Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

10. Direct LS Solution for the 3D Similarity
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The slides have been created by Wolfgang Förstner.

Direct solution
- No approximate values necessary
- Fixed computing time for given size of input data
- Allows for
  - Solving linear equation system
  - Finding roots of polynomial
  - Eigenvalues, eigenvectors
  - Singular value decomposition
  - Finding minimum of convex optimization problem*

*S. Boyd and L. Vandenberghe (2004): Convex Optimization

Direct solution for 3D Similarity
- Simplified stochastical model
- Direct solution
  - Optimal if model holds
  - Good approximation if no outliers
- Solution in 2D and 3D
  - Translation and scale estimation identical
  - Rotation estimation differs
- Uncertainty:
  → maximum likelihood estimation same stochastical model

Goal
- **Given:** pairs of corresponding points
  \[(x_i, x'_i), \quad i = 1, \ldots, I\]
  and \[D(x'_i) = \sigma^2_{x'_i} I_3\]
- **Assumption:** related by a similarity
  \[E(x'_i) = \lambda R x_i + t, \quad i = 1, \ldots, I\]
- **Task:** Estimate the parameters, provide uncertainty
  - If only \(x_i\) are noisy → invert model
  - If both, \(x_i\) and \(x'_i\) are noisy → more involving
    (Gauss-Helmert model for estimation)
Reformulation of model
Refer to centroid $x'_C$ of observed points

$$(x'_i + v_{x'_i}) - x'_C = \lambda R(x_i - u) \quad w_i = \frac{1}{\sigma^2 x'_i}, \quad i = 1, \ldots, I$$

Centroid known, fixed

$$x'_C = \frac{\sum_{i=1}^{I} x'_i w_i}{\sum_{i=1}^{I} w_i}$$

Unknown:
Rotation matrix $R$, scale parameter $\lambda$
modified translation vector $u$, corrections $v_{x'_i}$

Reformulation of model
Relation between original model

$$(x'_i + v_{x'_i}) = \lambda R x_i + t, \quad i = 1, \ldots, I$$

and centroid model

$$(x'_i + v_{x'_i}) - x'_C = \lambda R (x_i - u), \quad i = 1, \ldots, I$$

Rotation and scale parameter identical
Translation parameters
$$t = x'_C - \lambda Ru$$

Optimization function
Minimize

$$\Omega(R, u, \lambda) = \sum_i v_i^T v_i w_i$$

$$= \sum_i w_i (v_{x_i}^2 + v_{y_i}^2 + v_{z_i}^2)$$

$$= \sum_i [\lambda R(x_i - u) - (x'_i - x'_C)]^T [\lambda R(x_i - u) - (x'_i - x'_C)] w_i$$

Rewrite

$$\sum_i [\lambda R(x_i - u) - (x'_i - x'_C)]^T [\lambda R(x_i - u) - (x'_i - x'_C)] w_i$$

$$= \sum_i \lambda^2 (x_i - u)^T R^T R (x_i - u) w_i$$

$$- 2 \lambda \sum_i (x'_i - x'_C)^T R (x_i - u) w_i$$

$$+ \sum_i (x'_i - x'_C)^T (x'_i - x'_C) w_i$$

$$= \text{const.}$$

Partial derivatives w.r.t. parameters need to be zero
**Estimating the translation**

**Necessary condition**
\[
\frac{\partial \Omega}{\partial u} = 2 \sum_i \lambda^2 (x_i - u)w_i - 2 \sum_i \lambda R^T (x'_i - x'_C)w_i
\]
\[
= 2\lambda^2 \sum_i (x_i - u)w_i - 2\lambda R^T \sum_i (x'_i - x'_C)w_i = 0
\]

Since the centroid fulfils \( \sum (x'_i - x'_C)w_i = 0 \) →

**Estimating the scale parameter**

**Necessary condition**
\[
\frac{\partial \Omega}{\partial \lambda} = -2 \sum (x'_i - x'_C)^T R(x_i - x_C)w_i + 2\lambda \sum (x_i - x_C)^T (x_i - x_C)w_i = 0.
\]

Estimated scale parameter
\[
\lambda = \frac{\sum (x'_i - x'_C)^T R(x_i - x_C)w_i}{\sum (x_i - x_C)^T (x_i - x_C)w_i}
\]

• depending on rotation

**Estimating the scale parameter**

If noise is small (say below 1%)
\[
R(x_i - x_C) \approx \lambda^{-1}(x'_i - x'_C)
\]

\[\Rightarrow\] Symmetric solution
\[
\lambda^2 \approx \frac{\sum (x'_i - x'_C)^T (x'_i - x'_C)w_i}{\sum (x_i - x_C)^T (x_i - x_C)w_i}
\]

= ratio of
sum of weighted squares of distances to centroids
Estimation of rotation

