Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

7. Spatial Motions and Similarities

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The slides have been created by Wolfgang Förstner.

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Topics and notation

Topics
- 3D motions
- 3D similarities

Notation
- Non-homogeneous coordinates: 3-vector $\boldsymbol{x} = [x, y, z]^\mathsf{T}$
- Homogeneous coordinates: 4-vector $\mathbf{x} = [u, v, w, t]^\mathsf{T}$
- Motions $\mathcal{M}$
- Similarities (german: Ähnlichkeiten) $\mathcal{A}$

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4.1 Spatial similarities

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Spatial similarity: definition

Spatial similarity of 3D non-homogeneous coordinates

$$\mathcal{A}(R, \boldsymbol{t}, \lambda): \quad \boldsymbol{x}' = \lambda R \boldsymbol{x} + \boldsymbol{t}$$

- Scaling with $\mathcal{D}(\lambda)$
- Rotation with $\mathcal{R}(R)$
- Translation with $\mathcal{T}(\boldsymbol{t})$

Concatenation using homogeneous coordinates →

For this video see Förstner/Wrobel (2016), p. 209-211, 383-384,

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Homogeneous coordinates in 3D

For 3D point

$$\chi(\boldsymbol{x}) = \chi([x, y, z]^{\mathsf{T}})$$

homogeneous coordinates

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = t \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{with} \quad t \neq 0$$

and

$$x = \frac{u}{t} \qquad y = \frac{v}{t} \qquad z = \frac{w}{t} \quad \text{with} \quad t \neq 0$$

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Translation

Homogeneous for of translation

$$\begin{bmatrix} u' \\ v' \\ w' \\ t' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix}$$

Or compactly →

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Transformations

- Translation

$$\mathcal{T}(\boldsymbol{t}) : \chi \to \chi' : \quad \mathbf{x}' = \mathsf{T}\mathbf{x} \quad \text{with} \quad \mathsf{T} = \begin{bmatrix} I_3 & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

- Rotation

$$\mathcal{R}(R) : \chi \to \chi' : \quad \mathbf{x}' = \mathsf{R}\mathbf{x} \quad \text{with} \quad \mathsf{R} = \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

- Dilation

$$\mathcal{D}(\lambda) : \chi \to \chi' : \quad \boldsymbol{x}' = \mathsf{D}\mathbf{x} \quad \text{with} \quad \mathsf{D} = \begin{bmatrix} \lambda I_3 & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

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Spatial similarity

Spatial similarity in homogeneous coordinates

$$\mathcal{A}(R, \boldsymbol{t}, \lambda) : \chi \to \chi' : \quad \mathbf{x}' = \mathsf{A}\mathbf{x} = \begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x}$$

Since: 1. Scaling, 2. rotation, 3. translation

$$\mathsf{A} = \mathsf{TRD} = \begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} = \begin{bmatrix} I_3 & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \begin{bmatrix} \lambda I_3 & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

Remark: Scaling and rotation may be exchanged

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Inverse similarity

Inverse similarity with homogeneous coordinates

$$\mathcal{A}^{-1}(R, \boldsymbol{t}, \lambda): \quad \mathsf{A}^{-1} = \begin{bmatrix} \frac{1}{\lambda} R^{\mathsf{T}} & -\frac{1}{\lambda} R^{\mathsf{T}} \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

3D similarities

Properties

- build continuous group
- have 7 degrees of freedom
  (d.o.f., minimal number of parameters)
  3 rotations, 3 translation, 1 scale
- Free choice of representing rotation
  → allows to use more than 3 parameters
  → not independent parameters
  → take care of when estimating similarities

4.2 Spatial motions

Spatial motion

3D motion for nonhomogeneous coordinates

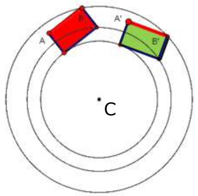
$$\mathcal{M}(R, \boldsymbol{t}): \quad \boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{t}$$

For homogeneous coordinates

$$\mathcal{M}(R, \boldsymbol{t}): \quad \mathbf{x}' = \mathsf{M}\mathbf{x} = \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x}$$

- build a continuous group
  special 3D Euclidean group $\mathrm{SE}(3)$
- 6 degrees of freedom (3 rotations, 3 translations)

