Photogrammetry & Robotics Lab

3D Coordinate Systems

(Bsc Geodesy & Geoinformation)

13. Evaluation of estimated parameters

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The slides have been created by Wolfgang Förstner.

Topics

1. How to characterize the uncertainty of the estimates
   1. What is the uncertainty of the input data?
   2. What is the uncertainty of the output data?
2. How to check the implementation?
3. Uncertainty of estimated parameters of 3D similarity

For this video see Förstner/Wrobel (2016), p. 86, 89-91, 137, 139-141

5.4.1 The uncertainty of the estimated parameters

Uncertainty of the estimated parameters

- Assumption: linear model → nonlinear model
- Estimated parameters: fixed values, using Σ^a_{ll}

$$\widehat{\boldsymbol{x}} \mid \Sigma^a_{ll} = \underbrace{(A^\mathsf{T} W^a_{ll} A)^{-1} A^\mathsf{T} W^a_{ll}}_{J_{\widehat{x}l}} \boldsymbol{l}$$

- Predicted covariance matrix of $\widehat{\boldsymbol{x}}$
  - model assuming $\boldsymbol{l}$ is sample from $\mathcal{N}(\boldsymbol{f}(\boldsymbol{x}), \Sigma_{ll})$
  - estimation with Σ^a_{ll}: $\widehat{\boldsymbol{x}}$ is sample from $\mathcal{N}(\boldsymbol{x}, \Sigma_{\widehat{x}\widehat{x}|\Sigma_{ll}})$

$$\begin{aligned}\Sigma_{\widehat{x}\widehat{x}|\Sigma^a_{ll}} &= J_{\widehat{x}l}\, \Sigma_{ll}\, J^\mathsf{T}_{\widehat{x}l} \\ &= (A^\mathsf{T} W^a_{ll} A)^{-1} A^\mathsf{T} W^a_{ll} \cdot \Sigma_{ll} \cdot W^a_{ll} A (A^\mathsf{T} W^a_{ll} A)^{-1}\end{aligned}$$

Uncertainty of the estimated parameters

- If $W^a_{ll} = \Sigma^{-1}_{ll}$, optimal choice

$$\boxed{\Sigma_{\widehat{\widetilde{x}\widetilde{x}}|\Sigma_{ll}} = N^{-1} = (A^\mathsf{T}\Sigma_{ll}^{-1}A)^{-1}}$$

- If approximate covariance matrix e.g. $W^a_{ll} = I$

$$\Sigma_{\widehat{\widetilde{x}\widetilde{x}}|I_N} = (A^\mathsf{T}A)^{-1}A^\mathsf{T} \cdot \Sigma_{ll} \cdot A(A^\mathsf{T}A)^{-1}$$

→ uncertainty of parameters from LS solution

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Example 1: Mean vs weighted mean

Mean of $a = 0$ and $b = 1$

Standard deviations

$$\sigma_a = \sigma_0/\sqrt{k}$$

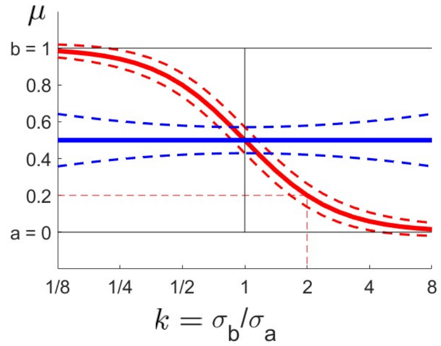
$$\sigma_b = \sigma_0\ \sqrt{k}$$

Blue: arithmetic mean

error band of estimate

Red: weighted mean

error band



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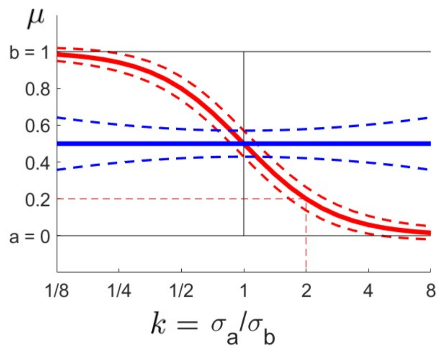
Deviation from weight ratio = 1

