Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

12. ML solution for 3D Similarity

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The slides have been created by Wolfgang Förstner.

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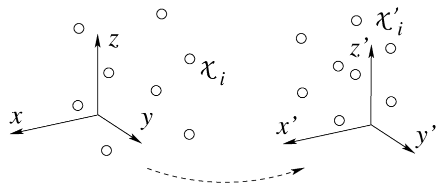
Outline

ML estimation of parameters of similarities

- Iterative solution for nonlinear Gauss-Markov model
- 3D similarity with simplified stochastical model
  for finding CovM of direct solution
- 3D similarity with full covariance matrices per point



For this video see Förstner/Wrobel (2016), p. 102-104

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Nonlinear Gauss-Markov model

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Non-linear Gauss-Markov problem

Model

$$\underline{\boldsymbol{l}} \sim \mathcal{N}(\boldsymbol{f}(\boldsymbol{x}), \Sigma_{ll})$$

Observed $\boldsymbol{l}$

Given $\boldsymbol{f}, \Sigma_{ll}$

Task Minimize w.r.t. $\boldsymbol{x}$

$$\Omega(\boldsymbol{x}) = (\boldsymbol{l} - \boldsymbol{f}(\boldsymbol{x}))^{\mathsf{T}} \Sigma_{ll}^{-1} (\boldsymbol{l} - \boldsymbol{f}(\boldsymbol{x}))$$

- Nonlinear problem → iterative
- Approximate values → linear substitute problem
- Estimates after convergence

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Gauss-Markov model: iterative solution

Principle

1. Provide approximate values $\boldsymbol{x}^a$ for parameters $\boldsymbol{x}$
2. Iteratively improve approximate values $\boldsymbol{x}^a$:
   1. Derive linear substitute problem $\underline{\Delta \boldsymbol{l}} \sim \mathcal{N}(A\Delta\boldsymbol{x}, \Sigma_{ll})$
   2. Solve for corrections, e.g. $\Delta\boldsymbol{x} := (A^{\mathsf{T}}\Sigma_{ll}A)^{-1}A^{\mathsf{T}}\Sigma_{ll}\Delta\boldsymbol{l}$
   3. Update approximate values $\boldsymbol{x}^a := u(\boldsymbol{x}^a, \Delta\boldsymbol{x})$
   4. If corrections too large → iterate, otherwise stop loop
3. Last approximate parameters as estimates $\widehat{\boldsymbol{x}} := \boldsymbol{x}^a$
4. Derive quality measures, e.g. $\mathbb{D}(\widehat{\underline{\boldsymbol{x}}})$

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Comments to iterative solution

- Approximate values sometimes hard to obtain
- Linear substitute problem by Taylor expansion
- Linear substitute problem = linear GM-model

$$\boldsymbol{l} + \boldsymbol{v} \approx \boldsymbol{f}(\boldsymbol{x}^a) + A\,\Delta\boldsymbol{x}$$

$$\overbrace{(\boldsymbol{l} - \boldsymbol{f}(\boldsymbol{x}^a))}^{\Delta\boldsymbol{l}} + \boldsymbol{v} \approx A\,\Delta\boldsymbol{x} \qquad \text{with} \quad A = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{x=x^a}$$

- Updates: usually additive, sometimes multiplicative
- Termination of iteration: use ratios $|\Delta x_u|/\sigma_{x_u}$
- Uncertainty: transfer from linear substitute problem

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Gauss-Markov model of 3D similarity model

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3D similarity

- Approximate values from direct solution
  if no outliers
- Derive substitute problem by Taylor expansion
  use multiplicative corrections for scaled rotation
- Use normal equation system (7x7)
- Update: (translation, scaled rotation)
- Number or iterations: usually 2-3
- Uncertainty by variance propagation

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Model of 3D similarity

- Functional model

$$\boldsymbol{l}_i + \boldsymbol{v}_i = \boldsymbol{f}_i(\boldsymbol{x}) \quad \text{or} \quad \boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = \boldsymbol{t} + \lambda R \boldsymbol{x}_i \quad i = 1, \ldots, I$$

