Photogrammetry & Robotics Lab

3D Coordinate Systems

(Bsc Geodesy & Geoinformation)

10. Direct LS Solution for the 3D Similarity

Wolfgang Förstner

The slides have been created by Wolfgang Förstner.

Direct solution

- No approximate values necessary
- Fixed computing time for given size of input data
- Allows for
  - Solving linear equation system
  - Finding roots of polynomial
  - Eigenvalues, eigenvectors
  - Singular value decomposition
  - Finding minimum of convex optimization problem*

*S. Boyd and L. Vandenberghe (2004): Convex Optimization

Direct solution for 3D Similarity

- Simplified stochastical model
- Direct solution
  - Optimal if model holds
  - Good approximation if no outliers
- Solution in 2D and 3D
  - Translation and scale estimation identical
  - Rotation estimation differs
- Uncertainty:
  →maximum likelihood estimation same stochastical model

For this video see Förstner/Wrobel (2016), 408-411

Goal

- **Given:** pairs of corresponding points
  $$(\boldsymbol{x}_i, \boldsymbol{x}'_i), \quad i = 1, \ldots, I \quad \text{and} \quad \mathbb{D}(\underline{\boldsymbol{x}}'_i) = \sigma^2_{x'_i} I_3$$
- **Assumption:** related by a similarity
  $$\mathbb{E}(\underline{\boldsymbol{x}}'_i) = \lambda R\, \boldsymbol{x}_i + \boldsymbol{t}, \quad i = 1, \ldots, I$$
- **Task:** Estimate the parameters, provide uncertainty

- If only $\boldsymbol{x}_i$ are noisy → invert model
- If both, $\boldsymbol{x}_i$ and $\boldsymbol{x}'_i$ are noisy → more involving (Gauss-Helmert model for estimation)

Reformulation of model

Refer to centroid $\boldsymbol{x}'_C$ of observed points

$$(\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i}) - \boldsymbol{x}'_C = \lambda R(\boldsymbol{x}_i - \boldsymbol{u}) \qquad w_i = \frac{1}{\sigma^2_{x'_i}}\,,\ i = 1, \ldots, I$$

Centroid known, fixed

$$\boldsymbol{x}'_C = \frac{\sum_{i=1}^{I} \boldsymbol{x}'_i w_i}{\sum_{i=1}^{I} w_i}$$

Unknown:

Rotation matrix R, scale parameter λ

modified translation vector $\boldsymbol{u}$, corrections $\boldsymbol{v}_{x'_i}$

Reformulation of model

Relation between original model

$$\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i} = \lambda R\,\boldsymbol{x}_i + \boldsymbol{t}\,, \quad i = 1, \ldots, I$$

and centroid model

$$(\boldsymbol{x}'_i + \boldsymbol{v}_{x'_i}) - \boldsymbol{x}'_C = \lambda R\,(\boldsymbol{x}_i - \boldsymbol{u})\,, \quad i = 1, \ldots, I$$

Rotation and scale parameter identical

Translation parameters

$$\boldsymbol{t} = \boldsymbol{x}'_C - \lambda R \boldsymbol{u} \qquad (1)$$

Optimization function

Minimze

$$\Omega(R, \boldsymbol{u}, \lambda) = \sum_i \boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{v}_i w_i$$

$$= \sum_i w_i (v^2_{x_i} + v^2_{y_i} + v^2_{z_i})$$

$$= \sum_i [\lambda R(\boldsymbol{x}_i - \boldsymbol{u}) - (\boldsymbol{x}'_i - \boldsymbol{x}'_C)]^{\mathsf{T}} [\lambda R(\boldsymbol{x}_i - \boldsymbol{u}) - (\boldsymbol{x}'_i - \boldsymbol{x}'_C)] w_i$$

Optimization function to be minimized

Rewrite $\sum_i [\lambda R(\boldsymbol{x}_i - \boldsymbol{u}) - (\boldsymbol{x}'_i - \boldsymbol{x}'_C)]^{\mathsf{T}} [\lambda R(\boldsymbol{x}_i - \boldsymbol{u}) - (\boldsymbol{x}'_i - \boldsymbol{x}'_C)] w_i$

$$= \sum_i \lambda^2 (\boldsymbol{x}_i - \boldsymbol{u})^{\mathsf{T}} \underbrace{R^{\mathsf{T}} R}_{=I} (\boldsymbol{x}_i - \boldsymbol{u}) w_i$$

$$- \; 2\lambda \sum_i (\boldsymbol{x}'_i - \boldsymbol{x}'_C)^{\mathsf{T}} R (\boldsymbol{x}_i - \boldsymbol{u}) w_i \qquad (1)$$

$$+ \; \left.\sum_i (\boldsymbol{x}'_i - \boldsymbol{x}'_C)^{\mathsf{T}} (\boldsymbol{x}'_i - \boldsymbol{x}'_C) w_i \quad \right\} = \text{const.}$$