Spatial motion as screw motion



Each 3D motion is a screw motion
=
1. Direction $\boldsymbol{r}$ of rotation axis (2)
2. In plane perpendicular to
   planar motion = rotation around point (3)
3. Translation by d in direction of $\boldsymbol{r}$ (1)

→ View along $\boldsymbol{r}$: rotation around special axis C
different depths of red and green object

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Spatial motion as screw motion

Screw motion w.r.t z-axis $[0,0,1]^\mathsf{T}$

$$\mathcal{M}_0 : \quad M_0(\theta, d) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Given screw axis $\mathcal{L}$, screw rotation and translation →
1. Rotation (2) + translation (2): $\mathcal{L}$ → z-axis
2. Screw $\mathcal{M}_0$ w.r.t. z-axis (2)
3. Undo 1.

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4.3 Small motions and similarities

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Parameters of motions and similarities

Similarity: 7 parameters (rotation, translation, scale)

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{t} \\ \lambda \end{bmatrix}$$

Mapping

$$\mathcal{A}(\boldsymbol{p}) : \chi \to \chi' \qquad \mathbf{x}' = \mathsf{A}(\boldsymbol{p})\mathbf{x}$$

$$\mathsf{A}(\boldsymbol{p}) = \mathsf{A}(\boldsymbol{r}, \boldsymbol{t}, \lambda) = \begin{bmatrix} \lambda R(\boldsymbol{r}) & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

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Parameters of motions and similarities

Motion: 6 parameters (rotation, translation)

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{t} \end{bmatrix}$$

Mapping

$$\mathcal{M}(\boldsymbol{s}) : \mathcal{X} \to \mathcal{X}' \qquad \mathbf{x}' = \mathsf{M}(\boldsymbol{s})\mathbf{x}$$

$$\mathsf{M}(\boldsymbol{s}) = \mathsf{M}(\boldsymbol{r}, \boldsymbol{t}) = \begin{bmatrix} R(\boldsymbol{r}) & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

Small similarity

Small changes around $\mathsf{A}^{(0)} = I_4$, equivalent to

$$\boldsymbol{p}^{(0)} = \begin{bmatrix} \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times1} \\ 1 \end{bmatrix} \qquad R^{(0)} = I_3 \qquad \boldsymbol{t}^{(0)} = \mathbf{0}_{3\times1} \qquad \lambda^{(0)} = 1$$

Small similarity with parameters $\mathrm{d}\boldsymbol{p}$

$$\mathsf{A}(\mathrm{d}\boldsymbol{p}) = \mathsf{T}(\mathrm{d}\boldsymbol{t})\ \mathsf{R}(\mathrm{d}\boldsymbol{r})\ \mathsf{D}(1 + \mathrm{d}\lambda)$$

Useful for obtaining proper similarity from parameters

Linearizing similarity at $\mathsf{A}^{(0)} = I_4$

Up to first order terms

$$\mathsf{A}(\mathrm{d}\boldsymbol{p}) = \mathsf{A}^{(0)} + \mathrm{d}\mathsf{A}_0(\mathrm{d}\boldsymbol{p})$$

with

$$\mathrm{d}R_0(\mathrm{d}\boldsymbol{r}) = S(\mathrm{d}\boldsymbol{r})$$

we have

$$\begin{aligned} \mathsf{A}(\mathrm{d}\boldsymbol{p}) &\approx \begin{bmatrix} (1 + \mathrm{d}\lambda)(I_3 + \mathrm{d}R_0(\mathrm{d}\boldsymbol{r})) & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \\ &\approx \begin{bmatrix} I_3 + \mathrm{d}\lambda I_3 + S_{\mathrm{d}\boldsymbol{r}} & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \end{aligned}$$

