Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

6. Small and Uncertain Rotations, Relations, Rotations from Point Pairs

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3.8 Small Rotations

Motivation

Role of small rotations

- Representing uncertain rotations
- Differential equations
- Estimating rotations if approximations are available

For this video see Förstner/Wrobel (2016), p. 337-340, 381-383

Differential rotations

Start: rotation matrix

$$R_{r,\theta} = I_3 + \sin\theta\, S_r + (1-\cos\theta)\, S_r^2$$

For small angles we have

$$\sin(\mathrm{d}\theta) \approx \mathrm{d}\theta \quad \text{and} \quad \cos(\mathrm{d}\theta) \approx 1$$

Differentially small rotation

$$\boxed{R_{r,\theta}(\mathrm{d}\theta) \approx I_3 + \mathrm{d}\theta S_r}$$

- First term of exponential series
- For small angles $\Delta\theta$ not a rotation matrix

Differential rotation matrices (1/2)

Differential rotation vector

$$d\boldsymbol{\theta} = \boldsymbol{r}\, d\theta = \begin{bmatrix} r_1 d\theta \\ r_2 d\theta \\ r_3 d\theta \end{bmatrix} = \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{bmatrix}$$

$$R(d\boldsymbol{\theta}) \approx I_3 + S_{d\theta} = \begin{bmatrix} 1 & -d\theta_3 & d\theta_2 \\ d\theta_3 & 1 & -d\theta_1 \\ -d\theta_2 & d\theta_1 & 1 \end{bmatrix}$$

$$= I_3 + S_r d\theta = I_3 + \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} d\theta$$

Differential rotation matrices (2/2)

For Euler, Rodriguez, and Cayley representation

$$d\boldsymbol{\theta} = \begin{bmatrix} d\alpha \\ d\beta \\ d\gamma \end{bmatrix} \qquad d\boldsymbol{m} = \begin{bmatrix} da \\ db \\ dc \end{bmatrix} \qquad d\boldsymbol{u} = \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}$$

$$R(d\boldsymbol{\theta}) \approx \begin{bmatrix} 1 & -d\gamma & d\beta \\ d\gamma & 1 & -d\alpha \\ -d\beta & d\alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & -dc & db \\ dc & 1 & -da \\ -db & da & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2}dw & \frac{1}{2}dv \\ \frac{1}{2}dw & 1 & -\frac{1}{2}du \\ -\frac{1}{2}dv & \frac{1}{2}du & 1 \end{bmatrix}$$

Differential rotation vectors

For differential angles, equivalence to Rodriguez

$$d\boldsymbol{\theta} = d\boldsymbol{\alpha} = d\boldsymbol{m} = \boldsymbol{r}\, d\theta \quad \text{or} \quad \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} d\alpha \\ d\beta \\ d\gamma \end{bmatrix} = \begin{bmatrix} da \\ db \\ dc \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} d\theta$$

Differentially small rotation with quaternion $\equiv$ Cayley

$$\begin{bmatrix} 1 \\ dq_1 \\ dq_2 \\ dq_3 \end{bmatrix} = \begin{bmatrix} 1 \\ d\boldsymbol{q} \end{bmatrix} = \begin{bmatrix} 1 \\ d\boldsymbol{u} \end{bmatrix}$$

Differentiation of rotation matrix

Starting from

$$RR^{\mathsf{T}} = I_3$$

with the product rule we have the total differential

$$dRR^{\mathsf{T}} + RdR^{\mathsf{T}} = 0$$

hence skew matrix depending on some small vector

$$S(d\boldsymbol{r}) = dR\, R^{\mathsf{T}} = -R\, dR^{\mathsf{T}}$$

independent of representation of R, or differential

$$dR = S(d\boldsymbol{r})R$$

? Meaning of $d\boldsymbol{r}$?

