Photogrammetry & Robotics Lab

3D Coordinate Systems

(Bsc Geodesy & Geoinformation)

9. Estimating Parameters
of Spatial Transformations

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The slides have been created by Wolfgang Förstner.

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5. Estimating parameters of transformations
5.1 Introduction

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Goal

- Estimate transformation parameters
- Characterize the uncertainty of the estimates
- Handle outliers in the given data

For this video see Förstner/Wrobel (2016), 75-84

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5.1.1 Task

- Given: pairs of corresponding points

$$(\mathcal{X}_i, \mathcal{X}'_i), i = 1, \ldots, I$$

- Assumption: related by a similarity

$$\mathcal{X}'_i = \mathcal{A}(\mathcal{X}_i)$$

- Task: Helmert-transformation *
  - Estimate the parameters of the similarity
  - Provide quality measures for the estimation

Friedrich Robert Helmert, geodesist 1843 -- 1917

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Task

Overconstraint problem

- Dimension $d=2$: $2I$ constrains, 4 parameters

  Redundancy = number of remaining constraints

  $$R = 2I - 4$$

  → minimum number of correspondences: $I=2$
- Dimension $d=3$: $3I$ constrains, 7 parameters

  Redundancy

  $$R = 3I - 7$$

  → minimum number of correspondences: $I=3$

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Task

Inconsistencies due to measurement deviations

$$\boldsymbol{x}_i' \neq \lambda\, R\, \boldsymbol{x}_i + \boldsymbol{t}, i=1,...,I$$

Here: assumption: only χ_i' are perturbated

Corrections* $\boldsymbol{v}_{x_i'}$ at coordinates $\boldsymbol{x}_i'$ → constraints

$$\boldsymbol{x}_i' + \boldsymbol{v}_{x_i'} = \lambda\, R\, \boldsymbol{x}_i + \boldsymbol{t}, i=1,...,I$$

→Underconstrained system:

unknown parameters, unknown corrections

* German: "**V**erbesserungen"

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Solution

Use principle of least squares (LS, regularization) following Carl Friedrich Gauss (1777—1855)

Determine corrections and parameters such that corrections are smallest → Minimize sum of squared corrections

$$\Omega = \sum_{i=1}^{I} \boldsymbol{v}'^{\mathsf{T}}_{x_i} \boldsymbol{v}_{x_i'} = \sum_{i=1}^{I} v'^2_{x_i} + v'^2_{y_i} + v'^2_{z_i}$$

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Solution?

There exists a direct LS solution for estimating

- 7 parameters of 3D similarity
- 6 parameters of 3D motion

Why is this not sufficient?

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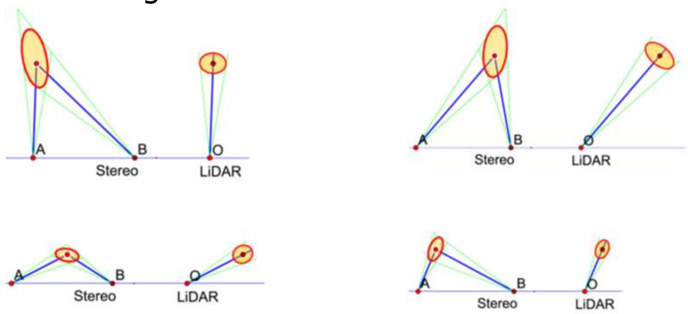
Point uncertainty highly varying

- Stereo images and LiDAR



- GPS: 3D accuracy = f(satellite configuration)

Solution?

- No uncertainty of observations is taken into account
  Given points may
  - have different accuracies
  - may have correlated coordinates
  - may be correlated, e.g. due to uncertainties of A, B and O
- No covariance matrix for resultant parameters

→ Take uncertainty of given coordinates into account
→ Derive uncertainty of estimated parameters

On notation

World of applications
e.g. coordinates $\boldsymbol{x}$, angles $\boldsymbol{\alpha}$
transformations M , parameters $\boldsymbol{p}$, etc.

