $\mathcal{B}$) and ${}_C\mathcal{M}^B$ (from $\mathcal{B}$ to $\mathcal{C}$) in the car coordinate system at the start of the motion. Determine the positions of $\mathcal{B}$ and $\mathcal{C}$ in the car system S_A at the beginning of the ride.

7. **(20)** A well known motion.

A robot can move in the plane. The x-axis of its right handed local coordinate system points forward.

The robot performs the following motions

- $\mathcal{M}_1$: forward motion by 1 m,
- $\mathcal{M}_2$: left rotation by 270°around the point p which is 1 m to the left of the robot,
- $\mathcal{M}_3$: forward motion by 2 m,
- $\mathcal{M}_4$: right rotation by 270°around the point p which is 1 m to the right of the robot, and
- $\mathcal{M}_5$: forward motion by 1 m.

(a) (**14**) Desrcibe each of the partial motions with homogeneous matrix (M_1 to M_5 referring to the start of the partial motion.

(b) (**4**) Determine the complete motion. Give reasons for your way of concatenating the partial motions.

(c) (**2**) How can you interprete the geometry of the complete motion? Which Figure describes the path?

8. **(34)** Two bumper cars on a fair.

Two bumper cars $\mathcal{A}$ and $\mathcal{B}$ wait for the beginning of their ride. Together with that of the cashier $\mathcal{C}$, their positions are shown in Fig. 3. Both drivers are in the centres of the coordinate systems of the two cars. The task is to determine the direction in which each driver sees the other one and the cashier.

Two bumper cars $\mathcal{A}$ and $\mathcal{B}$ wait for the beginning of their ride. Together with that of the cashier $\mathcal{C}$, their positions are shown in Fig. 3. Both drivers are in the centres of the coordinate systems of the two cars. The task is to determine the direction in which each driver sees the other one and the cashier.

(a) (**4**) Determine the displacements $\mathcal{M}_A$ and $\mathcal{M}_B$ of the reference coordinate system in the two car systems.

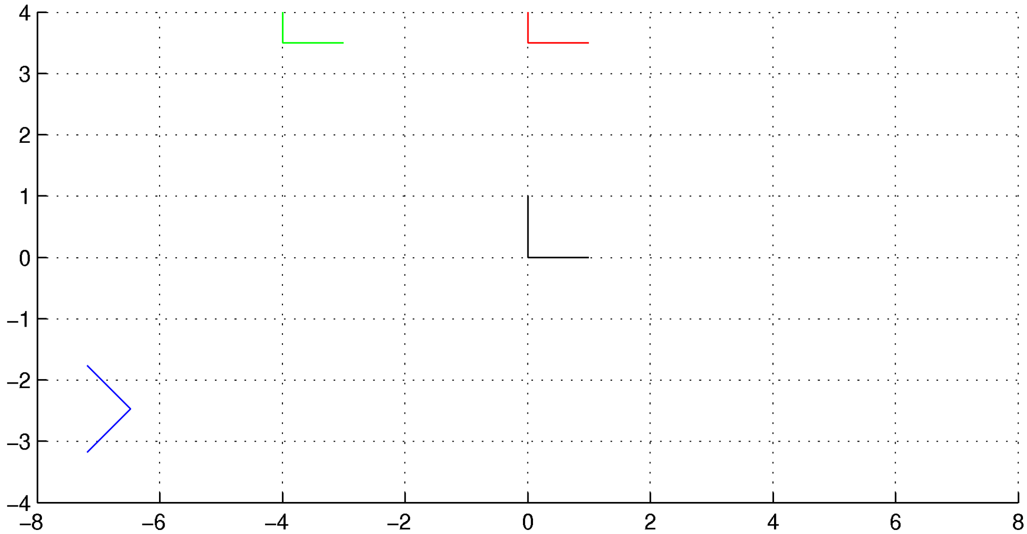

Figure 2: Result of ex. 6 using `plotScene`.

(b) (**4**) Determine the directions in which both drivers see the cashier, by expressing the coordinates of the cashier in the car systems. Compare the result with the directions in the figure.

Now both cars move. Turns are performed on the spot. Car $\mathcal{A}$ performs the following three moves: (1) 2 m ahead, (2) right turn by 45°, (3) 1 m ahead. Car $\mathcal{B}$ performs the following six moves: (1) right turn by 45°, (2) 1 m ahead, (3) right turn by 45°(4) 3 m backwards, (5) left turn by 90°, (6) 3 m ahead. Answer the following questions:

(c) (**15**) Determine the composite motion of both cars.

(d) (**4**) In what directions do the drivers see the cashier?

(e) (**4**) In what directions do the two drivers see each other?

(f) (**2**) What is the distance between the drivers now?

9. **(20)** Autonomeous Robot in a city with radially arranged streets.

A robot $\mathcal{R}$ can autonomously move in the area depicted in the ground plan of a city Fig. 4. The area allows the robot only to move along the paths given in the ground plan. The centre of $\mathcal{Z}$ of the scene at the same time is the coordinate system, whose x-axis shows towards east and its y-axis shows towards north. There are only paths radially and circularly. One step of the robot corresponds to moving to the next gridline.

The local x-axis of the robot is its viewing direction forward. The motion of the robot is either a translation from the centre or a circular motion around the centre. Furthermore the robot may rotate around its own axis.

At time t_0 the robot $\mathcal{R}$ is at the centre and the two coordinate systems of the robot and of the city are identical.

(a) **(2)** Determine the coordinates of the two lane crossings $\mathcal{A}$ and $\mathcal{B}$. in the city system.

(b) **(9)** The robot now moves towards $\mathcal{A}$. First it moves three steps forward, then it turns 90° left, adn finally moves 30°along the circular lane. Determine the three elementary transformations and concatenate them.

(c) **(2)** Which coordinates does the robot now have in the city system?

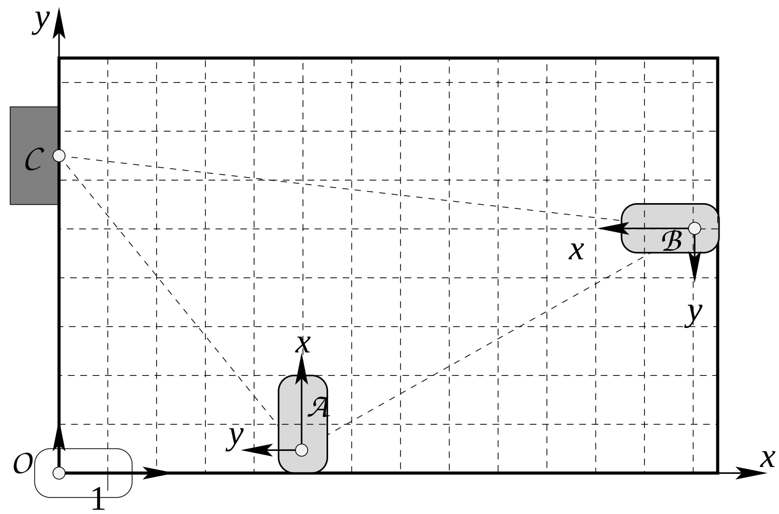

Figure 3: Two bumper cars $\mathcal{A}$ and $\mathcal{B}$ viewing the cashier $\mathcal{C}$ (in scale)

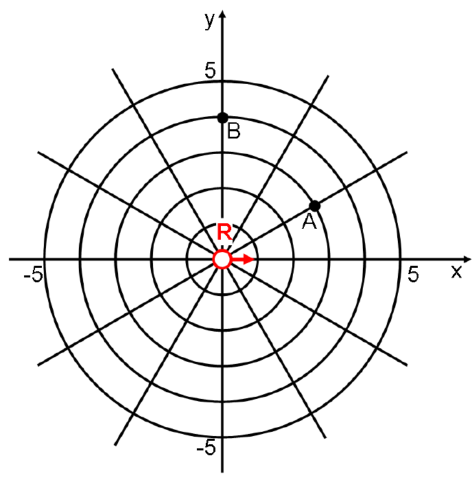

Figure 4: City plan with autonomous robot $\mathcal{R}$

(d) **(8)** Which coordinates do the two points $\mathcal{A}$ and $\mathcal{B}$ have in the local coordinate system of the robot?

10. **(11)** A turning car.