Linearizing similarity at $\mathsf{A}^{(0)} = I_4$

Differential change

$$\begin{aligned} \mathrm{d}\mathsf{A}_0(\mathrm{d}\boldsymbol{r}, \mathrm{d}\boldsymbol{t}, \mathrm{d}\lambda) &= \begin{bmatrix} S_{\mathrm{d}\boldsymbol{r}} & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 0 \end{bmatrix} + \begin{bmatrix} \mathrm{d}\lambda I_3 & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{d}\lambda I_3 + S_{\mathrm{d}\boldsymbol{r}} & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{d}\lambda & -\mathrm{d}r_3 & \mathrm{d}r_2 & \mathrm{d}t_1 \\ \mathrm{d}r_3 & \mathrm{d}\lambda & -\mathrm{d}r_1 & \mathrm{d}t_2 \\ -\mathrm{d}r_2 & \mathrm{d}r_1 & \mathrm{d}\lambda & \mathrm{d}t_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Linearizing motion at $\mathsf{M}^{(0)} = I_4$

Differential change

$$\begin{aligned} \mathrm{dM}_0(\mathrm{d}\boldsymbol{r}, \mathrm{d}\boldsymbol{t}) &= \begin{bmatrix} S(\mathrm{d}\boldsymbol{r}) & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\mathrm{d}r_3 & \mathrm{d}r_2 & \mathrm{d}t_1 \\ \mathrm{d}r_3 & 0 & -\mathrm{d}r_1 & \mathrm{d}t_2 \\ -\mathrm{d}r_2 & \mathrm{d}r_1 & 0 & \mathrm{d}t_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Linearizing at arbitrary point

Linearizing fully multiplicative

$$\mathsf{A} = \mathsf{A}(\mathrm{d}\boldsymbol{p})\ \mathsf{A}^{(0)} \quad \text{with} \quad \mathrm{d}\boldsymbol{p} = \begin{bmatrix} \mathrm{d}\boldsymbol{r} \\ \mathrm{d}\boldsymbol{t} \\ \mathrm{d}\lambda \end{bmatrix}$$

with

$$\begin{aligned} \mathsf{A}(\mathrm{d}\boldsymbol{p}) &= \mathsf{T}(\mathrm{d}\boldsymbol{t})\ \mathsf{R}(\mathrm{d}\boldsymbol{r})\ \mathsf{D}(1+\mathrm{d}\lambda) \\ &\approx I_4 + \mathrm{dA}_0(\mathrm{d}\boldsymbol{p}) = \begin{bmatrix} I_3 + \mathrm{d}\lambda I_3 + S_{\mathrm{d}r} & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \end{aligned}$$

Ex.: Linearizing a similarity transformation

Assume: **uncertain point**

$$\underline{\boldsymbol{x}} \sim \mathcal{N}(\boldsymbol{\mu}_x, \Sigma_{xx})$$

or

$$\underline{\mathbf{x}} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{x}}, \Sigma_{\mathrm{xx}}) \quad \text{with} \quad \Sigma_{\mathrm{xx}} = \begin{bmatrix} \Sigma_{xx} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}$$

Assume: **uncertain similarity**, mean $\overline{\mathsf{A}} = \mathbb{E}(\underline{\mathsf{A}})$

$$\underline{\mathsf{A}} = \mathsf{A}(\underline{\Delta \boldsymbol{p}})\ \overline{\mathsf{A}} \quad \text{with} \quad \underline{\Delta \boldsymbol{p}} \sim \mathcal{N}(\mathbf{0}, \Sigma_{\Delta p \Delta p})$$

Task: Derive covariance matrix of transformed point

$$\underline{\mathbf{x}}' = \underline{\mathsf{A}}\ \underline{\mathbf{x}}\,.$$

Linearizing a similarity

Total differential of similarity transformation

$$\mathrm{d}\mathbf{x}' = \mathrm{dA}\ \mathbf{x} + \mathsf{A}\ \mathrm{d}\mathbf{x}$$

or with $\mathsf{A} = \mathsf{A}(\mathrm{d}\boldsymbol{p})\ \mathsf{A}^{(0)}$, thus $\mathrm{dA} = \mathrm{dA}_0(\mathrm{d}\boldsymbol{p})\ \ \mathsf{A}|_{\mathsf{A}=\mathsf{A}^{(0)}}$

$$\mathrm{d}\mathbf{x}' = \mathrm{dA}_0(\mathrm{d}\boldsymbol{p})\ \mathsf{A}\ \mathbf{x} + \mathsf{A}\ \mathrm{d}\mathbf{x}$$

depending on 7-vector $\mathrm{d}\boldsymbol{p}$

$$\mathrm{d}\mathbf{x}' = \underset{4\times 7}{J_{\mathrm{x}'\Delta p}}\ \mathrm{d}\boldsymbol{p} + \underset{4\times 4}{J_{\mathrm{x}'\mathrm{x}}}\ \mathrm{d}\mathbf{x}$$