Partial derivatives w.r.t. parameters need to be zero

Estimating the translation

Necessary condition

$$\frac{\partial \Omega}{\partial \boldsymbol{u}} = 2\sum_i \lambda^2(\boldsymbol{x}_i - \boldsymbol{u})w_i - 2\sum_i \lambda R^{\mathsf{T}}(\boldsymbol{x}'_i - \boldsymbol{x}'_C)w_i$$
$$= 2\lambda^2 \sum_i (\boldsymbol{x}_i - \boldsymbol{u})w_i - 2\lambda R^{\mathsf{T}} \sum_i (\boldsymbol{x}'_i - \boldsymbol{x}'_C)w_i = \boldsymbol{0}$$

Since the centroid fulfils $\sum(\boldsymbol{x}'_i - \boldsymbol{x}'_C)w_i = 0$ →

Estimating the translation

Translation is

$$\boldsymbol{u} = \boldsymbol{x}_C = \frac{\sum_{i=1}^I \boldsymbol{x}_i w_i}{\sum_{i=1}^I w_i}$$

the centroid $\boldsymbol{x}_C$ of the given coordinates $\{\boldsymbol{x}_i\}$

- Determinable without knowing the numbering
- Centroids fulfill similarity, see (1) on slide 6

$$\boldsymbol{x}'_C = \lambda R \boldsymbol{x}_C + \boldsymbol{t}$$

Estimating the scale parameter

Necessary condition

$$\frac{\partial \Omega}{\partial \lambda} = -2\sum(\boldsymbol{x}'_i - \boldsymbol{x}'_C)^{\mathsf{T}} R(\boldsymbol{x}_i - \boldsymbol{x}_C)w_i$$
$$+ \; 2\lambda \sum(\boldsymbol{x}_i - \boldsymbol{x}_C)^{\mathsf{T}}(\boldsymbol{x}_i - \boldsymbol{x}_C)w_i = 0\,.$$

Estimated scale parameter

$$\lambda = \frac{\sum(\boldsymbol{x}'_i - \boldsymbol{x}'_C)^{\mathsf{T}} R(\boldsymbol{x}_i - \boldsymbol{x}_C)w_i}{\sum(\boldsymbol{x}_i - \boldsymbol{x}_C)^{\mathsf{T}}(\boldsymbol{x}_i - \boldsymbol{x}_S)w_i}$$

☹ depending on rotation

Estimating the scale parameter

If noise is small (say below 1%)

$$R(\boldsymbol{x}_i - \boldsymbol{x}_C) \approx \lambda^{-1}(\boldsymbol{x}'_i - \boldsymbol{x}'_C)$$

→Symmetric solution

$$\lambda^2 \approx \frac{\sum(\boldsymbol{x}'_i - \boldsymbol{x}'_C)^{\mathsf{T}}(\boldsymbol{x}'_i - \boldsymbol{x}'_C)w_i}{\sum(\boldsymbol{x}_i - \boldsymbol{x}_C)^{\mathsf{T}}(\boldsymbol{x}_i - \boldsymbol{x}_C)w_i}$$

= ratio of
sum of weighted squares of distances to centroids

Estimation of rotation

We use centred coordinates

$$ {}^c\boldsymbol{x}'_i = \boldsymbol{x}'_i - \boldsymbol{x}'_C \qquad \text{and} \quad {}^c\boldsymbol{x}_i = \boldsymbol{x}_i - \boldsymbol{x}_C $$

From (1) on slide 8, find matrix R **maximize**

$$ \Phi'(R) = \sum_{i=1}^{I} {}^c\boldsymbol{x}_i{}'^{\mathsf{T}}\, R\, {}^c\boldsymbol{x}_i\, w_i $$

under the constraint

$$ R^{\mathsf{T}} R = I_d $$

13

Estimation of rotation

We rewrite optimization function $\Phi'(R)$
using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, specifically

$$ \boldsymbol{a}^{\mathsf{T}} B \boldsymbol{c} = \mathrm{tr}(\boldsymbol{a}^{\mathsf{T}} B \boldsymbol{c}) = \mathrm{tr}(\boldsymbol{c}\boldsymbol{a}^{\mathsf{T}} B) $$