The position $\boldsymbol{p}$ of the co-driver in the car system is given, by e.g. ${}^0\boldsymbol{p}_0 = [0, -1]^\mathsf{T}$. The car performs the following two movements:

- forward motion by 6 m, followed by a
- right turn, which is a rotation by $\alpha = 45^o$ around the point $\mathcal{C}$ with radius $r = 4$ m.

(a) Give the two partial motions, the forward and the right turn, of the car referring to the start of each the partial motion as single homogeneous transformations.

(b) Determine the position ${}^0\boldsymbol{p}_t$ of the point $\boldsymbol{p}_t$ at time t in the original coordinate system S_0. Provide the derivation and give the reasoning for your solution.

total: 194

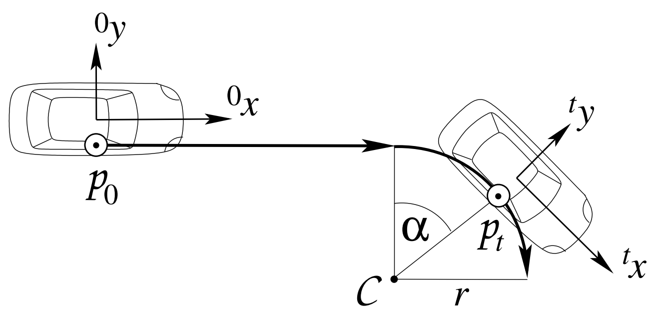

Figure 5: Motion of a car with co-driver at $\boldsymbol{p}$ at time 0 and time t

3: Orthogonal Matrices and Euler Angles

Learning objective: This set of exercises should enable you to reliably determine angles in 3D and to analyse, represent and use 3D rotations with rotation matrices and Euler angles.

A. Angles and special orthogonal transformations

1. **(16)** Angle between two vectors using the atan2-function.

   (a) **(6)** Empirical determine the range $[T_{c0}, T_{c1}]$ for an angle $\alpha \in [0°, +90°]$ such that for $\arccos(\cos(\alpha)) \neq 0$ the difference $|\arccos(\cos(\alpha)) - \alpha|$ is small, say below 10^{-14}. Report the software and the numerical resolution you used.

   (b) **(4)** Empirically determine the range $[T_{s0}, T_{s1}]$ for an angle $\alpha \in [0°, +90°]$ such that $\arcsin(\sin(\alpha)) \neq \pi/2$ the difference $|\arcsin(\sin(\alpha)) - \alpha|$ is small, say $< 10^{-14}$ and check whether it has the same length.

   (c) **(3)** Given are two 3-vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Derive an explicit expression for the angle $\angle(\boldsymbol{x}, \boldsymbol{y})$ using the atan2-function. *Hint:* Use the dot and the cross product.

   (d) **(3)** Show that the expression can be specialized for two 2-vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ to

$$\angle(\boldsymbol{u}, \boldsymbol{v}) = \operatorname{atan2}\left(|\boldsymbol{u}, \boldsymbol{v}|, \boldsymbol{u}^\mathsf{T}\boldsymbol{v}\right). \quad (13)$$

2. **(35)** Reflections.

   (a) **(3)** Give the three reflection matrices (see spanisch *espejo*) E_i $k = 1, 2, 3$ in the three coordinate planes. Chose i such that the coordinate x_i is changed.

   (b) **(6)** Show that a reflection at a plane $\mathcal{E}_n$ through the origin given by its normal $\boldsymbol{n}, |\boldsymbol{n}| = 1$, can be represented as

$$\mathsf{E}_n = I_3 - 2\boldsymbol{n}\boldsymbol{n}^\mathsf{T} \quad (14)$$

   Hint: Show that all points p with coordinates $\boldsymbol{p} = \boldsymbol{p}_0 + \lambda\boldsymbol{n}$, where $\boldsymbol{p}_0 \in \mathcal{E}_n$ and $\lambda \in \mathbb{R}$, are transformed to the point p' with coordinates $\boldsymbol{p}' = \boldsymbol{p}_0 - \lambda\boldsymbol{n}$, see http://mathworld.wolfram.com/Reflection.html.

   (c) **(6)** Show, that alternatively, a reflection at a plane $\mathcal{E}_n$ through the origin given by its normal $\boldsymbol{n}, |\boldsymbol{n}| = 1$, can be represented as

$$\mathsf{E}_n = \mathsf{R}_{ab}(\boldsymbol{e}_1, \boldsymbol{n})\ \mathsf{E}_1\ \mathsf{R}_{ab}(\boldsymbol{n}, \boldsymbol{e}_1), \quad (15)$$

   where the rotation matrix $\mathsf{R}_{ab}(\boldsymbol{a}, \boldsymbol{b})$ is a minimal rotation of $\boldsymbol{a}$ to $\boldsymbol{b}$. *Hint:* Show that all points p with coordinates $\boldsymbol{p} = \boldsymbol{p}_0 + \lambda\boldsymbol{n}$, where $\boldsymbol{p}_0 \in \mathcal{E}_n$ and $\lambda \in \mathbb{R}$, are transformed to the point p' with coordinates $\boldsymbol{p}' = \boldsymbol{p}_0 - \lambda\boldsymbol{n}$.

   (d) **(1)** Show that E_n is an orthogonal matrix with determinant -1.

   (e) **(13)** For two planes passing through the origin which have non-parallel normals $\boldsymbol{n}$ and $\boldsymbol{m}$, show for arbitrary dimensions:

      i. (**1**) The concatenation

$$\mathsf{R}(\boldsymbol{m}, \boldsymbol{n}) = \mathsf{E}_m \mathsf{E}_n \quad (16)$$

      is a proper rotation matrix.

      ii. (**1**) Show that a vector $\boldsymbol{a} \perp \boldsymbol{m}$ and $\boldsymbol{a} \perp \boldsymbol{n}$, perpendicular to both normals, is not changed by the rotation, i.e. $\mathsf{R}(\boldsymbol{m}, \boldsymbol{n})\,\boldsymbol{a} = \boldsymbol{a}$.

      iii. (**2**) Show, that $\mathsf{R}(\boldsymbol{m}, \boldsymbol{n})\,\boldsymbol{m} = \boldsymbol{n}$.

iv. (**9**) Show, that $R(\boldsymbol{m},\boldsymbol{n})$ is the smallest rotation between $\boldsymbol{m}$ and $\boldsymbol{n}$., thus

$$E_n E_m = R_{ab}(\boldsymbol{m},\boldsymbol{n})\,. \tag{17}$$

Hint: Show that for an arbitrary point $\boldsymbol{p} = A\boldsymbol{\lambda} + \nu\boldsymbol{n} + \mu\boldsymbol{m}$, with arbitrary $A \neq 0$ following $\boldsymbol{m}^\mathsf{T} A = \boldsymbol{0}$ and $\boldsymbol{n}^\mathsf{T} A = \boldsymbol{0}$, the angle between $\boldsymbol{p}$ and $\boldsymbol{p}' = R(\boldsymbol{m},\boldsymbol{n})\boldsymbol{p}$ is not larger than the angle between $\boldsymbol{m}$ and $\boldsymbol{m}$, and only identical if $\boldsymbol{\lambda} = \boldsymbol{0}$, i.e., if the point is in the plane spanned by $(\boldsymbol{m},\boldsymbol{n})$.

(f) **(2)** Given a rotation matrix R, is there a unique way to derive the two normals $\boldsymbol{m}$ and $\boldsymbol{n}$ in (16). Why? Explain the situation geometrically.

(g) **(4)** For a 2-vector $\boldsymbol{p}$, is $\boldsymbol{p}' = -\boldsymbol{p}$ a reflection? For a 3-vector $\boldsymbol{q}$, is $\boldsymbol{q}' = -\boldsymbol{q}$ a reflection? Explain the situation algebraically and geometrically.

B. Rotation matrices

3. **(3)** Given is a 3×3 matrix Q. How can you decide whether Q is a rotation matrix?

