Photogrammetry & Robotics Lab

3D Coordinate Systems

(Bsc Geodesy & Geoinformation)

11. ML solution for 2D similarity

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The slides have been created by Wolfgang Förstner.

Outline

ML estimation of parameters of similarities

- 1D: straight line fitting → principles, notation
- 2D similarity, two options
  (problems are linear in the parameters)

Assumption: Gaussian distribution

→ mean and covariances are sufficient

→ weighted LS with $W = \Sigma^{-1}$

Linear Gauss-Markov model

Mathematical model

$$\mathbb{E}(\underline{\boldsymbol{l}}) = A\boldsymbol{x}, \qquad \mathbb{D}(\underline{\boldsymbol{l}}) = \Sigma_{ll}$$

Optimization function

$$\Omega = (\boldsymbol{l} - A\boldsymbol{x})^{\mathsf{T}} W_{ll} (\boldsymbol{l} - A\boldsymbol{x}) \quad \text{with} \quad W_{ll} = \Sigma_{ll}^{-1}$$

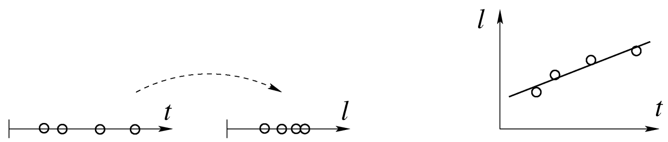
Necessary condition for estimates

$$\boldsymbol{0} = \frac{\partial \Omega(\boldsymbol{x})}{\partial \boldsymbol{x}} = -2(\boldsymbol{l} - A\boldsymbol{x})^{\mathsf{T}} W_{ll} A$$

Normal equations

$$N\widehat{\boldsymbol{x}} = \boldsymbol{h} \quad \text{with} \quad N = A^{\mathsf{T}} W_{ll} A \quad \text{and} \quad \boldsymbol{h} = A^{\mathsf{T}} W_{ll} \boldsymbol{l}$$

Straight line fitting



1D similarity = straight line fitting

Functional model: translation and scale, no rotation😉

$$\mathbb{E}(\underline{l}_i) = r + s t_i,\ i = 1, \dots, I \quad \text{with} \quad \{l_i, t_i, r, s\} \in \mathbb{R}$$

or

$$\mathbb{E}(\underline{\boldsymbol{l}}) = A\boldsymbol{x} \quad \text{with} \quad A = [\boldsymbol{1}, \boldsymbol{t}] \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} r \\ s \end{bmatrix}$$

Stochastical model

$$\mathbb{D}(\underline{\boldsymbol{l}}) = \Sigma_{ll} = \mathrm{Diag}(\{\sigma^2_{l_i}\}) = \mathrm{Diag}(\{1/w_i\}) = W_{ll}^{-1}$$

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Straight line fitting – normal equations

Normal equation matrix

$\propto t_C$

$$N = \begin{bmatrix} \boldsymbol{1}^{\mathsf{T}} \\ \boldsymbol{t}^{\mathsf{T}} \end{bmatrix} W_{ll} [\boldsymbol{1}, \boldsymbol{t}] = \begin{bmatrix} \sum_i w_i & \sum_i t_i\, w_i \\ \sum_i t_i\, w_i & \sum_i t_i^2\, w_i \end{bmatrix}$$

Right hand side

$\propto d^2$

$$\boldsymbol{h} = \begin{bmatrix} \boldsymbol{1}^{\mathsf{T}} \\ \boldsymbol{t}^{\mathsf{T}} \end{bmatrix} W_{ll} \boldsymbol{l} = \begin{bmatrix} \sum_i l_i\, w_i \\ \sum_i t_i\, l_i\, w_i \end{bmatrix} \longleftarrow \propto l_C$$