Here

$$ {}^c\boldsymbol{x}_i{}'^{\mathsf{T}} R\, {}^c\boldsymbol{x}_i w_i = \mathrm{tr}\left({}^c\boldsymbol{x}_i{}'^{\mathsf{T}} R\, {}^c\boldsymbol{x}_i w_i\right) = \mathrm{tr}\left({}^c\boldsymbol{x}_i\, {}^c\boldsymbol{x}_i{}'^{\mathsf{T}} R w_i\right) $$

therefore

$$ \sum_{i=1}^{I} {}^c\boldsymbol{x}_i{}'^{\mathsf{T}} R\, {}^c\boldsymbol{x}_i w_i = \mathrm{tr} \sum_{i=1}^{I} {}^c\boldsymbol{x}_i\, {}^c\boldsymbol{x}_i{}'^{\mathsf{T}} R w_i = \mathrm{tr}\left[\underbrace{\left(\sum_{i=1}^{I} {}^c\boldsymbol{x}_i\, {}^c\boldsymbol{x}_i{}'^{\mathsf{T}}\, w_i \right)}_{H} R \right] $$

14

Estimation of rotation

With asymmetric $d \times d$ matrix

$$ \underset{d\times d}{H} = \sum_{i=1}^{I} {}^c\boldsymbol{x}_i\, {}^c\boldsymbol{x}_i{}'^{\mathsf{T}} w_i = X W X'^{\mathsf{T}} $$

with

$$ \underset{d\times I}{X} = [{}^c\boldsymbol{x}_i], \quad \underset{d\times I}{X'} = [{}^c\boldsymbol{x}'_i], \quad \underset{I\times I}{W} = \mathrm{Diag}(\{w_i\}) $$

the solution is

$$ \widehat{R} = \mathrm{argmax}_R\, \mathrm{tr}(HR) \quad \text{under} \quad R^{\mathsf{T}} R = I_d $$

→ different solutions in 2D and 3D

15

Estimation of rotation in 2D

To find the rotation angle φ we need to maximize

$$ \begin{aligned} \mathrm{tr}(HR) &= \mathrm{tr}\left(\begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \right) \\ &= [\cos\varphi, \sin\varphi] \begin{bmatrix} H_{xx} + H_{yy} \\ H_{xy} - H_{yx} \end{bmatrix} =: \boldsymbol{r} \cdot \boldsymbol{h} \end{aligned} $$

With

$$ H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} \sum_i w_i {}^c x_i {}^c x'_i & \sum_i w_i {}^c x_i {}^c y'_i \\ \sum_i w_i {}^c y_i {}^c x'_i & \sum_i w_i {}^c y_i {}^c y'_i \end{bmatrix} $$

→ The vector $\boldsymbol{r}$ needs to be parallel to the vector $\boldsymbol{h}$

16

Estimation of rotation in 2D

With

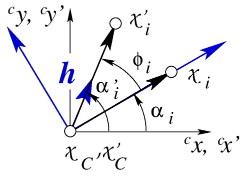
$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \sum_{i=1}^{I} \begin{bmatrix} ({}^c x_i {}^c x'_i + {}^c y_i {}^c y'_i) w_i \\ ({}^c x_i {}^c y'_i - {}^c y_i {}^c x'_i) w_i \end{bmatrix}$$

the rotation angle φ can directly be determined by

$$\boxed{\widehat{\varphi} = \operatorname{atan2}(h_2, h_1)}$$

Weighted mean $\boldsymbol{h}$ of direction vectors

$$\boldsymbol{h}_i = \begin{bmatrix} \cos\varphi_i \\ \sin\varphi_i \end{bmatrix} = \begin{bmatrix} \cos(\alpha'_i - \alpha_i) \\ \sin(\alpha'_i - \alpha_i) \end{bmatrix}$$



17

Estimation of rotation in 3D

Several solutions, see Eggert et al. (1997)

- With singular value decomposition
- With quaternions
- With orthonormal matrices
- With dual quaternions

Here: with singular value decomposition

18

Estimation of rotation in 3D

Solution with SVD, see Arun et al. (1987)

1. Perform SVD

   $H = USV^{\mathsf{T}} \qquad UU^{\mathsf{T}} = I \qquad VV^{\mathsf{T}} = I \qquad S = \operatorname{Diag}(s_i)$