- Approximate values (taking sequence of beginning)

$$\boldsymbol{x}^a \quad \text{or} \quad \{R^a, \boldsymbol{t}^a, \lambda^a\}$$

- Updates

$$\boldsymbol{x}^a := u(\boldsymbol{x}^a, \widehat{\Delta \boldsymbol{x}}) \quad \text{or} \quad \boldsymbol{p}^a = \left\{ \begin{array}{c} R^a \\ \boldsymbol{t}^a \\ \lambda^a \end{array} \right\} := \left\{ \begin{array}{c} \mathrm{e}^{S(\Delta r)} R^a \\ \boldsymbol{t}^a + \widehat{\Delta \boldsymbol{t}} \\ \mathrm{e}^{\widehat{\Delta \lambda}} \lambda^a \end{array} \right\}$$

or

$$\begin{bmatrix} \lambda^a R^a & \boldsymbol{t}^a \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix} := \begin{bmatrix} \mathrm{e}^{\widehat{\Delta \lambda}} \mathrm{e}^{S(\widehat{\Delta \boldsymbol{r}})} & \widehat{\Delta \boldsymbol{t}} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \lambda^a R^a & \boldsymbol{t}^a \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$$

Model of 3D similarity

- Linearized substitute model

$$\Delta \boldsymbol{l}_i + \boldsymbol{v}_i = A_i^\mathsf{T} \Delta \boldsymbol{x} \quad \text{or} \quad \Delta \boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = A_i^\mathsf{T} \Delta \boldsymbol{p}, \quad i = 1, \ldots, I$$

- With

$$\Delta \boldsymbol{x}'_i = \boldsymbol{x}'_i - (\underbrace{\boldsymbol{t}^a + \lambda^a R^a \boldsymbol{x}_i}_{\boldsymbol{x}'^a_i}) \quad \text{and} \quad \Delta \boldsymbol{p} = \begin{bmatrix} \Delta \boldsymbol{r} \\ \Delta \boldsymbol{t} \\ \Delta \lambda \end{bmatrix}$$

- And Jacobian, design matrix, coefficient matrix,…

$$\underbrace{A_i^\mathsf{T}(\boldsymbol{x}'^a_i)}_{3\times 7} = \left.\frac{\partial \boldsymbol{f}_i(\Delta \boldsymbol{p})}{\partial \Delta \boldsymbol{p}}\right|_{p=p^a} = [-S(\boldsymbol{x}'^a_i),\, I_3,\, \boldsymbol{x}'^a_i]$$

Estimation with simplified stochastical model

Isotropic uncertainty model

Assumption

Simplified uncertainty model of observed coordinates

$$\Sigma_{l_i l_i} := \mathbb{D}(\underline{\boldsymbol{x}}'_i) = \Sigma_{x'_i x'_i} = \sigma^2_{x'_i} I_3 = \frac{1}{w_i} I_3$$

→ use centred coordinates

$${}^c\boldsymbol{x}_i = \boldsymbol{x}_i - \boldsymbol{x}_C \quad \text{and} \quad {}^c\boldsymbol{x}'_i = \boldsymbol{x}'_i - \boldsymbol{x}'_C$$

with centroids

$$\boldsymbol{x}_C = \frac{\sum_i w_i \boldsymbol{x}_i}{\sum_i w_i} \quad \text{and} \quad \boldsymbol{x}'_C = \frac{\sum_i w_i \boldsymbol{x}'_i}{\sum_i w_i}$$

Model for centred coordinates

- Nonlinear model → centred nonlinear model

$$(\boldsymbol{x}'_i - \boldsymbol{x}'_C) + \boldsymbol{v}_{x'_i} = {}^c\boldsymbol{t} + \lambda R(\boldsymbol{x}_i - \boldsymbol{x}_C)$$

$${}^c\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = {}^c\boldsymbol{t} + \lambda R\, {}^c\boldsymbol{x}_i$$