We use centred coordinates

\[ \mathbf{c} x'_i = x'_i - x'_C \quad \text{and} \quad \mathbf{c} x_i = x_i - x_C \]

From (1) on slide 8, find matrix \( R \) maximize

\[ \Phi'(R) = \sum_{i=1}^{I} \mathbf{c} x_i^T R \mathbf{c} x_i w_i \]

under the constraint

\[ R^T R = I_d \]

Estimation of rotation

We rewrite optimization function \( \Phi'(R) \) using \( \text{tr}(AB) = \text{tr}(BA) \), specifically

\[ a^T B c = \text{tr}(a^T B c) = \text{tr}(c a^T B) \]

Here

\[ \mathbf{c} x_i^T R \mathbf{c} x_i w_i = \text{tr}\left( \mathbf{c} x_i^T R \mathbf{c} x_i w_i \right) = \text{tr}\left( \mathbf{c} x_i^T \mathbf{c} x_i R w_i \right) \]

therefore

\[ \sum_{i=1}^{I} \mathbf{c} x_i^T R \mathbf{c} x_i w_i = \text{tr}\left( \sum_{i=1}^{I} \mathbf{c} x_i^T \mathbf{c} x_i R w_i \right) = \text{tr}\left( \sum_{i=1}^{I} \mathbf{c} x_i^T \mathbf{c} x_i^T w_i \right) R \]

Estimation of rotation in 2D

To find the rotation angle \( \varphi \) we need to maximize

\[ \text{tr}(HR) = \text{tr}\left( \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \right) \]

\[ = \begin{bmatrix} \cos \varphi, \sin \varphi \end{bmatrix} \begin{bmatrix} H_{xx} + H_{yy} \\ H_{xy} - H_{yx} \end{bmatrix} = r \cdot h \]

With

\[ H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} \sum_i w_i \mathbf{c} x_i \mathbf{c} x_i' \\ \sum_i w_i \mathbf{c} y_i \mathbf{c} y_i' \end{bmatrix} \begin{bmatrix} \sum_i w_i \mathbf{c} x_i \mathbf{c} x_i' \\ \sum_i w_i \mathbf{c} y_i \mathbf{c} y_i' \end{bmatrix} \]

→ The vector \( r \) needs to be parallel to the vector \( h \)
**Estimation of rotation in 2D**

With

\[ h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \sum_{i=1}^{I'} \left[ (c_{x_i'} + c_{y_i'} + c_{z_i'}) w_i \right] \left[ (c_{x_i'} c_{y_i'} - c_{y_i'} c_{z_i'}) w_i \right] \]

the rotation angle \( \varphi \) can directly be determined by

\[ \hat{\varphi} = \text{atan2}(h_2, h_1) \]

Weighted mean \( h \) of direction vectors

\[ h_i = \begin{bmatrix} \cos \varphi_i \\ \sin \varphi_i \end{bmatrix} = \begin{bmatrix} \cos(\alpha'_i - \alpha_i) \\ \sin(\alpha'_i - \alpha_i) \end{bmatrix} \]

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**Estimation of rotation in 3D**

Several solutions, see Eggert et al. (1997)

- With singular value decomposition
- With quaternions
- With orthonormal matrices
- With dual quaternions

Here: with singular value decomposition

1. Perform SVD

\[ H = USV^T \quad UU^T = I \quad VV^T = I \quad S = \text{Diag}(s_i) \]

2. If mapping is known to be a proper rotation

\[ V := V \text{ Diag}([1, 1, s]) \quad \text{with} \quad s = \text{sign} (\det(UV^T)) \]

3. Determine best rotation matrix

\[ R = VU^T \]

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**Estimation of rotation in 3D**

- Unique, if singular values are distinct
- If \( \det(H) > 0 \) we obtain proper rotation matrix \( R \)
- If \( \det(H) < 0 \) we have a reflection, \( \det R = -1 \)
- If \( \det(H) = 0 \) points are coplanar \( \rightarrow \) special case

Mapping \( X' = RX \) for any noncoplanar triple \( X = [x_i, x_j, x_k] \)

\[ \text{sign of} \ ( \det(R) \) \text{ indicates reflection or no reflection} \]

Since \( H = XWX^T \) we have, with \( \text{sd}(\cdot) := \text{sign} \ (\det(\cdot)) \)

\[ \text{sd}(R) = \text{sd}(X)\text{sd}(X') = \text{sd}(H) = \text{sd}(UV^T) = \text{sd}(R) \geq 0 \]
Direct solution for similarity

- Simplified stochastical model
- Direct solutions in 2D and 3D
  - Optimal if model holds
  - Good approximation if no outliers
- Uncertainty: from maximum likelihood estimation

Next lecture

11. ML solution for spatial similarity