Photogrammetry & Robotics Lab

3D Coordinate Systems

(Bsc Geodesy & Geoinformation)

8. Denavit-Hartenberg Parametrization and Kinematic Chains

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The slides have been created by Wolfgang Förstner.

Motivation

Robots

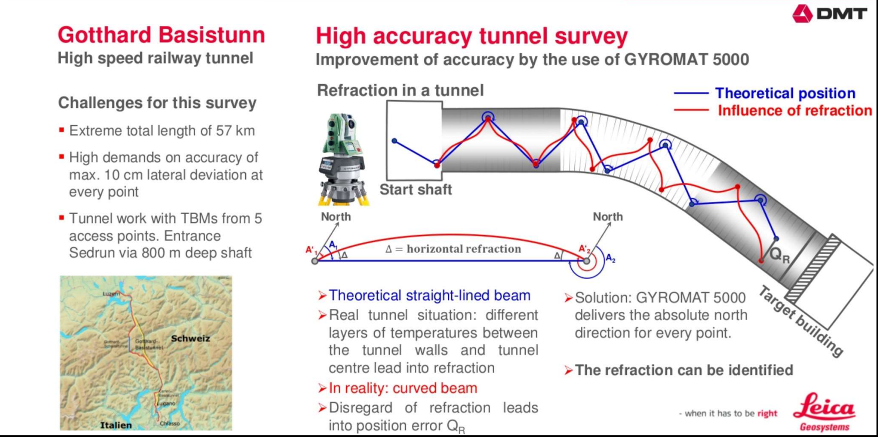
- Robot arms (industrial robots, SEAT, 2018, 1'00'')
- Walking running
  - humans (Motion capture, Hannover, 2011, 1'42'')
  - robots (Boston dynamics, 2019)
- Moving robots/frames →

For this video see Förstner/Wrobel (2016), p. 216-217, 264-265

Sequence of frames: tunnel surveying

- Long sequence of frames through tunnel
- Control of tunnel boring machine (TBM)
- High accuracy requirements
- Analysis of achievable accuracy

Sequence of frames: tunnel surveying



Kinematic chains

A kinematic chain is sequence of motions

- Kinematic = only geometry
  no forces → dynamic
- Sequence described by relative motions

→Tree as set of sequences

e.g.

Modelling a robot arm = special kinematic chain

Modelling a skeleton = tree of sequences

4.4 Denavit-Hartenberg Parametrization

Modeling a robot arm

Example: passive measuring arm

FARO Arm:

- 6 degrees of freedom
- Measuring volume appr. 1.5 m radius
- accuracy: 0.02 – 0.05 mm

→ demo

Modelling a robot arm

Basic types of joints

- Revolute joint: rotation around an axis
- Prismatic joint: translation along an axis

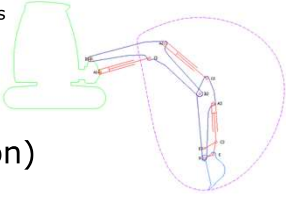
ball bearing

Others

- Spherical (like shoulder)
- Cylindrical (translation and rotation)

… we only discuss rotations

excavator: three hydraulic cylinders control of shovels



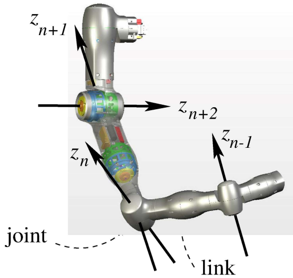
Images from wikipedia

Modelling revolute joints

Rotations axes: sequence of z-axes

How to describe
relative pose of axes?

How many parameters?



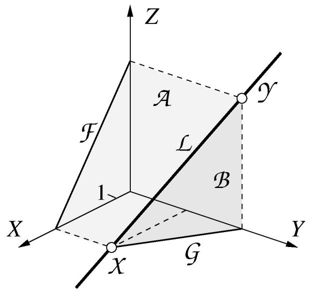
http://www.leros-f.iat.uni-bremen.de/

Modelling revolute joints

Modelling a 3D line

Degrees of freedom: 4

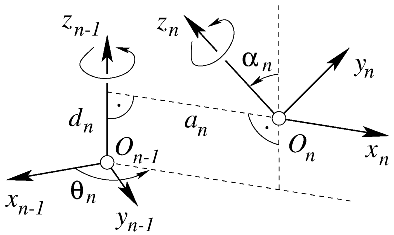
- Connection of two points
  in given planes $\mathcal{L} = \overline{\mathcal{X}\mathcal{Y}}$
- Intersection of two
  projecting planes $\mathcal{L} = \mathcal{A} \cap \mathcal{B}$
  perpendicular to two given planes