Choose centroids (t_C, l_C) → centred model

- Normal equation matrix diagonal → simpler equations
- Numerically more stable

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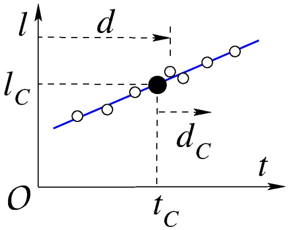
Centroids and average distance

- Centroids

$$t_C = \frac{\sum_i t_i w_i}{\sum_i w_i}$$

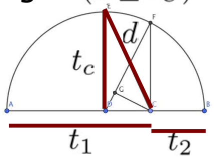
$$l_C = \frac{\sum_i l_i w_i}{\sum_i w_i}$$



- Mean squared distance of t_i to origin $(d \geq t_C)$

$$d = \sqrt{\frac{\sum_i t_i^2 w_i}{\sum_i w_i}}$$

From Wikipedia



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Straight line fitting – centred model

Centroids

$$l_C = \frac{\sum_i l_i w_i}{\sum_i w_i} \quad \text{and} \quad t_C = \frac{\sum_i t_i w_i}{\sum_i w_i}$$

Centred observations and coefficients

$$^c l_i = l_i - l_C \quad \text{and} \quad ^c t_i = t_i - t_C$$

Functional model

$$\mathbb{E}(\underline{l}_i - l_C) = {^c r} + s(t_i - t_C)$$

General relation, we expect $^c r$ is small

$$r = {^c r} + (l_C - s t_C)$$

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Straight line fitting – normal eqautions

Normal equation matrix

$$N = \begin{bmatrix} \mathbf{1}^\mathsf{T} \\ {}^c\boldsymbol{t}^\mathsf{T} \end{bmatrix} W_{ll} [\mathbf{1},\, {}^c\boldsymbol{t}] = \begin{bmatrix} \sum_i w_i & 0 \\ 0 & \sum_i {}^ct_i^2 w_i \end{bmatrix}$$

Right hand side

$$\boldsymbol{h} = \begin{bmatrix} \mathbf{1}^\mathsf{T} \\ {}^c\boldsymbol{t}^\mathsf{T} \end{bmatrix} W_{ll}\, {}^c\boldsymbol{l} = \begin{bmatrix} 0 \\ \sum_i {}^ct_i\, {}^cl_i\, w_i \end{bmatrix}$$

Estimates, see direct solution for similarity

$$\begin{bmatrix} {}^c\widehat{r} \\ \widehat{s} \end{bmatrix} = \begin{bmatrix} 0 \\ \sum_i {}^ct_i\, {}^cl_i\, w_i / \sum_i {}^ct_i^2 w_i \end{bmatrix}$$

Solution for straight line fitting

Given: weighted point pairs $\{l_i, t_i, w_i\}, i = 1, \ldots, I$

Centroids

$$t_C = \sum_i t_i w_i / \sum_i w_i \quad \text{and} \quad l_C = \sum_i l_i w_i / \sum_i w_i$$

Centred coordinates

$${}^ct_i = t_i - t_C \quad \text{and} \quad {}^cl_i = l_i - l_C$$

Estimated parameters

$$\widehat{s} \;=\; \sum_i {}^ct_i\, {}^cl_i\, w_i / \sum_i {}^ct_i^2\, w_i \qquad \widehat{r} \;=\; l_C - s\, t_C$$

→ Optimal line passes through centroids

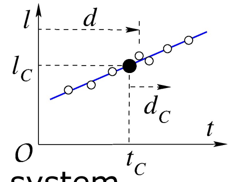
Numerical stability

- Condition number $\kappa \geq 1$ of N should be small
  $\kappa = \lambda_{\max} / \lambda_{\min}$
- with root mean square deviation d of t_i to **origin**

$$N = \sum_i w_i \begin{bmatrix} 1 & t_C \\ t_C & d^2 \end{bmatrix} \qquad \kappa = \frac{1 + d^2 + \sqrt{(1-d^2)^2 + (2t_C)^2}}{1 + d^2 - \sqrt{(1-d^2)^2 + (2t_C)^2}}$$