World of estimation
(measurements, unknown parameters)

$$(\boldsymbol{l}, \boldsymbol{x}) \quad \text{or} \quad (\boldsymbol{y}, \boldsymbol{\beta}) \quad \text{or} \quad (\boldsymbol{x}, \boldsymbol{\theta})$$

(depending on community !)
→ Estimation theory

On notation

→Leave notation in both worlds and **assign**

e.g. using estimation notation $(\boldsymbol{l}, \boldsymbol{x})$
1. observed coordinates, unknown parameters

$$\boldsymbol{l} := [\boldsymbol{x}_i]$$
$$\boldsymbol{x} := \boldsymbol{p}$$

2. After estimating parameters

$$\widehat{\boldsymbol{p}} := \widehat{\boldsymbol{x}}$$

Estimation principles

Given:

- Unknown parameters with some prior uncertainty

$$p(\boldsymbol{x})$$

- Observations with uncertainty given the parameters

$$p(\boldsymbol{l} \mid \boldsymbol{x})$$

Task:

estimate parameters best explaining the observations

$$\widehat{\boldsymbol{x}} = \operatorname{argmax}_x \, p(\boldsymbol{x} \mid \boldsymbol{l})$$

Bayesian estimation

Law of conditional probability

$p(\boldsymbol{l}, \boldsymbol{x}) = p(\boldsymbol{x} \mid \boldsymbol{l}) \, p(\boldsymbol{l}) = p(\boldsymbol{l} \mid \boldsymbol{x}) \, p(\boldsymbol{x})$

Bayesian theorem

$$p(\boldsymbol{x} \mid \boldsymbol{l}) = \frac{p(\boldsymbol{l} \mid \boldsymbol{x}) \, p(\boldsymbol{x})}{p(\boldsymbol{l})}$$

For given $\boldsymbol{l}$, fixed

- Prior distribution $p(\boldsymbol{x})$
- Likelihood $L(\boldsymbol{x}) = p(\boldsymbol{l} \mid \boldsymbol{x})$
- Posterior distribution $p(\boldsymbol{x} \mid \boldsymbol{l})$
- Probability to observe $\boldsymbol{l}$, fixed $p(\boldsymbol{l})$

MAP and ML estimation

1. Maximize posterior probability (**MAP**)
   → Maximum posterior estimate
   (also Bayesian estimate)

$$\widehat{\boldsymbol{x}}_{\mathrm{MAP}} = \operatorname{argmax}_x \, p(\boldsymbol{l} \mid \boldsymbol{x}) \, p(\boldsymbol{x}) = \operatorname{argmax}_x \, L(\boldsymbol{x}) \, p(\boldsymbol{x})$$

2. Maximum likelihood (**ML**) estimate, $p(\boldsymbol{x}) = \mathrm{const.}$

$$\widehat{\boldsymbol{x}}_{\mathrm{ML}} = \operatorname{argmax}_x \, p(\boldsymbol{l} \mid \boldsymbol{x}) = \operatorname{argmax}_x \, L(\boldsymbol{x})$$

Best linear unbiased estimates

Assume Gaussian distribution, linear dependency

$$p(\boldsymbol{l} \mid \boldsymbol{x}) \;\propto\; \exp\left(-\frac{1}{2}(\boldsymbol{l} - A\boldsymbol{x})^{\mathsf{T}} \Sigma_{ll}^{-1} (\boldsymbol{l} - A\boldsymbol{x})\right)$$

3. Best linear unbiased estimator
   a) Leads to smallest CovM of estimates
   b) Estimate is a linear function of observations
   c) Estimate is unbiased

$$\widehat{\boldsymbol{x}}_{BLUE} = \operatorname{argmin}_x \, (\boldsymbol{l} - A\boldsymbol{x})^{\mathsf{T}} \Sigma_{ll}^{-1} (\boldsymbol{l} - A\boldsymbol{x})$$

Least squares estimates

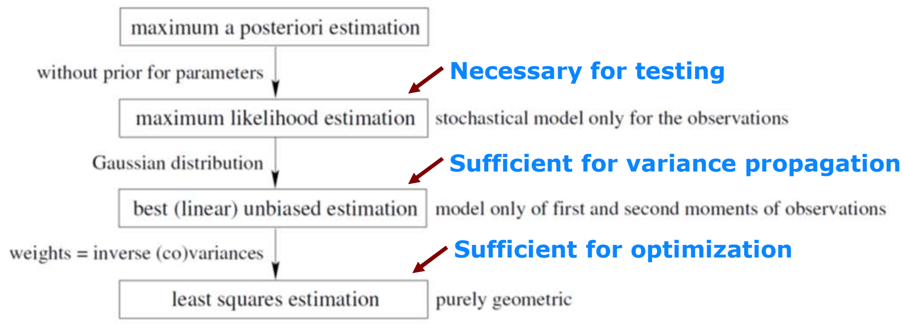
- Ordinary least squares

$$\widehat{\boldsymbol{x}}_{LS} = \text{argmin}_x \; (\boldsymbol{l} - A\boldsymbol{x})^{\mathsf{T}}(\boldsymbol{l} - A\boldsymbol{x})$$