Arithmetic mean

- Generally incorrect: same for all weights
- Actual uncertainty increases

Weighted mean

- Drastically changes
- Actual uncertainty decreases



$\sigma_b/\sigma_a = 2 \ \rightarrow\ w_a/w_b = 4 \ \rightarrow\ \mu = 0.2$

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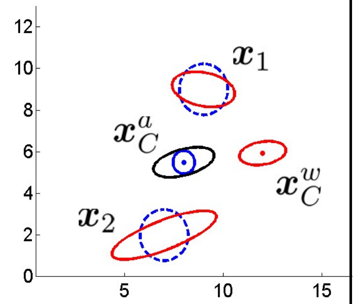
Example 2: Centroid with general model

Given

two points $\{\boldsymbol{x}_i, \Sigma_{ii}\}, i = 1, 2$

- Arithmetic mean $\boldsymbol{x}^a_C$
  - Using $\Sigma_{ii} = \sigma^2 I_2$, mean of sigmas
  - → midpoint on connecting line
- Weighted mean?
  - best point $\boldsymbol{x}^w_C$
  - highly depending on uncertainty
- real CovM of arithmetic mean (black)



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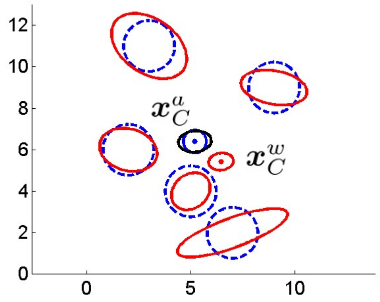
Arithmetic mean vs weighted centroid

For 5 points …

Arithmetic and weighted mean differ (possibly) more than standard deviations



High precision point attract

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5.4.2 The variance factor

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The variance factor

Assumption on uncertainty of input data

$$\mathbb{D}(\underline{\boldsymbol{l}}) = \Sigma_{ll}$$

Estimate independent on scaling of covariance matrix

$$\widehat{\boldsymbol{x}} = \left(A^{\mathsf{T}}(\mu\Sigma_{ll})^{-1}A\right)^{-1} A^{\mathsf{T}}(\mu\Sigma_{ll})^{-1}\boldsymbol{l}$$

If covariance matrix **approximately known** up to **variance factor** σ_0^2, close to 1, unit less

$$\boxed{\Sigma_{ll} = \sigma_0^2 \Sigma_{ll}{}^{a}}$$

Can we estimate the variance factor?

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Estimation of variance factor

Use weighted sum of squared residuals

$$\Omega = \widehat{\boldsymbol{v}}^{\mathsf{T}}(\sigma_0^2\Sigma_{ll}^a)^{-1}\widehat{\boldsymbol{v}}$$

Expectation with **redundancy** $R = N - U$

$$\begin{aligned}
\mathbb{E}(\underline{\Omega}) &= \mathbb{E}(\widehat{\underline{\boldsymbol{v}}}^{\mathsf{T}}(\sigma_0^2\Sigma_{ll}^a)^{-1}\widehat{\underline{\boldsymbol{v}}}) \\
&= \sigma_0^2 \operatorname{tr}(\Sigma_{ll}^a\, \mathbb{E}(\widehat{\underline{\boldsymbol{v}}}\,\widehat{\underline{\boldsymbol{v}}}^{\mathsf{T}})) \\
&\overset{\mathbb{E}(\widehat{\underline{v}}=0)}{=} \sigma_0^2 \underbrace{\operatorname{tr}(\Sigma_{ll}^a\, \Sigma_{\widehat{v}\widehat{v}}^a)}_{R} \quad \text{with} \quad R = N - U \\
&= R\,\sigma_0^2
\end{aligned}$$

See proof on extra slide +2

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Estimated variance factor

Estimate of variance factor

$$\widehat{\sigma}_0^2 = \frac{\Omega(\widehat{\boldsymbol{x}})}{N-U} = \frac{\widehat{\boldsymbol{v}}^{\mathsf{T}} (\Sigma_{ll}^a)^{-1} \widehat{\boldsymbol{v}}}{N-U} = \frac{\widehat{\boldsymbol{v}}^{\mathsf{T}} W_{ll}^a \widehat{\boldsymbol{v}}}{N-U}$$

weighted sum of residuals very useful !