- Approximate values, translation = 0

$$(R^a,\, \boldsymbol{0},\, \lambda^a)$$

- Linearized model

$$\Delta^c\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = \left[-S({}^c\boldsymbol{x}'^a_i),\, I_3,\, {}^c\boldsymbol{x}'^a_i\right] \begin{bmatrix} \widehat{\Delta \boldsymbol{r}} \\ \widehat{\Delta^c \boldsymbol{t}} \\ \widehat{\Delta\lambda} \end{bmatrix}$$

$$\Delta^c\boldsymbol{x}'_i = {}^c\boldsymbol{x}'_i - \lambda^a R^a\, {}^c\boldsymbol{x}_i = {}^c\boldsymbol{x}'_i - {}^c\boldsymbol{x}'^a_i$$

Normal equation system

- 7x7 Normal equation matrix

$$N = A^{\mathsf{T}} W_{ll} A := \sum_{i=1}^{I} \begin{bmatrix} S({}^c\boldsymbol{x}'^a_i) \\ I_3 \\ {}^c\boldsymbol{x}'^{a\mathsf{T}}_i \end{bmatrix} \left[-S({}^c\boldsymbol{x}'^a_i),\, I_3,\, {}^c\boldsymbol{x}'^a_i\right] w_i$$

$$= \begin{bmatrix} \sum_{i=1}^{I} S({}^c\boldsymbol{x}'^a_i) S^{\mathsf{T}}({}^c\boldsymbol{x}'^a_i) w_i & 0_{3\times 3} & \boldsymbol{0} \\ 0_{3\times 3} & \sum_{i=1}^{I} w_i I_3 & \boldsymbol{0} \\ \boldsymbol{0}^{\mathsf{T}} & \boldsymbol{0}^{\mathsf{T}} & \sum_{i=1}^{I} {}^c\boldsymbol{x}'^{a\mathsf{T}}_i\, {}^c\boldsymbol{x}'^a_i w_i \end{bmatrix}$$

→ block diagonal matrix, one 3x3 block for rotation

Normal equation system

- 7-vector of right hand sides

$${}^c\boldsymbol{x}'_i = \boldsymbol{x}'_i - \boldsymbol{x}'_C$$

$$\boldsymbol{h} = A^{\mathsf{T}} W_{ll}\, {}^c\Delta \boldsymbol{l} := \sum_{i=1}^{I} \begin{bmatrix} S({}^c\boldsymbol{x}'^a_i) \\ I_3 \\ {}^c\boldsymbol{x}'^{a\mathsf{T}}_i \end{bmatrix} ({}^c\boldsymbol{x}'_i - {}^c\boldsymbol{x}'^a_i)\, w_i$$

$${}^c\boldsymbol{x}'^a_i = \lambda^a R^a\, (\boldsymbol{x}_i - \boldsymbol{x}_C)$$

$$= \begin{bmatrix} \sum_{i=1}^{I} S({}^c\boldsymbol{x}'^a_i)({}^c\boldsymbol{x}'_i - {}^c\boldsymbol{x}'^a_i) w_i \\ \boldsymbol{0} \\ \sum_{i=1}^{I} {}^c\boldsymbol{x}'^{a\mathsf{T}}_i ({}^c\boldsymbol{x}'_i - {}^c\boldsymbol{x}'^a_i) w_i \end{bmatrix}$$

→ estimated translation = 0

Estimated parameters

- Update of scale factor

$$\lambda^a := (1 + \widehat{\Delta\lambda})\lambda^a$$

with

$$\widehat{\Delta\lambda} = \frac{\sum_{i=1}^{I} {}^c\boldsymbol{x}'^{a\mathsf{T}}_i ({}^c\boldsymbol{x}'_i - {}^c\boldsymbol{x}'^a_i) w_i}{\sum_{i=1}^{I} |{}^c\boldsymbol{x}'^a_i|^2\, w_i}$$