4. **(14)** The rotation matrix which rotates the unit 3-vector $\boldsymbol{e}_3$, e.g. the north pole of a sphere, into the normalized vector $\boldsymbol{x}$ with $\boldsymbol{x}_0^\mathsf{T} = [x_1, x_2]$ is given by

$$R_{ab}(\boldsymbol{e}_3,\boldsymbol{x}) = \begin{bmatrix} I_2 - \boldsymbol{x}_0\boldsymbol{x}_0^\mathsf{T}/(1+x_3) & \boldsymbol{x}_0 \\ -\boldsymbol{x}_0 & x_3 \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{x_1x_1}{1+x_3} & -\dfrac{x_1x_2}{1+x_3} & x_1 \\ -\dfrac{x_1x_2}{1+x_3} & 1 - \dfrac{x_2x_2}{1+x_3} & x_2 \\ -x_1 & -x_2 & x_3 \end{bmatrix}. \tag{18}$$

(a) (**6**) Derive the relation using a general equation for $R_{ab}(\boldsymbol{a},\boldsymbol{b})$.

(b) (**3**) For which points is the rotation matrix not defined? Why

(c) (**5**) Rotate the vector $\boldsymbol{y} = \mu\,[-x_2, x_1, 0]^\mathsf{T} \neq \boldsymbol{0}$. Interprete the result.

5. **(12)** Using properties of a rotation matrix.

The figure shows a desk in a rectangular room. The desk stands parallel to the walls. Four right handed

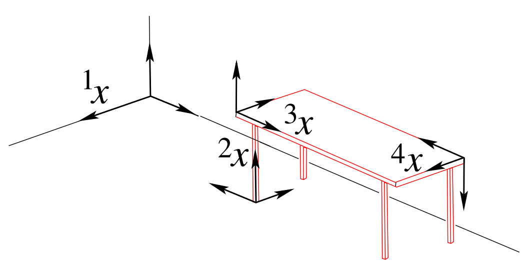

Figure 6: Desk in room

coordinate system $S_k, k = 1,2,3,4$ are indicated by their x-axes ^kx useful for describing objects in the room, for modeling one of the legs, for telling the pose of objects on the table, and for handling the table top. The task is to describe the relative rotations of the four coordinate systems:

(a) (**10**) For each of the four systems S_k give the rotational part $\mathcal{R}_k$ of the motion into the next system S_{k+1}, counting the systems cyclically.

(b) (**2**) Check the four relative rotations by concatenating them, starting from S_1 and ending at S_1.

6. **(32)** Finding and interpreting a relative rotation.

The figure shows the interior of a room with a dormer. The slope of the main roof is $\alpha = 130°$ w.r.t. vertical. The task is to determine the angle between the two blue 3D lines $\mathcal{L}_1$ and $\mathcal{L}_2$. We go through alternative solutions.

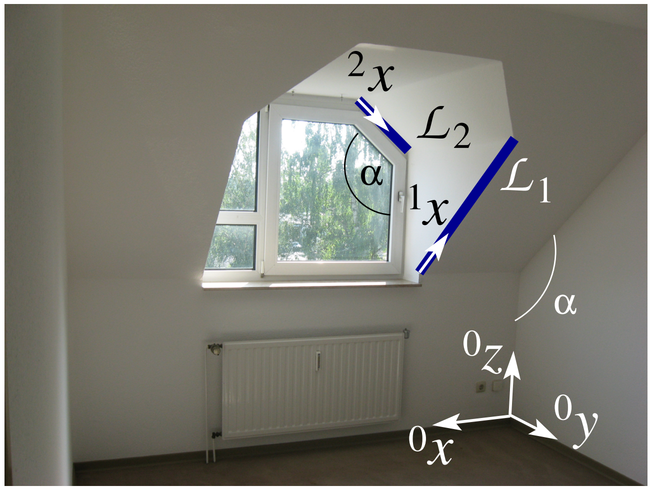

Figure 7: Room with dormer

(a) (**12**) Determine the relative rotation between the room system S_0 and the two local systems S_1 and S_2. Hint: You are free to choose the other axes of S_1 and S_2.

(b) (**4**) Determine the relative rotation between the two systems S_1 and S_2.

(c) (**2**) Determine the angle $\angle(\mathcal{L}_1, \mathcal{L}_2)$.

(d) (**4**) For checking the previous result, determine the angle between the two direction vectors of the two 3D lines in the room coordinate system.

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(e) (**6**) For a right angled spherical triangle with rectangle at C one of *Napier's rules for right spherical triangles* is

$$\cos(c) = \cos(a)\cos(b) \tag{19}$$

(see slide 30 in lecture 3, and `https://en.wikipedia.org/wiki/Spherical_trigonometry`, Sect. 3.6, Rule (R1)). Make a sketch of the geometric configuration of the two directions $^0\boldsymbol{x}_1$ and $^0\boldsymbol{x}_2$ on the unit sphere, identify the rectangular triangle ABC and the three sides a, b, and c. Which of the entities of the spherical triangle is the angle between the two 3D lines? Prove the previous result.

(f) (**4**) Linearize (19) and collect terms up to third order. Explain why the result is plausible.

7. **(18)** A special rotation of the cube.

Show, that a rotation around the axis $\boldsymbol{r} = [1, 1, 1]^\mathsf{T}/\sqrt{3}$ with 120° is given by

$$\mathsf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{20}$$

For this determine the following three rotations

(a) (**8**) rotation $\mathsf{R}(\boldsymbol{r}, \boldsymbol{a})$ from $\boldsymbol{r}$ into the one, say $\boldsymbol{a}$ of the three axes. *Hint:* Use the the rotation matrix from ex. 4.

(b) (**2**) rotation $\mathsf{R}_1(120°)$ by 120°around the axis $\boldsymbol{a}$.

(c) (**1**) rotation $\mathsf{R}(\boldsymbol{a}, \boldsymbol{r})$ from the axis $\boldsymbol{a}$ into the rotation axis $\boldsymbol{r}$

(d) (**7**) Concatenate the three rotations. Is the result plausible? Make a sketch using the unit cube in the first octant with a corner in the origin.

C. Euler angles

8. **(8)** Ambiguity of Euler angles.

   The determintion of the Euler angels from a rotation matrix is not unique, see slide 29 in lecture 3:

   $$R_3(\gamma)R_2(\beta)R_1(\alpha) = R_3(\gamma+\pi)R_2(\pi-\beta)R_1(\alpha+\pi) \tag{21}$$

   Prove this equivalence by determining the change of the sines and cosines of the angles α, β, and γ and check, that the entries in the rotation matrix $R_A(\alpha,\beta,\gamma)$ do not change.

9. **(40)** Effect on estimated angles when choosing the wrong rotation sequence.

   Given are the two rotation matrices generated according to case A (see slide 21 in lecture 3)

   $$R_1 = \begin{bmatrix} 0.998705872708108 & 0.025411959020922 & -0.044054649664433 \\ -0.026152033653421 & 0.999524999872461 & -0.016304777390287 \\ 0.043619387365336 & 0.017435795613492 & 0.998896061698712 \end{bmatrix} \tag{22}$$

   and according to case B (see slide 22)

   $$R_2 = \begin{bmatrix} 0.972580906061019 & 0.085089803642111 & 0.216439613938103 \\ -0.058266578364761 & 0.990134354608083 & -0.127432200289005 \\ -0.225147478358500 & 0.111326929091667 & 0.967943659438827 \end{bmatrix} \tag{23}$$

   (a) (**4**) Determine the three Euler angles $\boldsymbol{\alpha}_{1A}$ from R_1.

   (b) (**6**) Give explicit expressions for determining the angles for case B.

   (c) (**4**) Determine the three Euler angles $\boldsymbol{\alpha}_{2B}$ from R_2.

   (d) (**4**) Determine the Euler angles $\boldsymbol{\alpha}_{1B}$ from R_1 assuming it is a matrix constructed according to case B. Compare the angles $\boldsymbol{\alpha}'_1$ with the correct angles $\boldsymbol{\alpha}_1$. How large is the squared error $d_1^2 = |(\boldsymbol{\alpha}_{1B}-\boldsymbol{\alpha}_{1A})|^2$ in relation to the squared size of the rotation $\theta_1^2 = |\boldsymbol{\alpha}_{1A}|^2$?

   (e) (**4**) Similarly, determine the Euler angles $\boldsymbol{\alpha}_{2A}$ from R_2 assuming it is a matrix constructed according to case A. Compare the angles $\boldsymbol{\alpha}_{2A}$ with the correct angles $\boldsymbol{\alpha}_{2B}$. How large is the squared error $d_2^2 = |(\boldsymbol{\alpha}_{2A}-\boldsymbol{\alpha}_{2B})|^2$ in relation to the squared size of the rotation $\theta_2^2 = |\boldsymbol{\alpha}_{2B}|^2$?