Approximate standard deviations need to be updated

- Observations, uncertainty of the input data

$$\Sigma_{ll} := \widehat{\sigma}_0^2 \, \Sigma_{ll}^a \quad \text{or} \quad \sigma_{l_i} := \widehat{\sigma}_0 \, \sigma_{l_i}^a$$

- Estimated parameters, uncertainty of the output data

$$\Sigma_{\widehat{x}\widehat{x}} := \widehat{\sigma}_0^2 \, \Sigma_{\widehat{x}\widehat{x}}^a \quad \text{or} \quad \sigma_{\widehat{x}_u} := \widehat{\sigma}_0 \, \sigma_{\widehat{x}_u}^a$$

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Proof: Redundancy and estimated residuals

- Estimated/corrected observations

$$\widehat{\underline{\boldsymbol{l}}} \sim \mathcal{N}(A\widehat{\underline{\boldsymbol{x}}}, \Sigma_{\widehat{l}\widehat{l}}) \quad \text{with} \quad \Sigma_{\widehat{l}\widehat{l}} = A\,(AW_{ll}A)^{-1}\,A^{\mathsf{T}}$$

- Idempotent matrix, rank = number U of parameters

$$U = \Sigma_{\widehat{l}\widehat{l}} W_{ll} = A\,(AW_{ll}A)^{-1}\,A^{\mathsf{T}}\,W_{ll} \quad \text{with} \quad U^2 = U$$

$$\lambda_i \in \{0,1\} \quad \rightarrow \quad \mathrm{tr}(U) = \mathrm{rk}(U) = \mathrm{rk}(A) = U$$

- Estimated residuals

$$\widehat{\underline{\boldsymbol{v}}} \sim \mathcal{N}(\boldsymbol{0}, \Sigma_{\widehat{v}\widehat{v}}) \quad \text{with} \quad \Sigma_{\widehat{v}\widehat{v}} = \Sigma_{ll} - \Sigma_{\widehat{l}\widehat{l}}$$

Hence

$$\mathrm{tr}(\Sigma_{\widehat{v}\widehat{v}} W_{ll}) = \mathrm{tr}(I_N - U) = N - U = R$$

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5.4.3 Proving the Validity of a Method

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Proving the validity of a method

The procedure yields

- the estimates
- the covariance matrix
- the estimated variance factor

→

- Why validate?
- Checking the correctness of an estimation procedure

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Why validate a method?

Questions of the programmer/the user of a program

- Is the method implemented as intended?
- Does the method what it should?
- Can the user be sure the method works correctly?

(different flavor of the same question)

How to validate a method?

- theoretical validation (proofs*)
  - practitioners never believe in the usefulness of a theoretically proven method
- empirical validation
  - synthetic data
    practitioners never accept the relevance of the simulated data
  - real data
    theoreticians never trust a method proven with one (or many) example(s)

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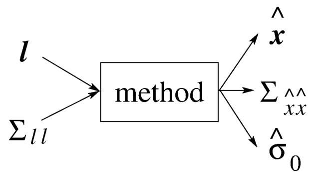
* This is why we study maths … which is never enough

Checking the correctness of a procedure

Estimation method



Three checks

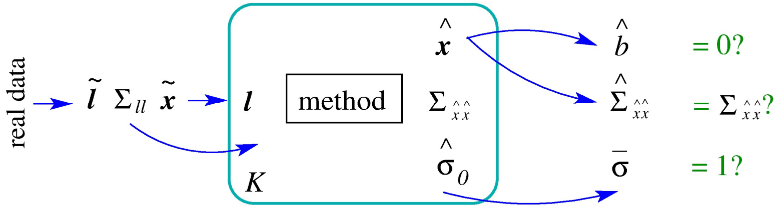
1. Estimated noise level $\widehat{\sigma}_0$ should be 1
2. Theoretical covariance matrix $\Sigma_{\widehat{x}\widehat{x}}$ should be correct
3. Bias of estimates $\boldsymbol{b} = \widehat{\boldsymbol{x}} - \tilde{\boldsymbol{x}}$ should be 0

see Förstner and Wrobel (2016, Sect. 4.7.2)