Remark: in order to guarantee positivity use $1 + x \approx \mathrm{e}^x$

$$\lambda^a := \mathrm{e}^{\widehat{\Delta\lambda}}\, \lambda^a$$

→ compare to

$$R^a := \mathrm{e}^{\mathsf{S}(\Delta r)}\, R^a$$

Estimated parameters

- Updated rotation matrix

$$R^a := R(\widehat{\Delta r})R^a$$

with parameters from 3x3 system

$$\underbrace{\left(\sum_{i=1}^{I} S({}^c\boldsymbol{x}_i'^a) S^{\mathsf{T}}({}^c\boldsymbol{x}_i'^a) w_i\right)}_{3\times3} \widehat{\Delta r} = \underbrace{\sum_{i=1}^{I} S({}^c\boldsymbol{x}_i'^a)({}^c\boldsymbol{x}_i' - {}^c\boldsymbol{x}_i'^a) w_i}_{3\times1}$$

- Estimated translation parameter

$$\boldsymbol{t}^a := \boldsymbol{x}'_C - \lambda^a R^a \boldsymbol{x}_C$$

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3D similarity: Matlab code (1/3)

```
function A_est = estimate_3d_similarity(X,Xt,R0,l0,niter)
%% xt_i = l * R * x_i + t, i=1,...,I, weights = 1
%  X, Xt     3xI matrices of given and observed points
%  R0, l0    3x3 matrix approximate R, lambda
%  niter     number of iterations
I    = size(X,2);                          % number I
X    = [X ;ones(1,I)];Xt =[Xt;ones(1,I)]; % -> homogeneous
Xmean  = mean(X,2);                        % centroids
Xtmean = mean(Xt,2);
cX   = X  - [  Xmean(1:3);0]*ones(1,I);   % centred data
cXt  = Xt - [ Xtmean(1:3);0]*ones(1,I);
A0   = [l0*R0,[0,0,0]';[0,0,0],1];         % initial A
...
```

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3D similarity: Matlab code (2/3)

```
...
for iter = 1:niter  % ======== begin of iterations ========
  cXt0   = A0 * cX;  dl = Xtq-Xtq0;  % appr. transf. points, corr.
  Norm   = zeros(3); hv = zeros(3,1);% initialize NE(dr), h
  suml_z = 0;  suml_d = 0;           % ... sums for lambda
  for i = 1:I    % ......... begin of NE building .........
    A    = -calc_S(cXt0(1:3,i));     % Jacobian for rotation
    Norm = Norm + A' * A;            % add contribution to N
    hv   = hv + A' * dl(1:3,i);      % ... to h, z, n
    suml_z = suml_z + cXt0(1:3,i)'*(cXt(1:3,i)-cXt0(1:3,i));
    suml_d = suml_d + cXt0(1:3,i)'* cXt0(1:3,i);
  end            %  .......... end of NE building .........
...
```

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3D similarity: Matlab code (3/3)

```
...
% solve normal equations and update
  dr    = Norm\hv;                       % corr. dr to u
  R0    = calc_Rot_r(dr) * R0;           % updated r
  l0    = exp(suml_z/suml_d) * l0;       % updated l
  A0    = [l0*R0,[0,0,0]';[0,0,0],1]; % new transform
end % ============ end of iterations ==============
% provide final results
t_est = Xtmean(1:3)-l0*R0*Xmean(1:3);  % translation
A_est = [l0*R0,t_est;[0,0,0],1];       % similarity
end
```

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Comments

- Add criterium for stopping iterations,
  estimated scale and rotation changes: unitless

$$\max(|\Delta\lambda|, |\Delta r_i|) < T \quad \text{with} \quad T = 10^{-6}$$

- Add output of residuals
- Add output of uncertainty measures

Estimation with full covariance matrices

General 3D similarity

- Normal equation matrix

$$N = A^{\mathsf{T}} W A := \sum_i A_i(\boldsymbol{x}_i'^a)\, W_{ii}\, A^{\mathsf{T}}(\boldsymbol{x}_i'^a)$$