- Generalized least squares with weight matrix W_{ll}

$$\widehat{\boldsymbol{x}}_{GLS} = \text{argmin}_x \; (\boldsymbol{l} - A\boldsymbol{x})^{\mathsf{T}} W_{ll} (\boldsymbol{l} - A\boldsymbol{x})$$

Hierarchy of estimates



Hierarchy of estimates

- ML estimate = MAP estimate if no prior
- Best linear unbiased estimate = ML estimate if
  - Distribution is Gaussian
  - Model is linear
- GLS estimate = best linear unbiased estimate if

$$W_{ll} = \Sigma_{ll}^{-1}$$

- LS estimate = GLS estimate if

$$W_{ll} = I$$

→ e. g. **LSE = MAP if model linear, Gaussian** $\Sigma_{ll} = I$

5.1.2 Non-linear Gauss-Markov Model

Basis: Maximum likelihood estimation (ML-estimation)

Given: observation vector $\boldsymbol{l}$, fixed
sample from

$$\underline{\boldsymbol{l}} \sim p(\boldsymbol{l} \mid \boldsymbol{x}) = \mathcal{N}(\mathbb{E}(\underline{\boldsymbol{l}} \mid \boldsymbol{x}), \mathbb{D}(\underline{\boldsymbol{l}} \mid \boldsymbol{x}))$$

Notation: stochastical variables are underscored

Mathematical model

Functional model:

generative model for $\boldsymbol{l}$, given parameters $\boldsymbol{x}$

model is non-linear

$$\mathbb{E}(\underline{\boldsymbol{l}} \mid \boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$$

Stochastical model

$$\mathbb{D}(\underline{\boldsymbol{l}} \mid \boldsymbol{x}) = \mathbb{D}(\underline{\boldsymbol{l}} - \boldsymbol{f}(\boldsymbol{x})) = \Sigma_{ll}$$

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Mathematical model

- Generative model with perturbations $\boldsymbol{e}$

$$\underline{\boldsymbol{l}} = \boldsymbol{f}(\boldsymbol{x}) + \underline{\boldsymbol{e}} \quad \text{with} \quad \mathbb{D}(\underline{\boldsymbol{e}}) = \Sigma_{ee}$$

or with corrections $\boldsymbol{v} = -\boldsymbol{e}$

$$\underline{\boldsymbol{l}} + \underline{\boldsymbol{v}} = \boldsymbol{f}(\boldsymbol{x}) \quad \text{with} \quad \mathbb{D}(\underline{\boldsymbol{v}}) = \Sigma_{vv} = \Sigma_{ee}$$

- Matrix Σ_{ll} refers to corrections or perturbations

$$\Sigma_{ll} = \mathbb{D}(\underline{\boldsymbol{l}} - \boldsymbol{f}(\boldsymbol{x})) = \mathbb{D}(\underline{\boldsymbol{e}}) = \mathbb{D}(\underline{\boldsymbol{v}})$$

→ the mean values $\mathbb{E}(\underline{\boldsymbol{l}} \mid \boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$ are fixed

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Maximum likelihood estimation

Maximize likelihood

$$L(\boldsymbol{x}) = p(\boldsymbol{l} \mid \boldsymbol{x})$$

Here

$$p(\boldsymbol{l} \mid \boldsymbol{x}) \propto \exp\left(-\frac{1}{2}(\boldsymbol{l} - \boldsymbol{f}(\boldsymbol{x}))^{\mathsf{T}}\Sigma_{ll}^{-1}(\boldsymbol{l} - \boldsymbol{f}(\boldsymbol{x}))\right)$$

Hence minimize

$$\Omega = (\boldsymbol{l} - \boldsymbol{f}(\boldsymbol{x}))^{\mathsf{T}}\Sigma_{ll}^{-1}(\boldsymbol{l} - \boldsymbol{f}(\boldsymbol{x}))$$

or

$$\Omega = \boldsymbol{v}^{\mathsf{T}}\Sigma_{ll}^{-1}\boldsymbol{v}$$

→ Best estimate, generalizes LS with $W_{ll} = \Sigma_{ll}^{-1}$

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5.1.3 Model for Helmert transformation

- Functional model
- Stochastical model
  - General
  - Simplified

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Observations and corrections