Remark: covariance matrices

- Theoretical/predicted covariance matrix (internal)
  $$\Sigma_{\widehat{x}\widehat{x}} = N^{-1} = (A^\mathsf{T}\Sigma_{ll}^{-1}A)^{-1}$$
- Empirical covariance matrix
  - From one sample (internal)
    $$\widehat{\Sigma}_{\widehat{x}\widehat{x}} = \widehat{\sigma}_0^2 N^{-1} = \widehat{\sigma}_0^2 \Sigma_{\widehat{x}\widehat{x}}$$
    → standard deviations of $\widehat{x}_u$ increase linearly with σ_l
  - From K samples (external)
    $$\widehat{\Sigma}_{\widehat{x}\widehat{x}} = \mathbb{C}\mathrm{ov}\left(\{\widehat{\boldsymbol{x}}_k\}, k = 1, ..., K\right)$$

Principle of checking



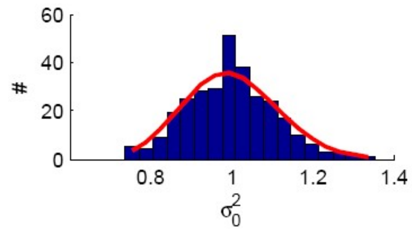
- Take fitted values from real data as true values
- Add noise K times
- Perform estimation
- Check noise level, covariance matrices, and bias

Checking the noise level

Mean of variance factor

$$s^2 = \frac{1}{K}\sum_{k=1}^{K}\widehat{\sigma}^2_{0k} \quad \text{with} \quad \frac{s^2}{\sigma_0^2} \sim F(KR, \infty)$$

Histogram with theoretical density



The reader learns:
I can rely on the estimated noise level

Checking the covariance matrix

Empirical covariance matrix

$$\widehat{\Sigma} = \mathbb{C}\text{ov}(\{\widehat{\boldsymbol{x}}_k\})$$

Test statistic (see Koch (1999), Sect. 2.8.7, 4.1.2)

$$\underline{X}(\widehat{\Sigma}, \Sigma_{\widehat{x}\widehat{x}}, K, U) \sim \chi^2_{U(U+1)/2}$$

e.g.

```
covariance matrix C(^x) ok  *
X(C) = 15.1763 in [6.4467,46.797]
```

→ The reader learns: I can rely on $\Sigma_{\widehat{x}\widehat{x}}$ **

* better: empirical data do not contradict $\Sigma_{\widehat{x}\widehat{x}}$

** better: reader needs not have doubt about $\Sigma_{\widehat{x}\widehat{x}}$

Checking the bias

Empirical bias = difference of estimate and true value

Test statistic:

$$\boldsymbol{m} = \frac{1}{K}\sum_{k=1}^{K}(\widehat{\boldsymbol{x}}_k - \tilde{\boldsymbol{x}}) \quad \text{with} \quad \underline{X} = K\underline{\boldsymbol{m}}^{\mathsf{T}}\Sigma^{-1}_{\widehat{x}\widehat{x}}\underline{\boldsymbol{m}} \sim \chi^2_U$$

e.g.

```
mean of parameters ^x ok:
X(^x) = 5.2384 < T = 22.4577
```

The reader learns:
Method has not shown to introduce systematic errors

Properties of checks

Extremely useful for checking implementations

- Check all three software parts
  - the simulation
  - the estimation
  - the checking
- Sensitive even to errors with small effect
- Checks linearization
- Use very small noise (say 10^{-8}) within simulations
  otherwise: effect of linearization is visible

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5.4.4 The uncertainty
in the centred model of 3D similarity

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Parameters in centred 3D similarity model

- Covariance matrix = inverse normal equation matrix
- Three independent estimates $\{\Delta\boldsymbol{r}, \Delta\lambda, \Delta\boldsymbol{t}\}$

$$N = \begin{bmatrix} \sum_{i=1}^{I} S({}^c\boldsymbol{x}_i'^a) S^{\mathsf{T}}({}^c\boldsymbol{x}_i'^a) w_i & 0_{3\times3} & \mathbf{0} \\ 0_{3\times3} & \sum_{i=1}^{I} w_i I_3 & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}} & \sum_{i=1}^{I} {}^c\boldsymbol{x}_i'^{a\mathsf{T}}\, {}^c\boldsymbol{x}_i'^a w_i \end{bmatrix}$$

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Uncertainty of scale parameter