Differentiation close to unit matrix

For linearization point $R^a = I_3$

$$\mathrm{d}R_0(\mathrm{d}\boldsymbol{r}) = S_{\mathrm{d}r} = \begin{bmatrix} 0 & -\mathrm{d}r_3 & \mathrm{d}r_2 \\ \mathrm{d}r_3 & 0 & -\mathrm{d}r_1 \\ -\mathrm{d}r_2 & \mathrm{d}r_1 & 0 \end{bmatrix}$$

or (see above)

$$R(\mathrm{d}\boldsymbol{r}) \approx I_3 + \mathrm{d}R_0(\mathrm{d}\boldsymbol{r}) = \begin{bmatrix} 1 & -\mathrm{d}r_3 & \mathrm{d}r_2 \\ \mathrm{d}r_3 & 1 & -\mathrm{d}r_1 \\ -\mathrm{d}r_2 & \mathrm{d}r_1 & 1 \end{bmatrix}$$

→ differential vector = differential **rotation** vector

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta\, \boldsymbol{r}$$

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Differentiation at arbitrary R^a

If $R^{(a)} \neq I_3$, assume differential **multiplicative** change

$$R = R(\mathrm{d}\boldsymbol{\theta})\; R^{(a)}$$

approximate rotation perturbed by small rotation →

$$\begin{aligned} R &\approx (I_3 + \mathrm{d}R_0(\mathrm{d}\boldsymbol{\theta}))\; R^{(a)} \\ &= R^{(a)} + S_{\mathrm{d}\theta} R^{(a)} \\ &= R^{(a)} + \mathrm{d}R_a(\mathrm{d}\boldsymbol{\theta}) \end{aligned}$$

Additive differential change (see above)

$$\mathrm{d}R_a(\mathrm{d}\boldsymbol{\theta}) = S_{\mathrm{d}\theta} R^{(a)}$$

→ $\mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta\, \boldsymbol{r}$ = differential disturbing rotation vector

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Differential equation for rotating point

Point $\boldsymbol{x}(t)$ rotating around $\boldsymbol{r}(t)$

with differential angle $\mathrm{d}\theta(t)$ during differential time $\mathrm{d}t$

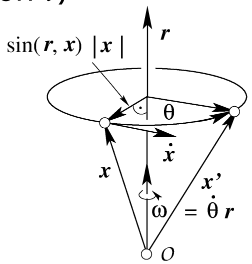
(in the figure: no dependency of $\boldsymbol{r}$ on t)

Angular velocity

$$\omega(t) := \dot{\theta}(t) = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$

Angular/rotational velocity vector

$$\boldsymbol{\omega}(t) = \omega(t)\, \boldsymbol{r}(t)$$



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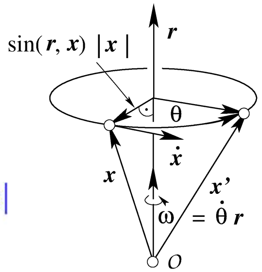
Differential equation for rotating point

Current velocity of point

$$\begin{aligned} \mathrm{d}x &= \sin(\boldsymbol{r},\boldsymbol{x})\, |\boldsymbol{x}|\;\; \mathrm{d}\theta \\ &= \sin(\boldsymbol{r},\boldsymbol{x})\, |\boldsymbol{x}|\;\; \omega\, \mathrm{d}t \end{aligned}$$

Velocity is perpendicular to $\boldsymbol{r}$ and $\boldsymbol{x}$

$$\dot{\boldsymbol{x}} = \underbrace{\sin(\boldsymbol{r},\boldsymbol{x})\; |\boldsymbol{x}|}_{|\boldsymbol{r}\times\boldsymbol{x}|}\; \underbrace{\omega \quad \boldsymbol{r}}_{\boldsymbol{\omega}} \times \boldsymbol{x} \;/\; |\boldsymbol{r} \times \boldsymbol{x}|$$



Differential equation (right hand rule)

$$\boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{x}(t)}$$

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Uncertain rotations

- Rotations may be uncertain
- Many representations are redundant
  i.e. > 3 parameters
- Degrees of freedom of 3D rotation = 3
  → rank of covariance matrix must be 3
  e.g. $(\alpha, \beta, \gamma) \rightarrow R(\alpha, \beta, \gamma)$ $\quad \underset{9\times 9}{\Sigma_{rr}} = \underset{9\times 3}{\left(\frac{\partial r}{\partial \alpha}\right)} \underset{3\times 3}{\Sigma_{\alpha\alpha}} \underset{3\times 9}{\left(\frac{\partial r}{\partial \alpha}\right)}^{\mathsf{T}}$
  → 9x9 covariance matrix of 9 elements: rank 3
- Domain of rotations bounded or repetitive

→ How to handle uncertain rotations?