   (f) (**6**) Explain the observed effects. Argue why the relative errors are in the order of the rotation angles in radians.

   (g) (**12**) Assume the three angles are small random variables of the same order following $\underline{\alpha} \sim \mathcal{N}(0,\sigma_\delta^2)$, $\underline{\beta} \sim \mathcal{N}(0,\sigma_\delta^2)$, $\underline{\gamma} \sim \mathcal{N}(0,\sigma_\delta^2)$.

      i. Show, that by restricting to second order terms the relative squared error $d^2 = |\boldsymbol{\alpha}_{1A}-\boldsymbol{\alpha}_{1B}|^2 = (\alpha\gamma)^2 + (\beta\gamma)^2 + (\alpha\beta)^2$.

      ii. Show, that the ratio of the expected square error and the expected square angle is $e_r^2 = \mathbb{E}(|\underline{\boldsymbol{d}}|^2)/\mathbb{E}(|\underline{\boldsymbol{\alpha}}|^2) = \sigma_\delta^2$, hence $e_r = \sigma_\delta$.

      Hint: For zero mean uncorrelated normally distributed variables $\underline{e}_i$ we have $\mathbb{E}(\sum_{1=1}^n \underline{e}_i^2/\sigma_i^2) = n$ and $\mathbb{E}(\underline{e}_1^2\underline{e}_2^2) = \sigma_1^2\sigma_2^2$.

D. Motions on the earth

10. **(27)** Flight direction and distance.

    Frankfurt $\mathcal{F}$ (Germany) has the geographic coordinates with $[\lambda_F, \phi_F] = [8°2', 50°34']$, San Francisco $\mathcal{S}$ (USA) has the geographical coordinates $[\lambda_S, \phi_S] = [-122°24', 37°367]^\mathsf{T}$.

    (a) (**7**) Transform the geographic coordinates $[\lambda, \phi]$ into Cartesian coordinates $[x, y, z]^\mathsf{T}$ and determine the distance of the two cities. Assume the radius of the earth is $r_E = 6371$ km.

(b) (**14**) Determine the azimuth under which the aeroplane needs to leave Frankfurt in order to fly to San Francisco on the shortest path. How long is the path?

Hint: Transform the original earth coordinate S_E into a system S_F, such that Frankfurt has coordinates ${}^F\boldsymbol{x}_F = [0, 0, 1]$.

(c) (**6**) Use the Haversine-Formula (see `https://en.wikipedia.org/wiki/Haversine_formula`) for determining the distance. Comment on the two ways (a) and (c) to determine the distance in this type of application.

11. **(15)** Intermediate stop of long flights.

Long international flights often are partitioned into two parts. As an example, take the 24-hour flight from Auckland (New Zeeland, $(37°00'$ south, $174°47'$ east)) to Frankfort (Germany, $(50°2'$ north, $8°34'$ east,)), which takes around 24 flying hours. Assume the flight would take the shortest path.

(a) (**5**) Given are the following airports with their geographic coordinates Use a world map, and guess

Airport	latitude	longitude
Bangkok	13°41'	100°45'
Hongkong	22°18'	113°56'
Osaka	34°26'	135°13'
Peking	40°04'	116°36'
Singapur	1°21'	103°59'
Tokyo	35°33'	139°47'

Table 1: Airports with geographic coordinates

which of these airports is best for an intermediate stop.

(b) (**6**) Which airport is best for a stop?

(c) (**3**) How do the distances between this airport and Frankfurt differ in time, if the complete flight from Auckland to Frankfurt, without break, requires 24 hours?

(d) (**4**) Determine the mid point of the path on the great circle through Auckland and Frankfurt and determine the distances of the mid point from the six airports. Compare the result with that from 11b. Discuss the result.

12. **(14)** The rocket.

Assume the earth is a sphere with a Cartesian coordinate system as shown in Fig. 8: the origin O in the centre of the sphere,the x-axis lying on the equator and the zero-meridian.

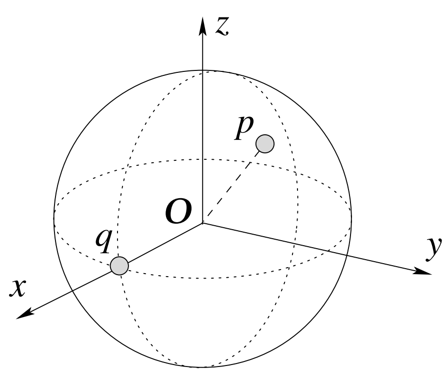

Figure 8: World coordinate system with point p

A rocket starts at $p(3682, 3682, 3682)$ km and flies into space following a very simplified motion.

- **Start:** A straight path of distance $d = 40000$ km in the direction $\boldsymbol{s} = \frac{1}{\sqrt{3}}[1\,,1\,,1]^{\mathsf{T}}$.
- **Circular motion:** around the axis $\boldsymbol{r} = \frac{1}{\sqrt{3}}[1\,,1\,,-1]^{\mathsf{T}}$.

(a) (**2**) Determine the position $\boldsymbol{p}'$ after the starting path using a homogeneous motion matrix M_s. Give M_s and the position $\boldsymbol{p}'$ in Cartesian coordiantes.

(b) (**2**) Assume the rocket circumnavigate the earth. Provide the homogeneous motion matrix M_e, which describes this path until the rocket has circumnavigated the earth by a third.

(c) (**3**) Determine the position $\boldsymbol{p}''$ of the rocket after the rotational motion. Provide the motion matrix M_1 of the complete path and the Cartesian coordinates of $\boldsymbol{p}''$ in km.

(d) (**3**) After the motion $\mathcal{M}_1$ the rocket performs a straight path to the point $\boldsymbol{a} = [-3250\,,1740\,,-5200]^{\mathsf{T}}$ where it is meant to be sunk. Determine this straight motion $\mathcal{M}_a$ and determine the motion $\mathcal{M}_2$ from the start to the end. Provide M_a and M_2.

(e) (**4**) Determine the geographic coordinates of the point $\boldsymbol{p}_a$ of impact.

13. **(17)** Parachutist.

A parachutist $\mathcal{P}$ at a certain time t_0 is at position $\boldsymbol{P} = [x_0, y_0, z_0]^{\mathsf{T}}$ in a reference system. His own, local coordinate system, positioned in his head, is defined as follows: the x-axis points downwards towards nadir in opposite direction to the z- axis of the reference system. The local z axis points in the opposite direction of the reference y axis. The local y axis completes the right handed local coordinate system.

At a later time t_1 the parachutist has fallen down by the distance d and turned his head by ϕ, such that the local coordinate system rotates around its y-axis.

The matrix $\mathcal{M}_P$ describes the motion of the global into the local coordinate system at time t_1.

(a) (**5**) Make a sketch which shows the relative pose of the reference and the local system at time t_1.

(b) (**12**) Give the individual homogeneous matrices M_i which define the complete motion $\mathcal{M}_P$. Explain how you concatenate the individual motions and provide arguments for your choices. – You need not determine the concatenated motion explicitly.

total: 251

4: Axis angle and exponential representation

Learning objective: This set of exercises should enable you to reliably determine angles in 3D and to analyse and represent 3D rotations using rotation matrices and Euler angles.

A. Skew symmetric and rotation matrices

1. **(15)** Refer to arbitrary vectors $\boldsymbol{a}$ and unit vectors $\boldsymbol{r}$ and prove the following

$$\begin{aligned} \mathsf{S}_r^2 &= \mathsf{D}_r - \mathsf{I}_3 \\ \mathsf{S}_a^2 &= \mathsf{D}_a - |\boldsymbol{a}^2|\,\mathsf{I}_3 \\ \mathsf{R}_{r,\theta}\mathsf{R}_{r,\theta}^{\mathsf{T}} &= \mathsf{I}_3 \end{aligned} \tag{24}$$

   Interprete the relation (24) geometrically by multiplying both sides with an arbitrary vector $\boldsymbol{x}$. Make a sketch.

B. Rotation matrices

2. **(12)** Axis and angle from rotation matrix.