Estimated scale parameter from

$$\Big(\sum_i \underbrace{{}^c\boldsymbol{x}_i^{a\mathsf{T}}\, {}^c\boldsymbol{x}_i'^a}_{d_i'^2=\lambda^2 d_i^2}\, w_i\Big)\widehat{\Delta\lambda} = \sum_i {}^c\boldsymbol{x}_i{}'^{\mathsf{T}} ({}^c\boldsymbol{x}_i' - {}^c\boldsymbol{x}_i'^a) w_i$$

Weighted squared average of distances d_i' to centroid

$$\overline{d'^2} = \frac{\sum_i d_i'^2 w_i}{\sum_i w_i} \quad \text{or} \quad \overline{d'^2} \overset{w_i=1}{=} \frac{\sum_i d_i'^2}{I}$$

Uncertainty= f(number of points, lever arms)

$$\sigma^2_{\widehat{\lambda}} = \frac{1}{\sum_i w_i}\, \frac{1}{\overline{d'^2}} \quad \text{or} \quad \sigma^2_{\widehat{\lambda}} = \frac{1}{I}\, \frac{1}{\overline{d'^2}}\, \sigma^2_{x'}$$

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Uncertainty of rotation

… from upper left submatrix, assuming $\sigma_0 = 1$

$$\Sigma_{\widehat{r}\widehat{r}}$$

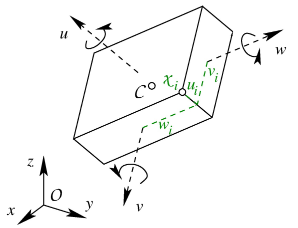
$$= \left(\sum_{i=1}^{I} S({}^c\boldsymbol{x}_i'^a) S^{\mathsf{T}}({}^c\boldsymbol{x}_i'^a) w_i \right)^{-1}$$

$$= \underbrace{\begin{bmatrix} \sum_{i=1}^{I} ({}^cy_i'^2 + {}^cz_i'^2) w_i & \sum_{i=1}^{I} -{}^cx_i'{}^cy_i' w_i & \sum_{i=1}^{I} -{}^cx_i'{}^cz_i' w_i \\ \sum_{i=1}^{I} -{}^cx_i'{}^cy_i' w_i & \sum_{i=1}^{I} ({}^cx_i'^2 + \overline{z}_i'^2) w_i & \sum_{i=1}^{I} -{}^cy_i'{}^cz_i' w_i \\ \sum_{i=1}^{I} -{}^cx_i'{}^cz_i' w_i & \sum_{i=1}^{I} -{}^cy_i'{}^cz_i' w_i & \sum_{i=1}^{I} ({}^cx_i'^2 + {}^cy_i'^2) w_i \end{bmatrix}}_{M}$$

= inverse **moment matrix** of points

Uncertainty of rotation

- Generally correlated
- If coordinate axes =
  principal moment axes
  (u, v, w) of point cloud $\{\boldsymbol{x}_i\}$
  → matrix diagonal



$$\sigma^2_{r_u} = \frac{\sigma^2_{x'}}{\sum_i v_i^2 + w_i^2} \quad \sigma^2_{r_v} = \frac{\sigma^2_{x'}}{\sum_i u_i^2 + w_i^2} \quad \sigma^2_{r_w} = \frac{\sigma^2_{x'}}{\sum_i u_i^2 + v_i^2}$$

Standard deviations proportional to lever arms

Uncertainty of translation

- Estimated translation vector from
$$\left(\sum_i W_{ii}\right) {}^c\widehat{\boldsymbol{t}} = \sum_i W_{ii} {}^c\boldsymbol{x}_i'$$
- Uncertainty
$$\mathbb{D}({}^c\underline{\widehat{\boldsymbol{t}}}) = \Sigma_{{}^c\widehat{t}{}^c\widehat{t}} = \left(\sum_i W_{ii}\right)^{-1}$$
- If $\mathbb{D}(\underline{\boldsymbol{l}}_i) = \sigma_l^2 I_3$, isotropic, homogeneous
$$\sigma_{{}^c\widehat{t}_x} = \sigma_{{}^c\widehat{t}_y} = \sigma_{{}^c\widehat{t}_z} = \frac{1}{\sqrt{I}} I_3$$

Plausible: expected precision of arithmetic mean

Meaning of covariance matrix?