Linearizing a similarity

Jacobian w.r.t. coordinates

$$\underset{4\times 4}{J_{\mathrm{x'x}}} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} = \mathsf{A}\Big|_{\mathrm{A}^{(0)}}$$

evaluated at the current approximate values

Linearizing a similarity

For derivative w.r.t. $\Delta \boldsymbol{p}$ we rearrange

$$\begin{aligned}
\mathrm{d}\mathbf{x}' &= \mathrm{d}\mathsf{A}_0(\mathrm{d}\boldsymbol{p})\ \mathsf{A}\ \mathbf{x} \\
&= \begin{bmatrix} \mathrm{d}\lambda I_3 + S(\mathrm{d}\boldsymbol{r}) & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \underbrace{\mathsf{A}\mathbf{x}}_{=\mathrm{x}'} \\
&= \begin{bmatrix} \mathrm{d}\lambda I_3 + S(\mathrm{d}\boldsymbol{r}) & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}' \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -S(\boldsymbol{x}') & I_3 & \boldsymbol{x}' \\ \mathbf{0}^\mathsf{T} & \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}\boldsymbol{r} \\ \mathrm{d}\boldsymbol{t} \\ \mathrm{d}\lambda \end{bmatrix}
\end{aligned}$$

Linearizing a similarity

Jacobian w.r.t parameters $\Delta \boldsymbol{p}$

$$\begin{aligned}
\underset{4\times 7}{J_{\mathrm{x}'\Delta p}} &= \frac{\partial \mathbf{x}'}{\partial(\Delta \boldsymbol{p})}\bigg|_{x=x^{(0)}} \\
&= \begin{bmatrix} -S(\boldsymbol{x}') & I_3 & \boldsymbol{x}' \\ \mathbf{0}^\mathsf{T} & \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}\bigg|_{x=x^{(0)}} \\
&= \begin{bmatrix} 0 & x_3' & -x_2' & 1 & 0 & 0 & x_1' \\ -x_3' & 0 & x_1' & 0 & 1 & 0 & x_2' \\ x_2' & -x_1' & 0 & 0 & 0 & 1 & x_3' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\Bigg|_{x=x^{(0)}}
\end{aligned}$$

Covariance matrix of transformed point

- Transformation

$$\underline{\mathbf{x}}' = \underline{\mathsf{A}}\ \underline{\mathbf{x}} \quad \text{with} \quad \underline{\mathsf{A}} \sim \mathcal{N}\left(\overline{\mathsf{A}}, \Sigma_{\Delta p \Delta p}\right) \quad \text{and} \quad \underline{\mathbf{x}} \sim \mathcal{N}\left(\overline{\mathbf{x}}, \Sigma_{\mathrm{xx}}\right)$$

- Linearization

$$\mathrm{d}\mathbf{x}' = J_{\mathrm{x}'\Delta p}\ \mathrm{d}\boldsymbol{p} + J_{\mathrm{x}'\mathrm{x}}\ \mathrm{d}\mathbf{x}$$

- Variance propagation

$$\Sigma_{\mathrm{x}'\mathrm{x}'} = J_{\mathrm{x}'p}\, \Sigma_{\Delta p \Delta p}\, J_{\mathrm{x}'p}^\mathsf{T} + J_{\mathrm{x}'\mathrm{x}}\, \Sigma_{\mathrm{xx}}\, J_{\mathrm{x}'\mathrm{x}}^\mathsf{T}$$

Options for linearizing at arbitrary point

Two options for linearization. Here: motions

1. Fully multiplicative (see above), parameters

$$\mathrm{d}\boldsymbol{s} = \begin{bmatrix} \mathrm{d}\boldsymbol{r} \\ \mathrm{d}\boldsymbol{t} \end{bmatrix}$$

update of motion

$$^{(s)}\mathsf{M} = \mathsf{M}(\mathrm{d}\boldsymbol{s})\mathsf{M}^{(0)} = \begin{bmatrix} R(\mathrm{d}\boldsymbol{r}) & \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} R^{(0)} & \boldsymbol{t}^{(0)} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\mathrm{d}\boldsymbol{r})R^{(0)} & R(\mathrm{d}\boldsymbol{r})\boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