Modelling revolute joints

Definition of coordinate system S_n

- Origin O_n of system S_n
  point on z_n-axis
  closest to z_{n-1}-axis
- x_n-axis perpendicular
  to z_{n-1}-axis, pointing
  away
- x_n-axis completes the right handed system

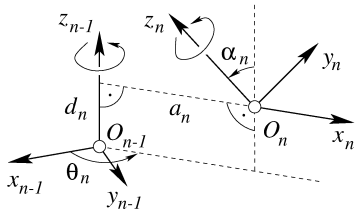


Modelling revolute joints

Four motions

1. Rotation of S_{n-1} by θ_n
   measurable/controllable

$$
{}^{1}\mathrm{M}(\theta_n) = \begin{bmatrix} R_z(\theta_n) & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}
$$



fixes direction of x_n -axis of S_n

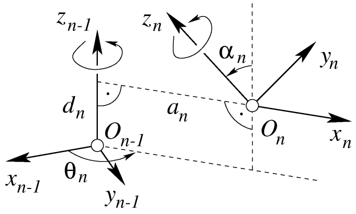
Modelling revolute joints

Four motions

2. Translation by d_n

along z_{n-1}-axis

$$^2\mathsf{M}(d_n) = \begin{bmatrix} I_3 & \boldsymbol{t}_z(d_n) \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$



fixes vertical position of origin O_n of S_n

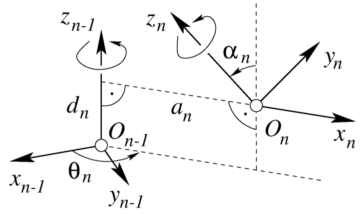
Modelling revolute joints

Four motions

3. Translation by a_n

along new x_n-axis

$$^3\mathsf{M}(a_n) = \begin{bmatrix} I_3 & \boldsymbol{t}_x(a_n) \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$



fixes origin O_n of system S_n

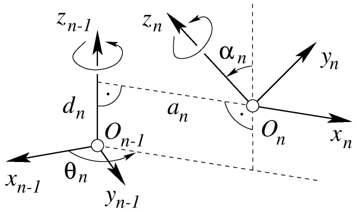
Modelling revolute joints

Four motions

4. Rotation by α_n

around x_n-axis

$$^4\mathsf{M}(\alpha_n) = \begin{bmatrix} R_x(\alpha_n) & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$



fixes pose of system S_n

Modelling revolute joints

Joint **active motion** ${}_n\mathsf{M}^{n-1}: S_{n-1} \to S_n$, case B

$$\begin{aligned} &\mathsf{M}_n(\theta_n; \alpha_n, d_n, a_n) \\ &= {}^1\mathsf{M}(\theta_n){}^2\mathsf{M}(d_n){}^3\mathsf{M}(a_n){}^4\mathsf{M}_4(\alpha_n) \\ &= \begin{bmatrix} \cos\theta_n & -\cos\alpha_n\sin\theta_n & \sin\alpha_n\sin\theta_n & a_n\ \cos\theta_n \\ \sin\theta_n & \cos\alpha_n\cos\theta_n & -\sin\alpha_n\cos\theta_n & a_n\ \sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Joint **passive transform**, due to $\mathsf{M}_n := {}^{n-1}\mathsf{M}_n = {}_n\mathsf{M}^{n-1}$

$$^{n-1}\mathbf{x} = {}^{n-1}\mathsf{M}_n\,{}^n\mathbf{x}$$

Modelling revolute joints

- Revolute joint: θ_n measurable/controllable
- Prismatic joint: d_n measurable/controllable

other **three parameters to be calibrated**

Many revolute joints with individual $\mathsf{M}_n(\theta_n) := {}_n\mathsf{M}^{n-1}$