   Given are the two rotation matrices

$$\mathsf{R}_1 = \frac{1}{7225}\begin{bmatrix} 7175 & 744 & -408 \\ 600 & -6905 & -2040 \\ -600 & 1992 & -6919 \end{bmatrix} \quad \text{and} \quad \mathsf{R}_2 = \frac{1}{169}\begin{bmatrix} -151 & -24 & 72 \\ -24 & -137 & -96 \\ 72 & -96 & 119 \end{bmatrix}$$

   (a) (**4**) Check whether the matrices are rotation matrices. Work with rational numbers if possible

   (b) (**4**) Determine the rotation angles $\theta_i, i = 1, 2$.

   (c) (**4**) Determine the rotation axes $\boldsymbol{r}_i, i = 1, 2$. Work with integers.

3. **(12)** Axis and angle of a rotation.

   Given is a rotation with angle $\phi = 125°$ and axis $\boldsymbol{r} = [3, 4, 12]/13$.

   (a) (**4**) Determine the rotation matrix using the axis angle and the exponential representation. Compare the results.

   (b) (**4**) From the rotation matrix $\mathsf{R}_{r,\theta}$ derive the angle and the axis and compare with the given values.

   (c) (**4**) Choose $\phi = 720°$ and repeat (3a) and (3b. Discuss the result.

4. **(26)** Rotational symmetry groups of a cube.

   We have seen a rotation by 120° which maps a unit cube into itself, in this case assuming a corner is in the origin. Repeating this motion again maps the cube into itself. Together with the zero rotation there are three rotations, which of course could be applied to other corners. The task is to find the rotation groups of the cube $[\pm 1, \pm 1, \pm 1]$ with a fixed axis which are different and finally provide the total number of all rotations mapping the cube into itself.

   (a) (**13**) How many rotation groups around a fixed axis having different cardinality do exist? For each group,

      i. (**2**) give the number of non-zero rotations,

      ii. (**2**) give the smallest angles for a non-zero rotation,

      iii. (**2**) describe the geometric position of the rotation axis,

      iv. (**2**) give the number of geometrically different rotation axes.

      v. (**5**) give an example for a rotation matrix of the cube with coordinates $[\pm 1, \pm 1, \pm 1]$.

(b) (**2**) Together with the zero-rotation, how many different rotations map the cube into itself?

(c) (**5**) How many rotation groups of the regular *tetrahedron* around a fixed axis exist? Give the smallest angles for a non-zero rotation. How many rotations map the tetrahedron into itself?

(d) (**6**)) If you take an image of a building in a general direction, you can infer the three directions of a rectangular coordinate system using vanishing points. Assume you are just given the three mutually

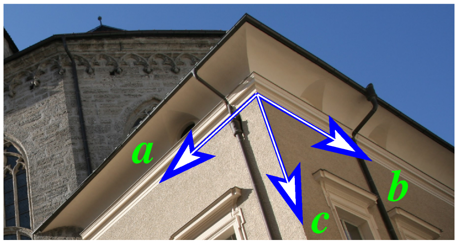

Figure 9: Image with inferred tripod with inferred normalized 3-vectors $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$

orthonormal 3-vectors $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ with $\det([\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]) = 1$, say, in the local (camera) coordinate system, not the image. You want to relate this tripod to the three axes of the global (world) system. How many possibilities for defining the three coordinate axes of a right handed coordinate system do you have? Give a geometric explanation.

See video `https://www.youtube.com/watch?v=Ch95sES5D9A` and lecture notes on symmetry groups `http://www-groups.mcs.st-andrews.ac.uk/~john/geometry/Lectures/L10.html` and following page.

total: 65

5: Quaternions and concatenation of parametric rotations

Learning objective: This set of exercises should enable you to work with quaternions, interprete them as rotations and build links the different representation fo rotations discussed in the lectures so far.

A. Using the algebra of quaternions

1. **(15)** Basics operations

   Given are the two quaternions

$$\mathbf{q}_1 = \begin{bmatrix} 4 \\ 2 \\ 5 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} 3 \\ 5 \\ 11 \\ 13 \end{bmatrix}$$

   Determine

   (a) **(3)** the product $\mathbf{q}_1\mathbf{q}_2$.

   (b) **(2)** the inverse $\mathbf{q}_2^{-1}$

   (c) **(2)** the product $\mathbf{q}_2\mathbf{q}_2^{-1}$.

   (d) **(8)** the ratio $\frac{\mathbf{q}_1}{\mathbf{q}_2}$. Clarify, what the task means. Give a full explanation and the corresponding solutions.

2. **(8)** Manipulating pure quaternions.

   Pure quaternions have scalar part zero, i.e. are of the form $\mathbf{q} = (0, \boldsymbol{q})$. Similarly to general quaternions, pure quaternions may be unit quaternions. Assume all quaternions $\mathbf{x}$ and $\mathbf{y}$ in this task are pure. Prove the following expressions (most proofs are not longer than one line)

Multiplication	$\mathbf{x}\,\mathbf{y} = (-\boldsymbol{x}\cdot\boldsymbol{y}, \boldsymbol{x} \times \boldsymbol{y})$	(25)
Square	$\mathbf{x}^2 = -\lvert\boldsymbol{x}\rvert^2$	(26)
Cube	$\mathbf{x}^3 = -\lvert\boldsymbol{x}\rvert^2\mathbf{x}$	(27)
Product of conjugates	$\overline{\mathbf{x}}\,\overline{\mathbf{y}} = \mathbf{x}\,\mathbf{y}$	(28)
Reverse product	$\mathbf{y}\,\mathbf{x} = \overline{\mathbf{x}\,\mathbf{y}}$	(29)
Constraint for Product of orthogonal quaternions	$0 = \mathbf{x}\,\mathbf{z} + \mathbf{z}\,\mathbf{x}$	(30)
Square and cube of unit quaternion	$\mathbf{x}^2 = -1\,, \quad \text{and} \quad \mathbf{x}^3 = -\mathbf{x}$	(31)
Product of two unit quaternions with angle α		(32)

$$\mathbf{x}\,\mathbf{y} = (-\cos\alpha, \sin\alpha\, \mathsf{N}(\mathbf{x} \times \mathbf{y})) \quad \text{with} \quad \lvert\mathbf{x}\,\mathbf{y}\rvert = 1$$

3. **(10)** A 3D point $\chi(\boldsymbol{x})$ is reflected at a plane $\mathcal{A}(\boldsymbol{n})$ with normal vector $\boldsymbol{n}$ going through the origin yielding the point $\chi'(\boldsymbol{x}')$. This reflection can be represented with the pure quaternions $\mathbf{x} = (0, \boldsymbol{x})$ and $\mathbf{n} = (0, \boldsymbol{n})$ as:

$$\mathbf{x}' = \mathbf{n}\mathbf{x}\mathbf{n}\,. \tag{33}$$

   Prove that the corresponding transformation of the coordinates is

$$\boldsymbol{x}' = \mathsf{E}_n \boldsymbol{x} \quad \text{with} \quad \mathsf{E}_n = \mathsf{I}_2 - 2\boldsymbol{n}\boldsymbol{n}^\mathsf{T}\,. \tag{34}$$

   See `https://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/transforms/index.htm`

and Video 5a on rotations as pairs of reflections with quaternions.

4. **(18)** Deriving the product rule of quaternions.

   R. W. Hamilton has defined quaternions as hypercomplex numbers $\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$ with the basic numbers i, j and k fulfilling the non-commutative multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1\,. \tag{35}$$

   A quaternion also can be defined by a pair $\mathbf{q} = (q, \boldsymbol{q})$ of a scalar $q = q_0$ and a vector $\boldsymbol{q} = [q_1, q_2, q_3]^{\mathsf{T}}$ with the multiplication rule for two quaternions $\mathbf{p}$ and $\mathbf{r}$

$$\mathbf{q} = \mathbf{pr} \quad \text{with} \quad q = pr - \boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{r} \quad \text{and} \quad \boldsymbol{q} = r\boldsymbol{p} + p\boldsymbol{r} + \boldsymbol{p} \times \boldsymbol{r}\,. \tag{36}$$

   The task of this exercise is to derive further multiplication rules from (35), namely

$$ij = k = -ji \tag{37a}$$

$$jk = i = -kj \tag{37b}$$

$$ki = j = -ik \tag{37c}$$

   and (36).

   (a) **(6)** Prove the relations in (37a).
   (b) **(6)** Show, that only one of the three equations, say (37a) needs to be proven.
   (c) **(6)** Using these rules prove (36).