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Options for linearizing at arbitrary point

Two options for linearization. Here: motions

2. Partially multiplicative

rotation multiplicatively, translation additively

$$^{(\zeta)}\mathsf{M} = \begin{bmatrix} R(\mathrm{d}\boldsymbol{\rho})R^{(0)} & \boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{\tau} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

other parameters

$$\mathrm{d}\boldsymbol{\zeta} = \begin{bmatrix} \mathrm{d}\boldsymbol{\rho} \\ \mathrm{d}\boldsymbol{\tau} \end{bmatrix}$$

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Comparing the two options

If approximate motion $\mathsf{M}^{(0)}$ and changed motion identical

$$\begin{bmatrix} R(\mathrm{d}\boldsymbol{r})R^{(0)} & R(\mathrm{d}\boldsymbol{s})\boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} R(\mathrm{d}\boldsymbol{\rho})R^{(0)} & \boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{\tau} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

→

- Rotations: same parameters $\mathrm{d}\boldsymbol{\rho} = \mathrm{d}\boldsymbol{r}$

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Comparing the two options

- Translation: different type of correction

$$^{(s)}\boldsymbol{t} = R(\mathrm{d}\boldsymbol{r})\,\boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{t} \qquad ^{(\zeta)}\boldsymbol{t} = \boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{\tau}$$

hence from $^{(s)}\boldsymbol{t} = {}^{(\zeta)}\boldsymbol{t}$, up to second order terms

$$\mathrm{d}\boldsymbol{\tau} = (R(\mathrm{d}\boldsymbol{r}) - I_3)\boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{t} \approx S(\mathrm{d}\boldsymbol{r})\boldsymbol{t}^{(0)} + \mathrm{d}\boldsymbol{t} = \mathrm{d}\boldsymbol{t} - S(\boldsymbol{t}^{(0)})\,\mathrm{d}\boldsymbol{r}$$

Relations between $\mathrm{d}\boldsymbol{t}$ and $\mathrm{d}\boldsymbol{\tau}$ depending on $\boldsymbol{t}^{(0)}$

$$\mathrm{d}\boldsymbol{\tau} \approx \mathrm{d}\boldsymbol{t} - S\left(\boldsymbol{t}^{(0)}\right)\,\mathrm{d}\boldsymbol{r} \quad \text{and} \quad \mathrm{d}\boldsymbol{t} \approx \mathrm{d}\boldsymbol{\tau} + S\left(\boldsymbol{t}^{(0)}\right)\,\mathrm{d}\boldsymbol{\rho}$$

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Comparing the two options

Linearized relations between parameter vectors

$$\mathrm{d}\boldsymbol{s} = \begin{bmatrix} I_3 & 0 \\ S(\boldsymbol{t}) & I_3 \end{bmatrix} \mathrm{d}\boldsymbol{\zeta} \quad \text{und} \quad \mathrm{d}\boldsymbol{\zeta} = \begin{bmatrix} I_3 & 0 \\ -S(\boldsymbol{t}) & I_3 \end{bmatrix} \mathrm{d}\boldsymbol{s}\,.$$

→ Generally: **estimated motions are the same**

- **CovMs of parameters differ** $\Sigma_{ss} \neq \Sigma_{\zeta\zeta}$
- CovMs of rotations are identical $\Sigma_{rr} = \Sigma_{\rho\rho}$
- CovMs of translations differ $\Sigma_{tt} \neq \Sigma_{\tau\tau}$
- CovMs of rotation and translation differ $\Sigma_{rt} \neq \Sigma_{\rho\tau}$

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Linearizing a similarity

Useful for

- Variance propagation
  - Single or many points
  - Single or concatenated transformations
    → kinematic chains
- Estimation of parameters from many point pairs

Observe the difference of two linearization options!

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Next lecture

4.4 Denavit-Hartenberg parameters for robot arms and kinematic chains

An application for concatenating motions

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References of video series

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