When using centred coordinates with

$$\Sigma_{ll} = \sigma_i^2 I_3 \quad \text{and} \quad \boldsymbol{t}_C = \frac{\sum_i w_i \boldsymbol{t}_i}{\sum_i w_i}$$

Then always

$${}^c\widehat{\boldsymbol{t}}_C = \boldsymbol{0}$$

For **repeated experiments with varying noise**
→ estimated covariance matrix = 0 !

→ contradiction?

Assume repeated experiments

Multiple K experiments, fixed $\boldsymbol{f}(\boldsymbol{x})$ (see above)

$$\boldsymbol{l}_k = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{e}_k, \quad \text{with} \quad \underline{\boldsymbol{e}}_k \sim \mathcal{N}(\boldsymbol{0}, \Sigma_{ll}) \qquad k = 1, ..., K$$

Resulting K estimates $\widehat{\boldsymbol{x}}_k,\ k = 1, ..., K$

Empirical mean and covariance matrix of estimates

$$\widehat{\boldsymbol{m}}_{\widehat{x}} = \frac{1}{K}\sum_{k=1}^{K} \widehat{\boldsymbol{x}}_k \qquad \widehat{\Sigma} = \frac{1}{K-1}\sum_{k=1}^{K}(\widehat{\boldsymbol{x}}_k - \widehat{\boldsymbol{m}}_{\widehat{x}})(\widehat{\boldsymbol{x}}_k - \widehat{\boldsymbol{m}}_{\widehat{x}})^\mathsf{T}$$

Should be consistent

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Interpretations

Observations refer to centroid $\boldsymbol{t}'_C$ of observations

→varying for each repeated experiment

→ enforces ${}^c\widehat{\boldsymbol{t}}_C = \boldsymbol{0}$

If centroid $\boldsymbol{t}'_C$ **fixed for all** K **experiments**

→ estimates $\widehat{\Delta^c \boldsymbol{t}_k}$ vary

→ simulation could be used to check CovM

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5.4.5 The uncertainty in the original model

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Relation to original model

Original and centred model

$$\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = \boldsymbol{t} + \lambda R \boldsymbol{x}_i$$

$$(\boldsymbol{x}'_i - \boldsymbol{x}'_C) + \boldsymbol{v}_{x'_i} = {}^c\boldsymbol{t} + {}^c\lambda\,{}^cR(\boldsymbol{x}_i - \boldsymbol{x}_C)$$

→ only translation different

$$\underline{\boldsymbol{r}} = {}^c\underline{\boldsymbol{r}}, \qquad \widehat{\underline{\boldsymbol{t}}} = {}^c\widehat{\underline{\boldsymbol{t}}} + \boldsymbol{x}'_C - {}^c\widehat{\underline{\lambda}}\,{}^c\widehat{\underline{R}}\,\boldsymbol{x}_C, \qquad \widehat{\underline{\lambda}} = {}^c\widehat{\underline{\lambda}}$$

Variance propagation using the differential relation

$$\begin{bmatrix} \mathrm{d}\widehat{\boldsymbol{r}} \\ \mathrm{d}\widehat{\boldsymbol{t}} \\ \mathrm{d}\widehat{\lambda} \end{bmatrix} = \begin{bmatrix} I_3 & 0 & \boldsymbol{0} \\ S(\widehat{\boldsymbol{x}}'_C) & I_3 & -\widehat{\boldsymbol{x}}'_C \\ \boldsymbol{0}^\mathsf{T} & \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathrm{d}^c\widehat{\boldsymbol{r}} \\ \mathrm{d}^c\boldsymbol{t} \\ \mathrm{d}^c\lambda \end{bmatrix}, \qquad \widehat{\boldsymbol{x}}'_C = {}^c\widehat{\lambda}\,{}^c\widehat{R}\,\boldsymbol{x}_C$$

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Summary

Generally

- Covariance of estimate = inverse normal NE matrix
- Extended model with variance factor

Proving the validity of the estimation method

- Estimated variance factor
- Theoretical/predicted covariance matrix
- Bias of estimate

Evaluation of 3D similarity

… Code will be published

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Next/final lecture

6. Outlier detection

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