- Right hand side

$$\boldsymbol{h} = A^{\mathsf{T}} W \Delta \boldsymbol{l} := \sum_i A_i(\boldsymbol{x}_i'^a)\, W_{ii}\, \Delta\boldsymbol{x}_i'$$

- Corrections

$$\widehat{\Delta\boldsymbol{x}} := \begin{bmatrix} \widehat{\Delta\boldsymbol{\theta}} \\ \widehat{\Delta^c\boldsymbol{t}} \\ \widehat{\Delta\lambda} \end{bmatrix}$$

General 3D similarity

Using centroids → numerically more stable

- Centroids, choose any, translation generally not 0

$$\boldsymbol{x}_C = \left(\sum_i W_{ii}\right)^{-1} \sum_i W_{ii}\boldsymbol{x}_i \quad \text{and} \quad \boldsymbol{x}_C' = \ldots$$

- Centred coordinates

$$^c\boldsymbol{x}_i = \boldsymbol{x}_i - \boldsymbol{x}_C \quad \text{and} \quad {}^c\boldsymbol{x}_i' = \boldsymbol{x}_i - \boldsymbol{x}_C$$

- In normal equations: replace

$$\begin{aligned} \boldsymbol{x}_i'^a &\mapsto {}^c\boldsymbol{x}_i'^a = \lambda^a R^a\, {}^c\boldsymbol{x}_i \\ \Delta\boldsymbol{x}_i' &\mapsto {}^c\Delta\boldsymbol{x}_i' = {}^c\boldsymbol{x}_i' - {}^c\boldsymbol{x}_i'^a \\ \widehat{\boldsymbol{t}} &\mapsto {}^c\widehat{\boldsymbol{t}} \end{aligned}$$

Updates and undoing centering

- Updates

$$\boldsymbol{x}^a := u(\boldsymbol{x}^a, \widehat{\Delta \boldsymbol{x}}) \qquad \begin{aligned} R^a &:= e^{S(\widehat{\Delta\theta})}\, R^a \\ {}^c\boldsymbol{t}^a &:= {}^c\boldsymbol{t}^a + \widehat{\Delta {}^c\boldsymbol{t}} \\ \lambda^a &:= e^{\widehat{\Delta\lambda}}\, \lambda^a \end{aligned}$$

- Estimates

$${}^c\widehat{\boldsymbol{x}} := \left\{ \begin{array}{c} \widehat{R} \\ {}^c\widehat{\boldsymbol{t}} \\ \widehat{\lambda} \end{array} \right\}$$

- Undo centering for translation

$$\widehat{\boldsymbol{t}} = {}^c\widehat{\boldsymbol{t}} + \boldsymbol{x}'_C - \widehat{\lambda}\widehat{R}\boldsymbol{x}_C$$

Couldn't we have done everything with quaternions?

Model of 3D similarity: quaternion form

Given pairs of pure quaternions (only vector part $\neq 0$)

$$\mathbf{x}_i = (0, \boldsymbol{x}_i) \quad \text{and} \quad \mathbf{x}'_i = (0, \boldsymbol{x}'_i)$$

Unknown parameters

translation: pure quaternion

scaled rotation: unconstrained quaternion

$$\mathbf{t} = (0, \boldsymbol{t}) \qquad \mathbf{q} = (q, \boldsymbol{q}) = \sqrt{\lambda}\, \underbrace{\frac{\mathbf{q}}{|\mathbf{q}|}}_{\mathcal{R}} = \sqrt{\lambda}\, (\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}\, \boldsymbol{r})$$

Model $\boxed{\mathbf{x}'_i = \mathbf{t} + \mathbf{q}\mathbf{x}_i\overline{\mathbf{q}}}$