5. **(18)** Analyse the multiplication matrix T_{q} on slide 9, video 5.

   (a) (**2**) Prove that $\mathsf{T}_{\overline{\mathsf{q}}} = \mathsf{T}_{\mathsf{q}}^{\mathsf{T}}$.
   (b) (**2**) Prove that $\mathsf{T}_{\mathsf{q}^{-1}} = \mathsf{T}_{\mathsf{q}}^{-1}$.
   (c) (**1**) Prove that for a unit quaternion $\mathbf{q}^e = \mathbf{q}/|\mathbf{q}|$ the matrix $\mathsf{T}_{\mathsf{q}^e}$ is orthogonal.
   (d) (**5**) Prove that for any two quaternions $\mathbf{p}$ and $\mathbf{q}$ the norm of the product is identical to the product of the norms

$$|\mathbf{pq}| = \mathbf{p}|\;|\mathbf{q}|\,. \tag{38}$$

6. **(26)** Rational unit quaternions.

   We have used the following quaternions

$$\mathbf{q}_0 = \frac{1}{2}[-1, 1, 1, 1]^{\mathsf{T}} \quad \text{and} \quad \mathbf{q}_1 = \frac{1}{85}[12, -84, -4, 3]^{\mathsf{T}} \tag{39}$$

   in lecture V5, slide 21 and exercise V4-2 for generating R_1. Both contain only rational numbers as elements, why we call them rational quaternions.

   (a) (**2**) Show, that $\mathbf{q}_0$ and $\mathbf{q}_1$ are unit quaternions.
   (b) (**4**) Show that for an arbitrary quaternion $\mathbf{t} = (u, \boldsymbol{v})$ the quaternion

$$\mathbf{q}(\mathbf{t}) = \left(\frac{u^2 - |\boldsymbol{v}|^2}{u^2 + |\boldsymbol{v}|^2}, \frac{2u\boldsymbol{v}}{u^2 + |\boldsymbol{v}|^2}\right) \tag{40}$$

   is a unit quaternion. *Hint:* Give a short expression for the function $\mathbf{q} = \mathbf{q}(\mathbf{t})$.
   (c) (**4**) Give two integer quaternions $\mathbf{t}_2$ and $\mathbf{t}_3$ leading to rational unit quaternions $\mathbf{q}_2$ and $\mathbf{q}_3$ with denominators in the range $[8, 10]$.
   (d) (**8**) Specify two integer quaternions $\mathbf{t}_0$ and $\mathbf{t}_1$ leading to two unit quaternions $\mathbf{q}_0$ and $\mathbf{q}_1$.
   (e) (**8**) Given is an integer quaternion $\mathbf{q}$ having an integer norm. Describe in your own words the relation between the elements of the quaternion $\mathbf{q}$, say (a, b, c, d) and its norm, say e. Derive an expression for an integer quaternion $\mathbf{t}$ leading to $\mathbf{q}$.

B. Rotations with quaternions

7. **(20)** Interpretation and use of quaternions for rotations.

   (a) **(6)** Two rotations are given by the corresponding quaternions

$$\mathbf{q}_1 = \begin{bmatrix} 7 \\ -11 \\ 8 \\ -16 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

   Describe the rotations.

   (b) **(14)** Given is the point $p([2, -1, 4])$ and the rotation the direction $\boldsymbol{d} = [4, -3, 12]$ of its axis and its angle $\theta = 75°$.

      i. **(2)** Give the unit quaternion $\mathbf{q}$ which represents this rotation?

      ii. **(10)** Use the equation

$$\begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} = \mathbf{q} \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \mathbf{q}^{-1} ,$$

      for determining the rotated point p'.

      iii. **(2)** Is the angle between the original and the rotated point identical to θ. Why?

8. **(10)** Interpreting the rotations of the three axes.

   The 24 elements of the octaheron group are the rotations represented by the following quaternions.

$$\begin{array}{lll}
\mathbf{q}_0 = [1,0,0,0]^\mathsf{T} & \mathbf{q}_1 = [0,1,0,0]^\mathsf{T} & \mathbf{q}_2 = [0,0,1,0]^\mathsf{T} \\
\mathbf{q}_3 = [0,0,0,1]^\mathsf{T} & \mathbf{q}_4 = [1,1,1,1]^\mathsf{T} & \mathbf{q}_5 = [1,-1,-1,-1]^\mathsf{T} \\
\mathbf{q}_6 = [1,-1,1,1]^\mathsf{T} & \mathbf{q}_7 = [1,1,-1,1]^\mathsf{T} & \mathbf{q}_8 = [1,1,1,-1]^\mathsf{T} \\
\mathbf{q}_9 = [1,1,-1,-1]^\mathsf{T} & \mathbf{q}_{10} = [1,-1,1,-1]^\mathsf{T} & \mathbf{q}_{11} = [1,-1,-1,1]^\mathsf{T} \\
\mathbf{q}_{12} = [1,1,0,0]^\mathsf{T} & \mathbf{q}_{13} = [1,0,1,0]^\mathsf{T} & \mathbf{q}_{14} = [1,0,0,1]^\mathsf{T} \\
\mathbf{q}_{15} = [0,1,1,0]^\mathsf{T} & \mathbf{q}_{16} = [0,1,0,1]^\mathsf{T} & \mathbf{q}_{17} = [0,0,1,1]^\mathsf{T} \\
\mathbf{q}_{18} = [1,-1,0,0]^\mathsf{T} & \mathbf{q}_{19} = [1,0,-1,0]^\mathsf{T} & \mathbf{q}_{20} = [1,0,0,-1]^\mathsf{T} \\
\mathbf{q}_{21} = [0,1,-1,0]^\mathsf{T} & \mathbf{q}_{22} = [0,1,0,-1]^\mathsf{T} & \mathbf{q}_{23} = [0,0,1,-1]^\mathsf{T}
\end{array}$$

   These rotations map an octahedron or a cube, or the three orthogonal axes of a coordinate system, onto itself. Choose two quaternion indices l and m as a function of the number n of your university licence or another document. For the first index take $l = \mod(n, 23) + 1$. For the second index use the number $\overleftarrow{n}$ with the reverse sequence of digits and take $m = \mod(\overleftarrow{n}, 23) + 1$. If $l = m$, then choose $l = \mod(n + 11, 13) + 1$.

   Interprete the two quaternions $\mathbf{q}_l$ and $\mathbf{q}_m$ as rotations of the cube with corners $[\pm 1, \pm 1, \pm 1]$.

C. Rotation parametrizations

9. **(25)** Double rotation

   A rotation $\boldsymbol{x}' = R\mathbf{x}$ is provided with the three Euler angles $\alpha = -10°$, $\beta = 20°$, $\gamma = 30°$ following the definition $R = R_3(\gamma)R_2(\beta)R_1(\alpha)$.

   (a) (**8**) Determine the parameters of the rotation

      i. R, rotation matrix,

ii. $(\boldsymbol{r}, \omega)$, axis and angle,
iii. $\boldsymbol{\theta}$, rotation vector
iv. $\boldsymbol{m}$, Rodriguez parameters
v. $\boldsymbol{u}$, Cayley parameters

(b) (**12**) If possible, start from the parameters of the individual representations and determine the parameters of double the rotation $R_d = R^2$, i.e. for the Euler angles, for the axis and angle representation $(\boldsymbol{r}_d, \omega_d)$, etc. Numerically check the parameters of the double rotations, e.g., . by $\boldsymbol{x}'' = R_d\boldsymbol{x} = R\boldsymbol{x}'$. For which of the representation is the determination of the parameters of the double rotation, easy (less than one line of proof), moderate (a few lines of proof), awkward (more that 10 lines of proof)

10. **(20)** Half a rotation

Given is the rotation matrix $R = R_R(\boldsymbol{m})$, by the Rodriguez parameters $\boldsymbol{m} = [1, 2, -3]^\mathsf{T}$:

$$R = \frac{1}{9}\begin{bmatrix} -4 & 8 & 1 \\ -4 & -1 & -8 \\ -7 & -4 & 4 \end{bmatrix} = \begin{bmatrix} -0.4444 & 0.8889 & 0.1111 \\ -0.4444 & -0.1111 & -0.8889 \\ -0.7778 & -0.4444 & 0.4444 \end{bmatrix}. \tag{42}$$

Determine the parameters of half the rotation.