1. For given size d centroid should be $t_C = 0$ →
   $\kappa = \max(1, d^2) \,/\, \min(1, d^2)$
2. For $t_C = 0$ the distance d_C of the ${}^ct_i$ should be $d_C = 1$

Conditioning of data

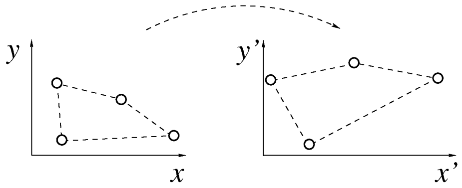


For numerically stable computation

1. Condition data = change coordinate system
   - Translation of data to centroid → $t_C = 0,\ (l_C = 0)$
   - Scaling of distance to centroid → $d_C = 1$
2. Perform computation
3. Undo conditioning after computation

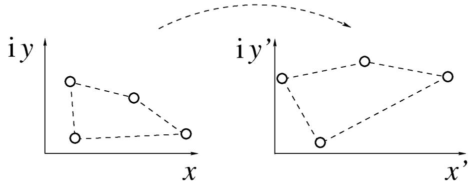
independent of application !

2D similarity



Reinterprete straight line fitting

Take all variables as complex numbers



Addition = translation

Multiplication = scaled rotation !

Reinterprete straight line fitting

All variables are complex numbers

- Observations $l_i := x'_i + \mathrm{i}y'_i$
- Given coordinates $t_i := x_i + \mathrm{i}y_i$
- Translation $r := t_x + \mathrm{i}t_y$
- Scaled rotation $s := \lambda \mathrm{e}^{\mathrm{i}\phi} = \lambda \cos\phi + \mathrm{i}\lambda \sin\phi = a + ib$
- Isotropic uncertainty

Model

$$l_i + v_i = r + st_i, \quad w_i$$

Use result of straight line fitting

2D similarity = complex straight line fitting

Functional model

$$\mathbb{E}(\underline{l}_i) = r + st_i,\ i = 1, \ldots, I \quad \text{with} \quad \{l_i, t_i, r, s\} \in \mathbb{C}$$

or

$$\mathbb{E}(\underline{\boldsymbol{l}}) = A\boldsymbol{x} \quad \text{with} \quad A = [\mathbf{1}, \boldsymbol{t}] \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} r \\ s \end{bmatrix}$$

Stochastical model

$$\mathbb{D}\left(\begin{bmatrix} \Re(l_i) \\ \Im(l_i) \end{bmatrix}\right) = \mathbb{D}\left(\begin{bmatrix} x'_i \\ y'_i \end{bmatrix}\right) = \sigma^2_{l_i} I_2 \quad \text{and} \quad \mathbb{V}(\underline{l}_i) = 2\sigma^2_{l_i}$$

$$\mathbb{D}(\underline{\boldsymbol{l}}) = \Sigma_{ll} = \mathrm{Diag}(\mathbb{V}(\underline{l}_i)) = \mathrm{Diag}(\{2\sigma^2_{l_i}\}) = \mathrm{Diag}(\{1/w_i\})$$

Comparison complex numbers vs vectors

Isomorphy https://en.wikipedia.org/wiki/Complex_number#Formal_construction

- Entities

$$\boldsymbol{s} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \leftrightarrow \quad s = a + \mathrm{i}b$$

- Multiplication

$$Z(\boldsymbol{s})\boldsymbol{t} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - by \\ ay + bx \end{bmatrix} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = Z(\boldsymbol{t})\boldsymbol{s}$$

$$s\,t = (a + \mathrm{i}b)(x + \mathrm{i}y) = (ax - by) + \mathrm{i}(ay + bx) = t\,s$$

or (twice the number of operations)

$$Z(\boldsymbol{s})Z(\boldsymbol{t}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} ax - by & -(ay + bx) \\ ay + bx & ax - by \end{bmatrix}$$