2. If mapping is known to be a proper rotation

   $$V := V\,\operatorname{Diag}([1,1,s]) \quad \text{with} \quad s = \operatorname{sign}(\det(UV^{\mathsf{T}}))$$

3. Determine best rotation matrix

   $$\boxed{R = VU^{\mathsf{T}}}$$

19

Estimation of rotation in 3D

- Unique, if singular values are distinct
- If $\det(H) > 0$ we obtain proper rotation matrix R
- If $\det(H) < 0$ we have a reflection, $\det R = -1$
- If $\det(H) = 0$ points are coplanar → special case

Mapping $X' = RX$ for any noncoplanar triple $X = [\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k]$

→ sign of $\det(R)$ indicates reflection or no reflection

Since $H = XWX'^{\mathsf{T}}$ we have, with $\operatorname{sd}(\cdot) := \operatorname{sign}\det(\cdot)$

$$\operatorname{sd}(R) = \operatorname{sd}(X)\operatorname{sd}(X') = \operatorname{sd}(H) = \operatorname{sd}(UV^{\mathsf{T}}) = \operatorname{sd}(\widehat{R}) \gtrless 0$$

20

Direct solution for similarity

- Simplified stochastical model
- Direct solutions in 2D and 3D
  - Optimal if model holds
  - Good approximation if no outliers
- Uncertainty: from maximum likelihood estimation

Next lecture

5.3. ML solution for spatial similarity

References of video series

Arun, K. S., T. S. Huang, and S. B. Blostein (1987). Least-Squares Fitting of Two 3D Point Sets. IEEE T-PAMI 9 (5), 698-700.

Bishop, C. (2006). Pattern Recognition and Machine Learning. Springer.

Boyd, S. and L. Vandenberghe (2004). Convex optimization. Cambridge University Press.

Cayley, A. (1846). Sur quelques proprétés des déterminants gauches. Journal für die reine und angewandte Mathematik 32, 119-123

Cover, T. and J. A. Thomas (1991). Elements of Information Theory. John Wiley & Sons.

H. S. M. Coxeter (1946): Quaternions and Reflections, The American Mathematical Monthly, 53(3), 136-146

Fallat, S. M. and M. J. Tsatsomeros (2002). On the Cayley Transform of Positivity Classes of Matrices. Electronic Journal of Linear Algebra 9, 190-196.

Fischler, M. A. and R. C. Bolles (1981). Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. Communications of the ACM 24 (6), 381-395.

Förstner, W. and B. P. Wrobel (2016). Photogrammetric Computer Vision Statistics, Geometry, Orientation and Reconstruction. Springer.

Horn, B. K. B. (1987). Closed-form Solution of Absolute Orientation Using Unit Quaternions. Optical Soc. of America.

Howell, T. D. and J.-C. Lafon (1975). The Complexity of the Quaternion Product. Technical Report TR75-245, Cornell University.

Kanatani, K. (1990). Group Theoretical Methods in Image Understanding. New York: Springer.

Koch, K.-R. (1999). Parameter Estimation and Hypothesis Testing in Linear Models (2nd ed.). Springer.

Li, S. Z. (2000). Markov random field modeling in computer vision. Springer.

McGlone, C. J., E. M. Mikhail, and J. S. Bethel (2004). Manual of Photogrammetry (5th ed.). Maryland, USA: American Society of Photogrammetry and Remote Sensing.

Mikhail, E. M. and F. Ackermann (1976). Observations and Least Squares. University Press of America

Mikhail, E. M., J. S. Bethel, and J. C. McGlone (2001). Introduction to Modern Photogrammetry. Wiley.

Palais, B. and R. Palais (2007). Euler's fixed point theorem: The axis of a rotation. J. fixed point theory appl. 2, 215-220.

Raguram, R., O. Chum, M. Pollefeys, J. Matas, and J.-M. Frahm (2013). USAC: A Universal Framework for Random Sample Consensus. IEEE Transactions on Pattern Analysis and Machine Intelligence 35 (8), 2022-2038.

Rao, R. C. (1973). Linear Statistical Inference and Its Applications. New York: Wiley.

Rodriguez, O. (1840). Des lois géometriques qui regissent les deplacements d'un système solide independament des causes qui peuvent les produire. Journal de mathématiques pures et appliquées 1 (5), 380440.

Sansò, F. (1973). An Exact Solution of the Roto-Translation Problem. Photogrammetria 29 (6), 203-206.

Vaseghi, S. V. (2000). Advanced Digital Signal Processing and Noise Reduction. Wiley.

Weber, M. (2003). Rotation between two vectors. Personal communication, Bonn.