(a) Determine the parameters of the full rotation $\boldsymbol{x}' = R\boldsymbol{x}$ in the various representations and give the parameters of half the rotation R_h $(\boldsymbol{x}' = R_h^2\boldsymbol{x})$ in the same representation

i. R_h, rotation matrix,
ii. $(\boldsymbol{r}_h, \omega_h)$, axis and angle,
iii. $\boldsymbol{\theta}_h$, rotation vector
iv. $\boldsymbol{m}_h$, Rodriguez parameters
v. $\boldsymbol{u}_h$, Cayley parameters

Hint. Possibly use basic relations from trigonometry.

(b) For which of the representations is this easy (less than one line of proof), medium (several lines of proof), awkward (more than 10 lines of proof)?

(c) Is there a unique solution? Are there multiple solutions? Discuss in detail.

11. **(18)** Square root of a quaternions and half rotations.

The previous exercise required to determine the square root $\mathbf{q}_h$ of the quaternion $\mathbf{q}$ since $\mathbf{q} = \mathbf{q}_h\mathbf{q}_h$. The task here is to find the number of quaternions representing half of a rotation. Given are the four quaternions

$$\mathbf{q}_1 = \left(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}\boldsymbol{e}_1\right), \quad \mathbf{q}_2 = \left(-\sqrt{2-\sqrt{2}}, \sqrt{2+\sqrt{2}}\boldsymbol{e}_1\right), \tag{43}$$

$\mathbf{q}_3 = -\mathbf{q}_1$ and $\mathbf{q}_4 = -\mathbf{q}_2$, with $\boldsymbol{e}_1 = [1, 0, 0]^\mathsf{T}$.

(a) (**2**) How many rotations $\mathcal{R}_i(\mathbf{q}_i)$ do the four quaternions $\mathbf{q}_i$ represent?

(b) (**8**) Determine the squares of the four quaternions.

(c) (**1**) How many different squares $\mathbf{q}_i^2$ do you obtain?

(d) (**1**) How many different rotations do the squares $\mathbf{q}_i^2$ represent?

(e) (**1**) Do the square roots of a single quaternion represent two different rotations?

(f) (**5**) Give an algorithm to determine the square root of a quaternion.

12. **(21)** Interpolation of rotations.

A camera is to be panned to smoothly move from one pose at time $t = 0$ to another pose at time $t = 1$. Given are the two sets of Euler angles

$$(\alpha, \beta, \gamma)_0 = (1, -45, +60)^\circ \qquad (\alpha, \beta, \gamma)_1 = (1, 45, +60)^\circ \tag{44}$$

referring to the reference system with rotations given in this system.

(6) Provide the quaternions for both poses (rotations).

The interpolation for the parameter value $t = 0.7$ is to be performed in three different manners:

(a) **(1)** Element wise linear interpolation of the angles, e.g., $\alpha(t) = (1 - t)\alpha_0 + t\alpha_1$.

(b) **(7)** Element wise linear interpolation of the unit quaternions.

(c) **(7)** Linear interpolation of the unit quaternion via the rotation angle $\phi = \arccos(\mathbf{q}_0^\mathsf{T}\mathbf{q}_t)$ between these quaternions:

$$\mathbf{q}(t) = \frac{1}{\sin\phi}(\mathbf{q}_0 \sin((1-t)\phi) + \mathbf{q}_1 \sin(t\phi)) \tag{45}$$

For each of the three cases give the three Euler angles $(\alpha, \beta, \gamma)_{0.7}$. How do you evaluate the three interpolation methods?

13. **(20)** Interpolation of directions.

Show, that 45 actually is an interpolation of unit direction vectors, for the general case $\mathbf{q}_0, \mathbf{q}_1 \in \mathbb{R}^n$.

$$\text{(a): } (\mathbf{1}) \qquad \mathbf{q}(t) = \mathbf{q}_t,\ t = (0, 1) \tag{46}$$

$$\text{(b): } (\mathbf{3}) \qquad \mathbf{q}(t),\ \mathbf{q}(0) \text{ and } \mathbf{q}(1) \text{ are coplanar} \tag{47}$$

$$\text{(c): } (\mathbf{8}) \qquad |\mathbf{q}(t)| = 1 \tag{48}$$

$$\text{(d): } (\mathbf{8}) \qquad \arccos(\mathbf{q}(t)^\mathsf{T}\mathbf{q}_0) = t\phi \tag{49}$$

Eqs. (47) and (48) state, that the path $\mathbf{q}(t)$ is circular in $\mathbb{R}^n$. The last equation (49) states, that the motion has a constant velocity along the circle.

total: 229

6: Small rotations and rotations from vector pairs

Learning objective: This set of exercises should enable you to handle small rotations and uncertain rotations and determine rotations from vector pairs.

A. Small rotations

1. **(9)** Approximate rotation matrix
   (a) Determine the matrix $M(\boldsymbol{\theta}) = I_3 + S_{\boldsymbol{\theta}}$ with $S_{\boldsymbol{\theta}} := \theta S_{\boldsymbol{r}}$ and $\boldsymbol{r} = [0,0,1]^\mathsf{T}, \theta = 60°$. Is $M(\boldsymbol{\theta})$ a proper rotation matrix? Provide arguments.
   (b) How can you determine a proper rotation matrix R for a given approximate rotation matris M? Determine the best approximating rotation matrix $R(\boldsymbol{\theta})$ for $M(\boldsymbol{\theta})$.
   (c) Is the matrix $M(\boldsymbol{\theta})$ a good aproximation for $R(\boldsymbol{\theta})$? Provide arguments.

2. **(12)** Rotation axis and angle for small angles
   Given are the three small angles $\alpha = 0.1°$, $\beta = 0.3°$, $\gamma = -0.6°$.
   (a) Determine the rotation matrix for $R = R_3(\gamma)R_2(\beta)R_1(\alpha)$.
   (b) Give the rotation angle θ in degrees. How could you determine an approximation of θ from the three given angles?
   (c) Give the rotation axis $\boldsymbol{r}$. How could you determine the axis approximately from the three angles.

   Discuss the results.

3. **(9)** Concatenation of small angles
   Show, that for small rotations the concatenation $R(\boldsymbol{\theta}) = R(\boldsymbol{\theta}_2)R(\boldsymbol{\theta}_1)$ can be achieved by direct addition of the rotation vectors $\boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$. Under which conditions is this an approximation? How good is the approximation, say $\boldsymbol{\theta}^a$, i.e., how large is the relative error w.r.t. rigorously determined rotation vector $\boldsymbol{\theta}$? Give reasons. *Hint:* Take as relative error the value $|\boldsymbol{\theta} - \boldsymbol{\theta}^a|/|\boldsymbol{\theta}|$.

B. Uncertain rotations

4. **(24)** Generating random rotations.
   Generate I random 3-vectors $\underline{\boldsymbol{\theta}}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2 I_3)$.
   (a) (**12**) Choose $I = 100$ and $\sigma = 3$ and generate rotation matrices $R_i = R_\theta(\boldsymbol{\theta}_i)$ using a function for the exponential of a matrix. Derive the estimated skew symmetric matrices $\widehat{S}_i = S(\widehat{\boldsymbol{\theta}}_i)$ by using a function for calculating the logarithm of a matrix and determine the estimated rotations vectors $\widehat{\boldsymbol{\theta}}_i$. How do the estimated rotation vectors differ from the generated ones? Report on the results.
   (b) (**12**) Repeat the experiment by choosing $I = 1000$ and $\sigma = 0.001$.
      i. Determine the estimated means $\widehat{\boldsymbol{\mu}}_\theta$ and $\widehat{\boldsymbol{\mu}}_{\widehat{\theta}}$ of the I generated and the I estimated rotation vectors. Comment on the result.
      ii. Determine the covariance matrices $\widehat{\Sigma}_{\theta\theta}$ and $\widehat{\Sigma}_{\widehat{\theta}\widehat{\theta}}$. Comment on the result.

      Hints: In MATLAB use the functions `expm.m` and `logm.m`.. Collect the generated and the estimated rotation vectors in 1000×3 matrices and use functions for estimating the 3-vector of the mean of the columns of a matrix and the 3×3 covariance matrix. In MATLAB these are the function `mean.m` and `cov.m` .