Classical setup

Classical representation of uncertain entities

- Random entities (underscored)
  - First **m**oment of density: mean vector
  - Second central **m**oment: covariance matrix (two indices)

$$\underline{\boldsymbol{x}} \sim \mathcal{M}(\mathbb{E}(\underline{\boldsymbol{x}}), \mathbb{D}(\underline{\boldsymbol{x}})) = \mathcal{M}(\boldsymbol{\mu}_x, \Sigma_{xx})$$

$\mathbb{D}(\underline{\boldsymbol{x}}) = \mathbb{C}\mathrm{ov}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}})$

$\mathbb{C}\mathrm{ov}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \Sigma_{xy}$
$= \mathbb{E}((\underline{\boldsymbol{x}} - \boldsymbol{\mu}_x)(\underline{\boldsymbol{y}} - \boldsymbol{\mu}_y)^{\mathsf{T}})$

  - or

$$\boxed{\underline{\mathcal{X}} : \quad \{\boldsymbol{\mu}_x, \Sigma_{xx}\}}$$

- If this is the only information
  → plausible: Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}_x, \Sigma_{xx})$

Uncertain rotations

Classical interpretation
→model for generating the i-th random sample

$$\boldsymbol{x}_i = \mathbb{E}(\underline{\boldsymbol{x}}) + \boldsymbol{e}_i\,, \quad \underline{\boldsymbol{e}}_i \sim \mathcal{M}(\boldsymbol{0}, \Sigma_{xx})$$

With mean vector

$$\overline{\boldsymbol{x}} := \mathbb{E}(\underline{\boldsymbol{x}})$$

alternative writing

$$\boldsymbol{x}_i = \overline{\boldsymbol{x}} + \boldsymbol{e}_i\,, \quad \underline{\boldsymbol{e}}_i \sim \mathcal{M}(\boldsymbol{0}, \Sigma_{xx})$$

Uncertain rotations

Model for uncertain rotation: multiplicative ($\underline{\boldsymbol{r}} := \underline{\boldsymbol{\theta}}$)

$$\underline{R} = R(\underline{\boldsymbol{r}})\,\overline{R}\,, \quad \underline{\boldsymbol{r}} \sim \mathcal{M}(\boldsymbol{0}, \Sigma_{rr})$$

Random distortion of mean rotation matrix
Uncertainty of rotation vector = 3x3 covariance matrix

$$\boxed{\underline{\mathcal{R}} : \quad \{\overline{R}, \Sigma_{rr}\}}$$

Usually: the stochastic vector $\underline{\boldsymbol{r}} := \underline{\boldsymbol{\theta}}$ is small
Small, measured in radians, e.g. 1° = 0.017 rad

Concatenation of uncertain rotations

Given two uncertain rotations

$$\underline{\mathcal{R}}_1 : \quad \{\overline{R}_1, \Sigma_{r_1 r_1}\}, \quad \text{und} \quad \underline{\mathcal{R}}_2 : \quad \{\overline{R}_2, \Sigma_{r_2 r_2}\}$$

then uncertain concatenated rotation

$$\underline{\mathcal{R}} = \underline{\mathcal{R}}_2 \underline{\mathcal{R}}_1 : \quad \{\overline{R}, \Sigma_{rr}\}$$

with

$$\overline{R} = \overline{R}_2 \overline{R}_1 , \qquad \Sigma_{rr} = \Sigma_{r_2 r_2} + \overline{R}_2 \Sigma_{r_1 r_1} \overline{R}_2^{\mathsf{T}}$$