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Comparison

- Transposition corresponds to conjugation

$$Z^{\mathsf{T}}(\boldsymbol{s}) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \leftrightarrow \quad \overline{s} = \overline{a + \mathrm{i}b} = a - \mathrm{i}b$$

- Transposition of complex matrices includes conjugation (C. **H**ermite, 1822 – 1901)

$$[Z(\boldsymbol{n}_{ij})]^{\mathsf{T}} = [Z^{\mathsf{T}}(\boldsymbol{n}_{ji})] \quad \leftrightarrow \quad N^{\mathsf{H}} = [n_{ij}]^{\mathsf{H}} = [\overline{n}_{ji}] = \overline{N}^{\mathsf{T}}$$

- Norm

$$Z^{\mathsf{T}}(\boldsymbol{s})Z(\boldsymbol{s}) = (a^2 + b^2)I_2 \quad \leftrightarrow \quad |s|^2 = \overline{s}s = (a - \mathrm{i}b)(a + \mathrm{i}b) = a^2 + b^2$$

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Complex straight line – normal equations

Normal equation matrix

$$N = \begin{bmatrix} \mathbf{1}^{\mathsf{H}} \\ \boldsymbol{t}^{\mathsf{H}} \end{bmatrix} W_{ll}[\mathbf{1},\, \boldsymbol{t}] = \begin{bmatrix} \sum_i w_i & \sum_i t_i\, w_i \\ \sum_i \overline{t}_i\, w_i & \sum_i \overline{t}_i\, t_i\, w_i \end{bmatrix}$$

Right hand side

$$\boldsymbol{h} = \begin{bmatrix} \mathbf{1}^{\mathsf{H}} \\ \boldsymbol{t}^{\mathsf{H}} \end{bmatrix} W_{ll}\boldsymbol{l} = \begin{bmatrix} \sum_i l_i\, w_i \\ \sum_i \overline{t}_i\, l_i\, w_i \end{bmatrix}$$

Choose centroids → centred model

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Solution for 2D similarity

Given: weighted point pairs $\{x_i, y_i, x'_i, y'_i, w_i\}, i = 1, \ldots, I$

$$t_i := x_i + \mathrm{i}y_i \quad \text{and} \quad l_i := x'_i + \mathrm{i}y'_i, \quad i = 1, \ldots, I$$

Centroids

$$t_C = \sum_i t_i w_i / \sum_i w_i \quad \text{and} \quad l_C = \sum_i l_i w_i / \sum_i w_i$$

Centred coordinates

$${}^c t_i = t_i - t_C \quad \text{and} \quad {}^c l_i = l_i - l_C, \quad i = 1, \ldots, I$$

Estimated parameters

$$\widehat{a} + \mathrm{i}\widehat{b} := \widehat{s} = \sum_i {}^c\overline{t}_i\, {}^c l_i\, w_i / \sum_i {}^c\overline{t}_i\, {}^c t_i\, w_i$$

$$\widehat{t}_x + \mathrm{i}\widehat{t}_y := \widehat{r} = l_C - s\, t_C$$

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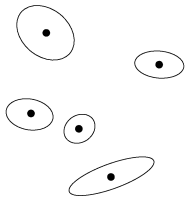
Matlab Code for linear regression/similarity

```
function est_x = estimate_linear_regression_d(l,t,d,sigma_s)
% l_i + v_i = r + s * t_i, dim=d, sigma_i -> [s;r]
w     = 1./(d*sigma_s.^2);                       % weights
sum_w = sum(w);                                  % sum of weights
tC    = sum(t.*w)/sum_w; lC=sum(l.*w)/sum_w; % centroids
ct    = t-tC;            cl=l-lC;                % centred data
N22   = sum(conj(ct).*ct.*w);                    % NE(2,2)
s     = sum(conj(ct).*cl.*w)/N22;                % estimated s
r     = lC-s*tC;                                 % estimated r
est_x = [s;r];                                   % column vector
end
```