5. **(20)** Uncertain rotation from uncertain Euler angles.
   Given are the Euler angles $\boldsymbol{\alpha} = [\omega, \phi, \kappa]$ with their covariance matrix $\Sigma_{\alpha\alpha}$. The task is to determine the covariance matrix Σ_{rr} of the uncertain rotation matrix $R = R_3(\kappa)R_2(\phi)R_1(\omega)$.

(a) Verify that with the unit 3-vectors $\boldsymbol{e}_i$ the required Jacobian is

$$J_{r\alpha} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\alpha}} = [R_3(\kappa)R_2(\phi)\boldsymbol{e}_1 \mid R_3(\kappa)\boldsymbol{e}_2 \mid \boldsymbol{e}_3]\,. \tag{50}$$

Hint: Evaluate the total differential. Confirm and use, e.g.,

$$\mathrm{d}R_1(\omega) = S(\boldsymbol{e}_1)R_1(\omega)\,\mathrm{d}\omega\,. \tag{51}$$

(b) Show, that the determinant if this Jacobian is

$$|J_{r\alpha}| = \cos\phi\,. \tag{52}$$

Interprete the result. Especially comment on the rank of the covariance matrix Σ_{rr} as function of the angle ϕ.

C. Rotations from pairs of directions

6. **(18)** Shortest rotation between two directions

Cid and Tojo (2018) show that for two unit n-vectors $\boldsymbol{x}, \boldsymbol{y} \in S^n$

$$R_{xy} = I + K + \frac{1}{1+\boldsymbol{x}\cdot\boldsymbol{y}}\,K^2 \quad \text{with} \quad K = \boldsymbol{y}\boldsymbol{x}^\mathsf{T} - \boldsymbol{x}\boldsymbol{y}^\mathsf{T} \tag{53}$$

performs the smallest rotation from $\boldsymbol{x}$ to $\boldsymbol{y}$.

(a) (**3**) Interpret the matrix K for 3D vectors in the context of the desired rotation.

(b) (**3**) Interpret R_{xy} in 3D by comparing it with the axis and and angle representation $R_{r,\theta}$ of a rotation matrix.

(c) (**12**) Show that the matrix R_{xy} performs the same rotation as R_{ab} presented in the lecture, independent of the dimension.

7. **(45)** Plate tectonics

The earth's continents move relative to each other. These motion can be interpreted as small rotation of spherical plates on the earth. For different positions, see Fig. 10, the coordinates and the velocities are given in [mm] and [mm/year], respectively, in the IRTF2000 system (International Terrestrial Reference Frame) with its origin at the centre of the earth.

For January 1st, 2015 we find the following data for EUSK (Euskirchen, Germany), SHAO (Shanghai, China) and NRIL (Norilsk, Russia) in the eurasian plate.

EUSK	$x = 4\,022\,106\,084.755$	$y = 477\,011\,254.387$	$z = 4\,910\,840\,843.753$
	$v_x = -15.522$	$v_y = 16.572$	$v_z = 9.689$
SHAO	$x = -2\,831\,733\,809.639$	$y = 4\,675\,665\,817.043$	$z = 3\,275\,369\,306.163$
	$v_x = -30.371$	$v_y = -10.949$	$v_z = -11.257$
NRIL	$x = 64\,536\,969.633$	$y = 2\,253\,782\,897.416$	$z = 5\,946\,363\,512.268$
	$v_x = -22.037$	$v_y = 3.248$	$v_z = 1.009$

(a) **(8)** Determine the positions of the three stations at January 1st, 2016. Then, determine the angular velocities ω_i for all stations and the rotation vector $\boldsymbol{r}_i$w.r.t. the centre of the earth. Take care of the numerical accuracy.

(b) i. **(12)** Determine the motion of the eurasian plate from the motions of the three stations in 2015. Provide the best approximating rotation matrix $\widehat{R}_1$. What is the rotational velocity of this motion in arcsec/year.

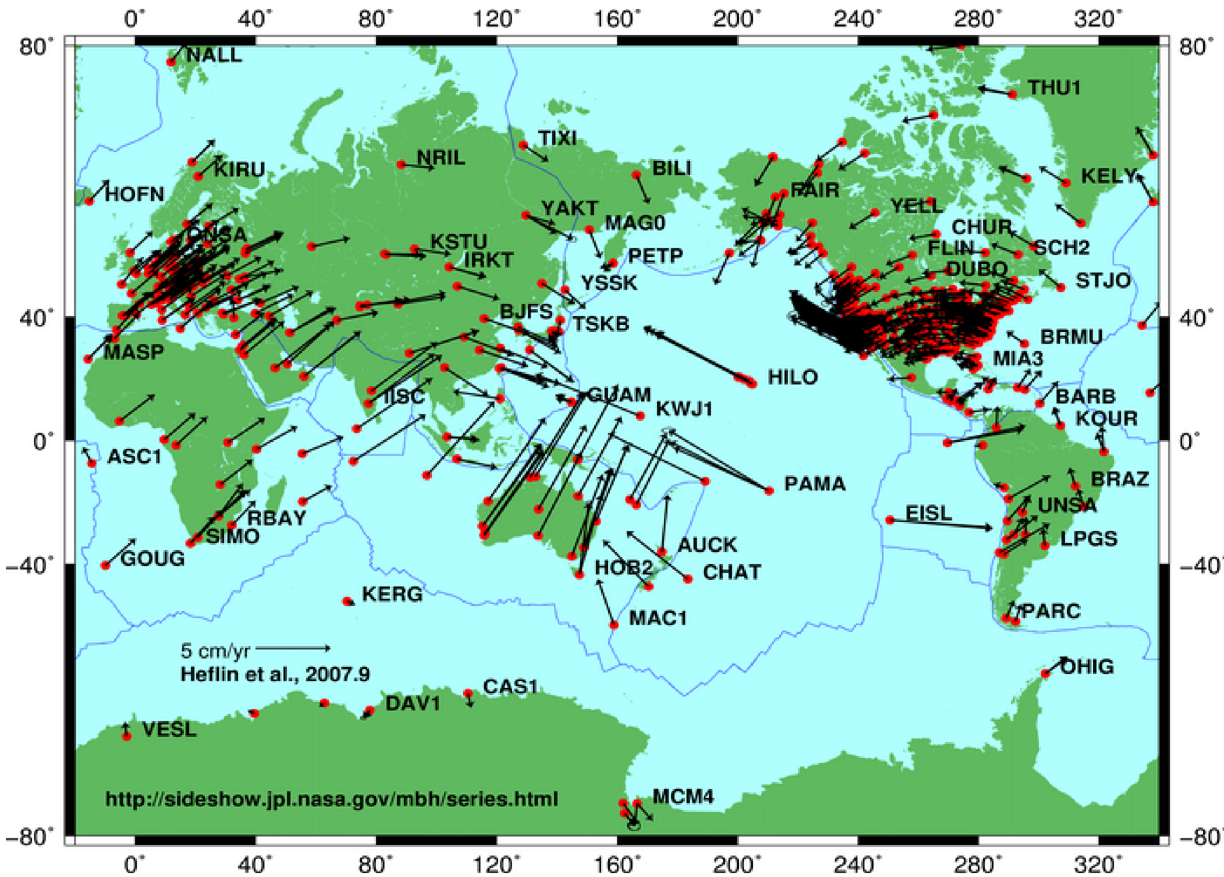

Figure 10: Velocity vectors of plate tectonics

ii. **(10)** Determine the motion of the eurasian plate only from the motions of the two stations Euskirchen and Norilsk in 2015. Determine the best approximating rotation matrix $\widehat{R}_2$. Again, determine the rotational velocity in arcsec/year. Compare the result with the one of the previous task.

(c) **(7)** Give the geographic coordinates of the fixed point on the earth, derived from $\widehat{R}_1$. Compare the result with the map in Fig. 10.

(d) **(8)** The new „Bundeskanzlerplatz", the geographic centre of Bonn, has geographic coordiantes BONN $= [+50° \, 43' \, 9'', +7° \, 7' \, 3'']^{\mathsf{T}}$. How many mm does it move annually in the IRTF2000 system?

total: 137

References

Cid, J. Á. and F. A. F. Tojo (2018). A Lipschitz condition along a transversal foliation implies local uniqueness for ODE. *Electronic Journal of Qualitative Theory of Differential Equations* (13), 1–14.