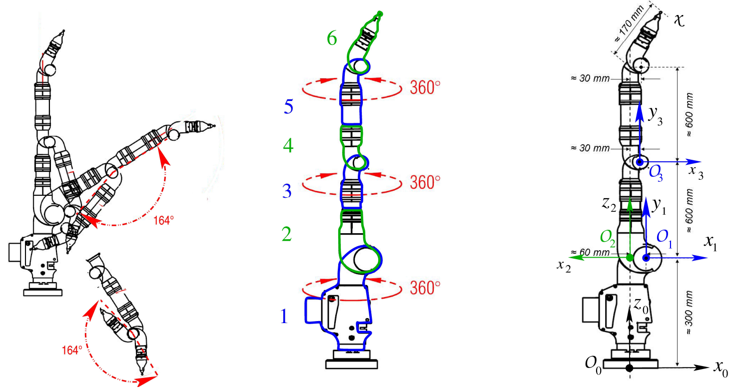
→ Combined **active motion** of coordinate frames

$$\begin{aligned} {}_N\mathsf{M}^0 &:= \mathsf{M}(\theta_1, ..., \theta_n, ..., \theta_N) \\ &= \mathsf{M}_1(\theta_1) \cdot ... \cdot \mathsf{M}_n(\theta_n) \cdot ... \cdot \mathsf{M}_N(\theta_N) \end{aligned}$$

→ Combined (**passive**) coordinate transformation

$$ {}^0\mathbf{x} = \mathsf{M}_1(\theta_1) \cdot ... \cdot \mathsf{M}_n(\theta_n) \cdot ... \cdot \mathsf{M}_N(\theta_N)\ {}^N\mathbf{x} $$

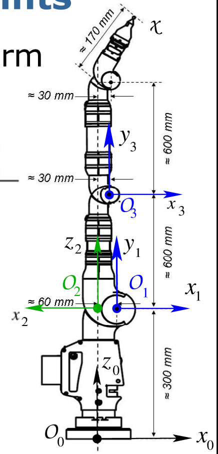
FARO measuring arm



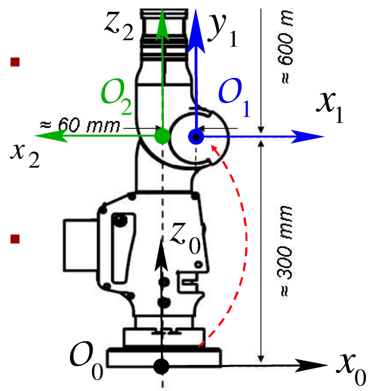
Modelling an arm with revolute joints

Denavit-Hartenberg parameters of robot arm in position of figure right

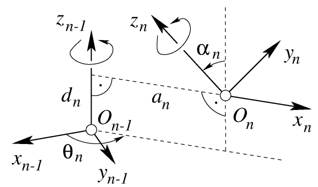
System S_n	θ_n [°]	d_n [mm]	a_n [mm]	α_n [°]
0→1	0	300	60	90
1→2	180	0	60	90
2→3	180	600	30	90
3→4				
4→5				
5→6				



Denavit-Hartenberg parameters



$$\begin{aligned} \theta_1 &= 0° & \theta_2 &= 180° \\ d_1 &= 300 \text{ mm} & d_2 &= 0 \text{ mm} \\ a_1 &= 60 \text{ mm} & a_2 &= 60 \text{ mm} \\ \alpha_1 &= 90° & \alpha_2 &= 90° \end{aligned}$$



4.5 Kinematic Chains

Kinematic chains

A kinematic chain is sequence of motions

Sequence described by relative motions

Tree as set of sequences

Modelling a kinematic chain

- End effector of arm $\chi\left({}^{T}\mathbf{x}\right)$ in last link
- Modelling passive motions backwards $t = T \to t = 0$
- Recursive definition coordinates in following frames

$$\mathcal{M}_t : \quad {}^{t-1}\mathbf{x} = \mathsf{M}_t \; {}^{t}\mathbf{x} \quad \text{with} \quad \mathsf{M}_t := {}^{t-1}\mathsf{M}_t$$

- Point at time $t = 0$, $\mathcal{M}_t$ described in local frame

$${}^{0}\mathbf{x} = \underbrace{\mathsf{M}_1 \ldots \mathsf{M}_{t-1} \;\; \mathsf{M}_t \;\; \mathsf{M}_{t+1} \ldots \;\; \mathsf{M}_{T-1} \, \mathsf{M}_T}_{{}^{0}\mathsf{M}_T} \, {}^{T}\mathbf{x}$$

→ Simple **variance propagation**

Variance propagation in kinematic chains

Jacobian of coordinates ${}^{0}\mathbf{x}$ in reference frame w.r.t.