General 2D similarity with real numbers

- 2-vectors instead of complex numbers
- Allow for arbitrary covariance matrix per point
  (inhomogeneous, anisotropic)



General 2D similarity with real numbers

- Observations $\boldsymbol{l}_i := [x'_i,\, y'_i]^\mathsf{T}$
- Given coordinates $\boldsymbol{t}_i := [x_i,\, y_i]^\mathsf{T}$
- Translation $\boldsymbol{r} := [t_x, t_y]^\mathsf{T}$
- Scaled rotation $\boldsymbol{s} := [\lambda\cos\phi,\, \lambda\sin\phi]^\mathsf{T} = [a,\, b]^\mathsf{T}$
- Isotropic uncertainty $W_{ii} = \Sigma_{ii}^{-1} := W_{l_i l_i}$

$$\boldsymbol{l}_i \quad + \quad \boldsymbol{v}_i \quad = \quad \boldsymbol{r} \quad + \quad Z(\boldsymbol{s}) \qquad \boldsymbol{t}_i \quad , \quad W_{ii}$$

Scaled rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix}_i + \begin{bmatrix} v'_x \\ v'_y \end{bmatrix}_i = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_i , \quad W_{ii}$$

General 2D similarity with real numbers

Model for I observations $\boldsymbol{l}_i := [x'_i,\, y'_i]^\mathsf{T}$ etc.

$$\boldsymbol{l}_i + \boldsymbol{v}_i = \boldsymbol{r} + Z(\boldsymbol{s})\boldsymbol{t}_i \quad \text{with} \quad Z(\boldsymbol{s})\boldsymbol{t}_i = Z(\boldsymbol{t}_i)\boldsymbol{s}$$

Compactly

$$\boldsymbol{l} + \boldsymbol{v} = A\boldsymbol{x} \quad \text{with} \quad W = \mathrm{Diag}(\{W_{ii}\})$$

with

$$\boldsymbol{l} := [\boldsymbol{l}_i], \quad \boldsymbol{v} := [\boldsymbol{v}_i], \quad A := [\{I_2,\, Z(\boldsymbol{t}_i)\}], \quad \boldsymbol{x} := \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{s} \end{bmatrix}$$

Solution

$$\widehat{\boldsymbol{x}} = (A^\mathsf{T} W A)^{-1} A^\mathsf{T} W \boldsymbol{l}$$

General 2D similarity: ML estimation

- Normal equation matrix

$$N = A^{\mathsf{T}} W A = \begin{bmatrix} \sum_i W_{ii} & \sum_i W_{ii} Z(\boldsymbol{t}_i) \\ \sum_i Z^{\mathsf{T}}(\boldsymbol{t}_i) W_{ii} & \sum_i Z^{\mathsf{T}}(\boldsymbol{t}_i) W_{ii} Z(\boldsymbol{t}_i) \end{bmatrix}$$

- Right hand side

$$\boldsymbol{h} = A^{\mathsf{T}} W \boldsymbol{l} = \begin{bmatrix} \sum_i W_{ii} \boldsymbol{l}_i \\ \sum_i Z^{\mathsf{T}}(\boldsymbol{t}_i) W_{ii} \boldsymbol{l}_i \end{bmatrix}$$

- Using centroids → numerically more stable
- Arbitrary centroids: small translation ${}^c\widehat{\boldsymbol{r}} \neq \boldsymbol{0}$ → sl. 8

$$\widehat{\boldsymbol{r}} = {}^c\widehat{\boldsymbol{r}} + \boldsymbol{l}_C - Z(\boldsymbol{t}_C)\boldsymbol{s}$$

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Centroid?