Model of 3D similarity: matrix form

- **Scaled rotation** matrix

$$Q = \lambda R$$

- Use quaternion for rotation $R_Q(\mathbf{q})$, scale $\lambda = |\mathbf{q}|^2$

$$Q(\mathbf{q}) := |\mathbf{q}|^2 R_Q(\mathbf{q}) = \left[\begin{array}{ccc} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\,(q_1 q_2 - q_0 q_3) & 2\,(q_1 q_3 + q_0 q_2) \\ 2\,(q_2 q_1 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\,(q_2 q_3 - q_0 q_1) \\ 2\,(q_3 q_1 - q_0 q_2) & 2\,(q_3 q_2 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{array} \right]$$

- Nonlinear model

see video 5

$$\boxed{\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = \boldsymbol{t} + Q(\mathbf{q})\boldsymbol{x}_i}$$

→ 2D similarity with scaled rotation $Z(s) = \left[\begin{array}{cc} a & -b \\ b & a \end{array} \right]$

$$\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = \boldsymbol{r} + Z(\mathbf{s})\boldsymbol{x}_i$$

Linearize 3D similarity

- Multiplicative update of quaternions/scaled rotation
  small quaternion s , to be estimated
  $$\mathbf{q} = (\mathbf{1}+\mathrm{s})\mathbf{q}^a \quad \text{or} \quad Q(\mathbf{q}) = Q(\mathbf{1}+\mathrm{s})Q(\mathbf{q}^a)$$
- General scaled rotation (! No normalization)
  $$Q(\mathbf{q}) = ((q^2 - \boldsymbol{q}^\mathsf{T}\boldsymbol{q})I_3 + 2qS_q + 2\boldsymbol{q}\boldsymbol{q}^\mathsf{T})$$
- Small scaled rotation
  $$Q(\mathbf{1}+\mathrm{s}) = ((1+s)^2 - \boldsymbol{s}^\mathsf{T}\boldsymbol{s})I_3 + 2(1+s)S_s + 2\boldsymbol{s}\boldsymbol{s}^\mathsf{T}$$
- → linearized, omit quadratic terms
  $$\approx \quad I_3 + 2sI_3 + 2S_s$$

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Linearized model

- Approximately transformed points
  $$\boldsymbol{x}_i'^a = \boldsymbol{t}^a + Q(\mathbf{q}^a)\,\boldsymbol{x}_i$$
- Linearized observations
  $$\Delta\boldsymbol{x}_i' = \boldsymbol{x}_i' - \boldsymbol{x}_i'^a$$
- Linearized model
  $$\Delta\boldsymbol{x}_i' + \boldsymbol{v}_{x_i'} = \Delta\boldsymbol{t} + (2\Delta sI_3 + 2S_{\Delta s})\boldsymbol{x}_i'^a = A_i^\mathsf{T}\begin{bmatrix}\Delta\boldsymbol{t}\\ \Delta\mathrm{s}\end{bmatrix}$$

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Linearized model

- Linearized model
  $$\Delta\boldsymbol{x}_i' + \boldsymbol{v}_{x_i'} = \Delta\boldsymbol{t} + (2\Delta sI_3 + 2S_{\Delta s})\boldsymbol{x}_i'^a = A_i^\mathsf{T}\begin{bmatrix}\Delta\boldsymbol{t}\\ \Delta\mathrm{s}\end{bmatrix}$$
- Design matrix for translation and scaled rotation
  $$A_i^\mathsf{T} := \left[\left\{I_3 \mid Y^\mathsf{T}(\boldsymbol{x}_i'^a)\right\}\right] \quad \text{with} \quad Y^\mathsf{T}(\boldsymbol{x}) = 2\,[\boldsymbol{x},\ -S(\boldsymbol{x})]$$

→see 2D similarity (linear)
$${}^2A_i^\mathsf{T} := [\{I_2 \mid Z(\boldsymbol{x}_i'^a)\}] \qquad \text{with} \quad Z(\boldsymbol{x}) = [\boldsymbol{x}, \boldsymbol{x}^\perp]$$

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… comparison to Jacobian with $(\lambda, \boldsymbol{\theta})$