- Coordinates ${}^{T}\mathbf{x}$ of end effector in last system
- All intermediate motion parameters $\boldsymbol{s}_t, t = 1, \ldots, T$

Jacobian w.r.t. ${}^{T}\mathbf{x}$ easy

$$J_{{}^{0}\mathbf{x}\,{}^{T}\mathbf{x}} = \frac{\partial\left({}^{0}\mathbf{x}\right)}{\partial\left({}^{T}\mathbf{x}\right)} = {}^{0}\mathsf{M}_T$$

Variance propagation in kinematic chains

Jacobian w.r.t. motion parameters

$$\mathrm{d}\boldsymbol{s} = \begin{bmatrix} \mathrm{d}\boldsymbol{r} \\ \mathrm{d}\boldsymbol{t} \end{bmatrix}$$

Total differential w.r.t. M_t

$$\mathrm{d}(^0\mathbf{x}) = \underbrace{\mathsf{M}_1 \ldots \mathsf{M}_{t-1}}_{^0\mathsf{M}_{t-1}} \; \color{blue}{\mathrm{d}\mathsf{M}_t} \; \underbrace{\mathsf{M}_{t+1} \ldots \; \mathsf{M}_{T-1}\, \mathsf{M}_T \; {}^T\mathbf{x}}_{^t\mathbf{x} \quad \text{lever arm}} \qquad (1)$$

with

$$\mathrm{d}\mathsf{M}_t = \mathrm{d}\mathsf{M}_0(\boldsymbol{s}_t)\;\; \mathsf{M}_t \quad \text{and} \quad \mathrm{d}\mathsf{M}_0(\boldsymbol{s}_t) = \begin{bmatrix} S(\mathrm{d}\boldsymbol{r}_t) & \mathrm{d}\boldsymbol{t}_t \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

Variance propagation in kinematic chains

Rearranging (1): lever arm $^{t-1}\mathbf{x}$ from $(t-1) \to T$

$$\begin{aligned} \mathrm{d}(^0\mathbf{x}) &= {}^0\mathsf{M}_{t-1}\;\; \mathrm{d}\mathsf{M}_0(\mathrm{d}\boldsymbol{s}_t)\;\; {}^{t-1}\mathbf{x} \\ &= {}^0\mathsf{M}_{t-1} \begin{bmatrix} S(\mathrm{d}\boldsymbol{r}_t) & \mathrm{d}\boldsymbol{t}_t \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} ^{t-1}\boldsymbol{x} \\ 1 \end{bmatrix} \\ &= {}^0\mathsf{M}_{t-1} \begin{bmatrix} -S(^{t-1}\boldsymbol{x}) & I_3 \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}\boldsymbol{r}_t \\ \mathrm{d}\boldsymbol{t}_t \end{bmatrix} \end{aligned}$$

hence

$$J_{^0\mathbf{x}\boldsymbol{s}_t} = \frac{\partial(^0\mathbf{x})}{\partial \boldsymbol{s}_t} = {}^0\mathsf{M}_{t-1} \begin{bmatrix} \underbrace{\color{blue}{-S(^{t-1}\boldsymbol{x})}}_{\color{blue}{\text{lever arm}}} & I_3 \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}.$$

Variance propagation in kinematic chains

Covariance matrix of
end effector coordinates $^0\mathbf{x}$ in reference system S_0

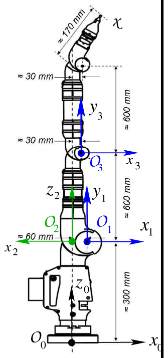
Assumption

- relative motions are mutually independent
- relative motions independent of starting point
- Calibration is perfect or errors negligible

$$\Sigma_{^0\mathrm{x}^0\mathrm{x}} = {}^0\mathsf{M}_T\;\; \Sigma_{^T\mathrm{x}^T\mathrm{x}}\;\; \left(^0\mathsf{M}_T\right)^\mathsf{T} + \sum_{t=1}^{T} J_{^0\mathrm{x}s_t}\; \Sigma_{s_t s_t}\; J^\mathsf{T}_{^0\mathrm{x}s_t}\,.$$