- If $\sum_i W_{ii} Z(\boldsymbol{t}_i) = \mathit{0}$ and $\sum_i W_{ii} Z(\boldsymbol{l}_i) = 0$ then
  - Block diagonal normal equation matrix
  - Estimated translation $\widehat{\boldsymbol{r}} = 0$
- Do centroids exist generally

  such that normal equation matrix is block diagonal?

No!

- Four constraints $\sum_i W_{ii} Z(\boldsymbol{t}_i - \boldsymbol{t}_C) = \mathit{0}$ for $\boldsymbol{t}_C$
- Only solvable if all W_{ii} have the structure of $Z(\boldsymbol{\cdot})$

→ Since W_{ii} is symmetric it needs to be $W_{ii} = c_i I_2$

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2D similarity with *isotropic* uncertainty

Given: point pairs $\{x_i, y_i, x'_i, y'_i, \sigma_i\}, i = 1, \ldots, I$

$$\boldsymbol{t}_i := \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad \boldsymbol{l}_i := \begin{bmatrix} x'_i \\ y'_i \end{bmatrix}, \quad w_i = \frac{1}{\sigma_i^2}, \quad i = 1, \ldots, I$$

Centroids

$$\boldsymbol{t}_C = \left(\sum_i w_i\right)^{-1} \sum_i w_i \boldsymbol{t}_i \quad \text{and} \quad \boldsymbol{l}_C = \left(\sum_i w_i\right)^{-1} \sum_i w_i \boldsymbol{l}_i$$

Centred coordinates

$${}^c\boldsymbol{t}_i = \boldsymbol{t}_i - \boldsymbol{t}_C \quad \text{and} \quad {}^c\boldsymbol{l}_i = \boldsymbol{l}_i - \boldsymbol{l}_C, \quad i = 1, \ldots, I$$

Estimated parameters

$$\widehat{\boldsymbol{s}} = \sum_i Z^{\mathsf{T}}({}^c\boldsymbol{t}_i)\, {}^c\boldsymbol{l}_i w_i \Big/ \sum_i |{}^c\boldsymbol{t}_i|^2\, w_i \quad \text{and} \quad \widehat{\boldsymbol{r}} = \boldsymbol{l}_C - Z(\widehat{\boldsymbol{s}})\, \boldsymbol{t}_C$$

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Matlab code 2D similarity, isotropic

```
function est_x =estimate_2D_similarity(lv,tv,sigma_s)
% l_i + v_i = r + s * t_i, wi = 1/sigma_i^2 -> [a,b,tx,ty]
I     = length(sigma_s);                       % number of data
w     = 1./sigma_s.^2;                         % weights
sum_w = sum(w);                                % sum of weights
tC    = sum(tv.*(w*[1,1]))/sum_w;              % centroids
lC    = sum(lv.*(w*[1,1]))/sum_w;
ct    = tv-repmat(tC,I,1); cl=lv-repmat(lC,I,1); % centred data
N22   = sum((ct(:,1).^2+ct(:,2).^2).*w);       % NE element (2,2)
s     = sum([ct(:,1).*cl(:,1)+ct(:,2).*cl(:,2), ...
              -ct(:,2).*cl(:,1)+ct(:,1).*cl(:,2)] ...
             .*(w*[1,1]))'/N22;                % est a, b
t     = lC'-[s(1) -s(2);s(2) s(1)]*tC';        % est tx, ty
est_x = [s;t];                                 % estimated x
end
```

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Numerical comparison

- Since same mathematical model
  → results are identical
- Computing time (Matlab realization)
  CPU(real)/CPU(complex) appr. 2

Lessons learned

- 2D similarity **=** line fitting with complex numbers (slightly faster)
- Centroid coordinates → **numerically more stable**
- **Conditioning of data** (translation, scale)
- Centroid coordinates **generally** not simplifying
- Simplified stochastical model $\Sigma_{ii} = \sigma^2_{x'_i} I_2$
  → allows to use centred coordinates
  → simplifies estimation
- **Evaluation of estimates later** …

Next video

5.3.2 Estimating parameters of 3D similarity

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