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Concatenation of uncertain rotations

Proof: Linearize $R = R_2 R_1$

Total differential

$$\mathrm{d}R = \mathrm{d}R_2 \; R_1 + R_2 \; \mathrm{d}R_1$$

Using $\mathrm{d}R = S(\mathrm{d}\boldsymbol{r}) R$

$$S(\mathrm{d}\boldsymbol{r}) R = S(\mathrm{d}\boldsymbol{r}_2) R_2 \;\; R_1 + R_2 \;\; S(\mathrm{d}\boldsymbol{r}_1) R_1$$

or after right multiplication with $R^{\mathsf{T}} = R_1^{\mathsf{T}} R_2^{\mathsf{T}}$

$$S(\mathrm{d}\boldsymbol{r}) = S(\mathrm{d}\boldsymbol{r}_2) + R_2 \; S(\mathrm{d}\boldsymbol{r}_1) R_2^{\mathsf{T}}$$

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Concatenation of uncertain rotations

Using (see video 4, slide 6)

$$R \; S(\boldsymbol{a}) \; R^{\mathsf{T}} = S(R \; \boldsymbol{a})$$

yields

$$S(\mathrm{d}\boldsymbol{r}) = S(\mathrm{d}\boldsymbol{r}_2) + S(R_2 \; \mathrm{d}\boldsymbol{r}_1)$$

or finally

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}\boldsymbol{r}_2 + R_2 \; \mathrm{d}\boldsymbol{r}_1$$

Remark: same setup for quaternions

$$\underline{\mathbf{q}} = \mathsf{N} \begin{bmatrix} 1 \\ \underline{\boldsymbol{u}} \end{bmatrix} \overline{\mathbf{q}}, \quad \underline{\boldsymbol{u}} \sim \mathcal{M}(\mathbf{0}, \Sigma_{uu}) \quad \rightarrow \quad \boxed{\underline{\mathcal{R}} : \quad \{\overline{\mathbf{q}}, \Sigma_{\mathrm{uu}}\}}$$

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Uncertain rotations with quaternions

… only a sketch (one of several approaches)

- Use unit quaternions = unit 4-vectors $\mathbf{q}\,, \; |\mathbf{q}| = 1$
- Use 4x4 covariance matrix $\Sigma_{\mathrm{qq}} = \mathbb{D}(\underline{\mathbf{q}})$
- Impose length constraint → $\mathrm{rk}\,(\Sigma_{\mathrm{qq}}) = 3$
- Variance propagation: exploit linearity → Jacobians

$$\mathbf{p} = \mathbf{rs} = \underbrace{\mathsf{M}_{\mathrm{r}}}_{\partial \mathrm{p}/\partial \mathrm{s}} \mathbf{s} = \underbrace{\overline{\mathsf{M}}_{\mathrm{s}}}_{\partial \mathrm{p}/\partial \mathrm{r}} \mathbf{r}$$

requires to impose length constraint

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3.9 Relation between representations

Relations between representations

- Orthonormal matrix R
- Euler angles (α, β, γ)
- Axis and angle $(\boldsymbol{r}, \theta)$
- Exponential of skew matrix $\exp(S(\boldsymbol{\theta}))$
- Quaternion $\mathbf{q}$
- Rodriguez $\boldsymbol{m}$
- Cayley $\boldsymbol{u}$