Variance propagation for robot arm

Intermediate motions = f(special parameters)
e.g. for robot arm with only revolute joints



$$J_{^{n-1}\mathsf{M}_n,\theta} = \begin{bmatrix} -\sin\theta_n & -\cos\alpha_n\cos\theta_n & \sin\alpha_n\cos\theta_n & -a_n\;\sin\theta_n \\ \cos\theta_n & -\cos\alpha_n\sin\theta_n & \sin\alpha_n\sin\theta_n & a_n\;\cos\theta_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the 4-vector from $\;{}^{n-1}\mathbf{x} = {}^{n-1}\mathsf{M}_n(\theta_n)\;{}^n\mathbf{x}$

$$J_{^0\mathrm{x},\theta_n} = \frac{\partial\left(^0\mathbf{x}\right)}{\partial\theta_n} = {}^0\mathsf{M}_{n-1}\;\; J_{^{n-1}\mathsf{M}_n,\theta}\;\; {}^n\mathbf{x}$$

demo

Example 1

Assumption

$\sigma_\theta = 0.0022°$

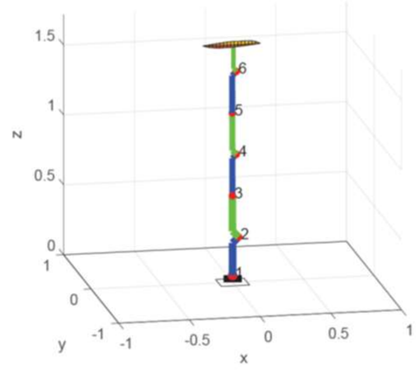
Predicted accuracy

- Standard deviation

  $\sigma_x = 0.063$ mm

- Condition ($\kappa = a^2/c^2$)

  $\sqrt{\kappa} = 8.3$



Example 2

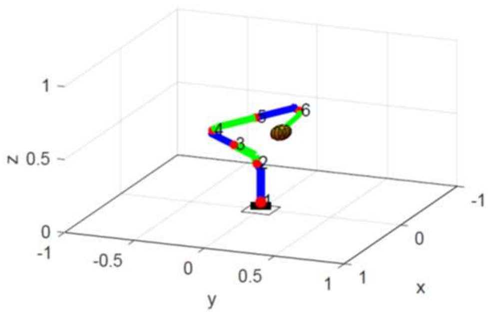
Predicted accuracy

- Standard deviation

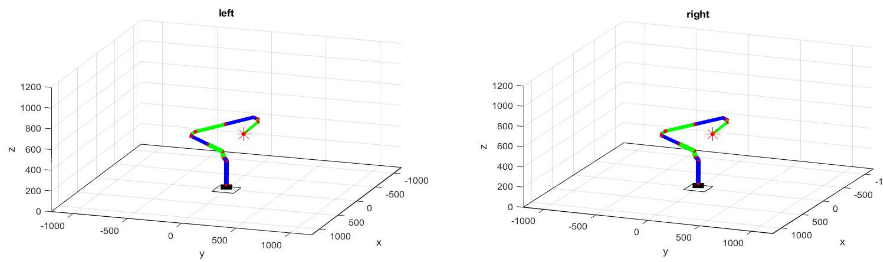
  $\sigma_x = 0.036$ mm

- Condition ($\kappa = a^2/c^2$)

  $\sqrt{\kappa} = 1.48$



Example 2, parallel eye stereo (LR)



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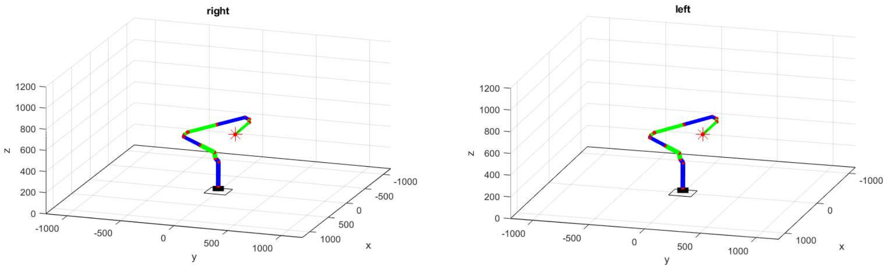
Example 2, cross eye stereo (RL)



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Modeling robot arms with kinematic chains

Summary

- Denavit-Hartenberg parametrization
  four parameters per link, one observable
- Variance propagation through kinematic chains

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Next lecture

5. Estimating Parameters of Spatial Transformations

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References of video series

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