- Scale factor, small factor
  $$\lambda = |\mathbf{q}|^2 \quad \text{and} \quad 1 + \mathrm{d}\lambda = |\mathbf{1}+\mathrm{ds}|^2$$
- Relation to small factor $1+\mathrm{d}\lambda$ and rotation vector $\mathrm{d}\boldsymbol{\theta}$
  $$\begin{aligned}\mathbf{1}+\mathrm{ds} &= (\sqrt{1+\mathrm{d}\lambda}, \mathbf{0})(1, \tfrac{1}{2}\mathrm{d}\boldsymbol{\theta}) \approx \\ &\approx (1+\tfrac{1}{2}\mathrm{d}\lambda)(1, \tfrac{1}{2}\mathrm{d}\boldsymbol{\theta}) \\ &\approx (1+\tfrac{1}{2}\mathrm{d}\lambda, \tfrac{1}{2}\mathrm{d}\boldsymbol{\theta})\end{aligned}$$
- Differential quaternion = ½ scale and rotation
  $$\mathrm{ds} = \tfrac{1}{2}(\mathrm{d}\lambda, \mathrm{d}\boldsymbol{\theta})$$

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… comparison to Jacobian with $(\lambda, \boldsymbol{\theta})$

- Linearized model with **scale** / **rotation** parameters

$$\Delta \mathbf{x}'_i + \mathbf{v}_{x'_i} = J_{x'_i \Delta p} \begin{bmatrix} \Delta \boldsymbol{t} \\ \Delta \boldsymbol{s} \end{bmatrix} \quad \text{with} \quad \Delta \boldsymbol{s} = \begin{bmatrix} \Delta \lambda \\ \Delta \boldsymbol{\theta} \end{bmatrix}$$

- Design matrix w.r.t. $(\mathrm{d}\boldsymbol{t}, \mathrm{d}\lambda, \mathrm{d}\boldsymbol{\theta})$

$$J_{x'_i \Delta p} = \begin{bmatrix} I_3 & \boldsymbol{x}'^a_i & -S(\boldsymbol{x}'^a_i) \\ \mathbf{0}^\mathsf{T} & 0 & \mathbf{0}^\mathsf{T} \end{bmatrix}$$

Equivalence to classical scale and rotation

Compare $\underset{4\times 7}{J_{x'\Delta p}} = \left.\dfrac{\partial \mathbf{x}'}{\partial(\Delta \boldsymbol{p})}\right|_{x'=x'^a}$ here $\Delta \boldsymbol{p} = \begin{bmatrix} \Delta \boldsymbol{\theta} \\ \Delta \lambda \\ \Delta \boldsymbol{t} \end{bmatrix}$

$$= \left.\begin{bmatrix} -S(\boldsymbol{x}') & I_3 & \boldsymbol{x}' \\ \mathbf{0}^\mathsf{T} & \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}\right|_{x'=x'^a}$$

$$= \left.\begin{bmatrix} 0 & x'_3 & -x'_2 & 1 & 0 & 0 & x'_1 \\ -x'_3 & 0 & x'_1 & 0 & 1 & 0 & x'_2 \\ x'_2 & -x'_1 & 0 & 0 & 0 & 1 & x'_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right|_{x'=x'^a}$$

from video 7 (! Sorting)

with (nonhomogeneous coordinates)

$$A_i^\mathsf{T} = \left[I_3 \mid Y^\mathsf{T}(\boldsymbol{x}'^a_i) \right] \quad \text{with} \quad Y^\mathsf{T}(\boldsymbol{x}) = [\underset{\lambda}{\boldsymbol{x}} \mid \underset{\theta}{-S(\boldsymbol{x})}]$$

Summary

1. General solution for arbitrary covariance matrix
   anisotropic uncertainty of coordinates
   - use centroids for improving numerical stability
   - Required for testing of residuals, deformations, …

2. Specific solution for isotropic uncertainty
   → Useful for deriving covariance matrix for direct solution

3. Model and linearization using quaternions

Next lecture

5.4 Uncertainty analysis

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