Collected in N-vectors,
for dimension d we have $N = dI$

$$\begin{bmatrix} l_1 \\ \dots \\ l_n \\ \dots \\ l_N \end{bmatrix} = \underset{N\times 1}{\boldsymbol{l}} := \begin{bmatrix} \boldsymbol{x}'_1 \\ \dots \\ \boldsymbol{x}'_i \\ \dots \\ \boldsymbol{x}'_I \end{bmatrix} \qquad \underset{N\times 1}{\boldsymbol{v}} := \begin{bmatrix} \boldsymbol{v}_{x'_1} \\ \dots \\ \boldsymbol{v}_{x'_i} \\ \dots \\ \boldsymbol{v}_{x'_I} \end{bmatrix}$$

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Parameters and functional model

Unknown parameters, collected in U-vector

$$\underset{U\times 1}{\boldsymbol{x}} := \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{t} \\ \lambda \end{bmatrix}$$

Functional model

$$\mathbb{E}(\boldsymbol{l}) = \boldsymbol{f}(\boldsymbol{x}) \quad \text{or} \quad \mathbb{E}\begin{bmatrix} \boldsymbol{x}'_1 \\ \dots \\ \boldsymbol{x}'_i \\ \dots \\ \boldsymbol{x}'_I \end{bmatrix} = \begin{bmatrix} \lambda R\boldsymbol{x}_1 + \boldsymbol{t} \\ \dots \\ \lambda R\boldsymbol{x}_i + \boldsymbol{t} \\ \dots \\ \lambda R\boldsymbol{x}_I + \boldsymbol{t} \end{bmatrix}$$

Representation of rotation: open

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Stochastical model

Stochastical model:

- All points may be mutually dependent
- Coordinates of one point may be dependent

$$\underset{N\times N}{\Sigma_{ll}} := \begin{bmatrix} \Sigma_{x'_1x'_1} & \dots & \Sigma_{x'_1x'_i} & \dots & \Sigma_{x'_1x'_j} & \dots & \Sigma_{x'_1x'_I} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma_{x'_ix'_1} & \dots & \Sigma_{x'_ix'_i} & \dots & \Sigma_{x'_ix'_j} & \dots & \Sigma_{x'_ix'_I} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma_{x'_jx'_1} & \dots & \Sigma_{x'_jx'_i} & \dots & \Sigma_{x'_jx'_j} & \dots & \Sigma_{x'_jx'_I} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Sigma_{x'_Ix'_1} & \dots & \Sigma_{x'_Ix'_i} & \dots & \Sigma_{x'_Ix'_j} & \dots & \Sigma_{x'_ix'_I} \end{bmatrix}$$

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Stochastical model

Simplified stochastical model

$$\Sigma_{ll} := \mathrm{Diag}\left(\sigma^2_{x_i} I_3\right) = \begin{bmatrix} \sigma^2_{x_1} I_3 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \sigma^2_{x_i} I_3 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \sigma^2_{x_I} I_3 \end{bmatrix}$$

- Points are mutually independent
- Coordinates of one point are mutually independent
- Accuracy/weight may vary from point to point

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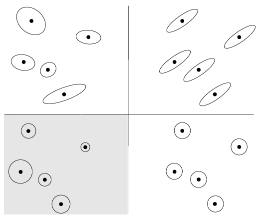
Visualization of stochastical model

- Isotropic/anisotropic (lower/upper row):
  rotation invariant/variant
- Homogeneous/inhomogeneous (right/left column):
  translation invariant/variant

$w_i := \frac{1}{\sigma^2_{x'_i}}$ →



← $w_i = w := \frac{1}{\sigma^2_{x'}}$

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Simplified optimization problem

If uncertainty is isotropic and points are independent estimates are

$$(\widehat{R}, \widehat{\boldsymbol{t}}, \widehat{\lambda}) = \operatorname{argmin}_{R,t,\lambda}\, \Omega(R, \boldsymbol{t}, \lambda)$$

with

$$\Omega(R, \boldsymbol{t}, \lambda) := \sum_{i=1}^{I} (\boldsymbol{x}'_i - \lambda R \boldsymbol{x}_i - \boldsymbol{t})^{\mathsf{T}} (\boldsymbol{x}'_i - \lambda R \boldsymbol{x}_i - \boldsymbol{t}) w_i$$

and weights

$$w_i := \frac{1}{\sigma^2_{x'_i}}$$

Non-linear least squares problem

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Solutions

- Optimal solution for general stochastical model
  - ☺ Yields covariance matrix of estimated parameters
  - ☺ Stochastical model can be simplified
  - ☹ Requires approximate values
  - ☹ Solution is iterative
- Direct solution for simplified stochastical model
- Fast direct solution for 3 points

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Next lecture

5.2 Direct solution for spatial similarity

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