Relations between representations

	rotation vector $\boldsymbol{\theta}$	axis + angle $\begin{bmatrix} \boldsymbol{r} \\ \theta \end{bmatrix} \equiv \begin{bmatrix} -\boldsymbol{r} \\ -\theta \end{bmatrix}$	quaternion $\mathbf{q} = \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix} \equiv -\mathbf{q}$	Rodriguez $\theta \neq 180°$ $\boldsymbol{m}$	Cayley $\theta \neq 180°$ $\boldsymbol{u}$
$\boldsymbol{\theta} =$	$\boldsymbol{\theta}$	$\boldsymbol{r}\theta$	$2\mathsf{N}(\boldsymbol{q})\,\mathrm{atan2}(\lvert\boldsymbol{q}\rvert,\, q_0)$	$2\mathsf{N}(\boldsymbol{m})\,\mathrm{atan}(\lvert\boldsymbol{m}\rvert/2)$	$2\mathsf{N}(\boldsymbol{u})\,\mathrm{atan}(\lvert\boldsymbol{u}\rvert)$
$\begin{bmatrix} \boldsymbol{r} \\ \theta \end{bmatrix} =$	$\begin{bmatrix} \mathsf{N}(\boldsymbol{\theta}) \\ \lvert\boldsymbol{\theta}\rvert \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{r} \\ \theta \end{bmatrix}$	$\begin{bmatrix} \mathsf{N}(\boldsymbol{q}) \\ 2\,\mathrm{atan2}(\lvert\boldsymbol{q}\rvert,\, q_0) \end{bmatrix}$	$\begin{bmatrix} \mathsf{N}(\boldsymbol{m}) \\ 2\,\mathrm{atan}(\lvert\boldsymbol{m}\rvert/2) \end{bmatrix}$	$\begin{bmatrix} \mathsf{N}(\boldsymbol{u}) \\ 2\,\mathrm{atan}(\lvert\boldsymbol{u}\rvert) \end{bmatrix}$
$\mathbf{q} =$	$\begin{bmatrix} \cos\frac{\lvert\boldsymbol{\theta}\rvert}{2} \\ \mathsf{N}(\boldsymbol{\theta})\,\sin\frac{\lvert\boldsymbol{\theta}\rvert}{2} \end{bmatrix}$	$\begin{bmatrix} \cos\frac{\theta}{2} \\ \boldsymbol{r}\,\sin\frac{\theta}{2} \end{bmatrix}$	$\mathbf{q}$	$\begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{m} \end{bmatrix}$	$\begin{bmatrix} 1 \\ \boldsymbol{u} \end{bmatrix}$
$\boldsymbol{m} =$	$2\mathsf{N}(\boldsymbol{\theta})\,\tan\frac{\lvert\boldsymbol{\theta}\rvert}{2}$	$2\,\boldsymbol{r}\tan\frac{\theta}{2}$	$2\,\boldsymbol{q}/q_0$	$\boldsymbol{m}$	$2\,\boldsymbol{u}$
$\boldsymbol{u} =$	$\mathsf{N}(\boldsymbol{\theta})\,\tan\frac{\lvert\boldsymbol{\theta}\rvert}{2}$	$\boldsymbol{r}\tan\frac{\theta}{2}$	$\boldsymbol{q}/q_0$	$\frac{1}{2}\boldsymbol{m}$	$\boldsymbol{u}$

Relations between representations

- Axis and angle
  - Rotation vector $\boldsymbol{\theta} = \boldsymbol{r}\theta$
  - Unit quaternion $(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\boldsymbol{r})$
- Quaternion and Cayley representation $\mathbf{q} = (1, \boldsymbol{u})$

- Rotations with differential angles
  - Rodriguez parameters $\mathrm{d}\boldsymbol{\theta} = \mathrm{d}[a,\ b,\ c]^{\mathsf{T}}$
  - Euler angles $= \mathrm{d}[\alpha,\ \beta,\ \gamma]^{\mathsf{T}}$
  - Quaternions $\mathbf{q} = (1, \frac{1}{2}\mathrm{d}\boldsymbol{\theta})$

Usefulness of representations

- Rotation matrix R — exchange format
- Euler angles $\boldsymbol{\alpha}$ — interpretable for small angles
- Axis and angle $(\boldsymbol{r}, \theta)$ — interpretable
- Exponential $\exp(S(\boldsymbol{\theta}))$ — interpretable compact form
- Unit quaternions $\mathbf{q}$ — interpretable, no singularities
- Rodriguez $\boldsymbol{m}$ — no trigonometry, $\mathrm{d}\boldsymbol{m} = \mathrm{d}\boldsymbol{\theta} = \mathrm{d}\boldsymbol{\alpha}$
- Cayley $\boldsymbol{u}$ — rational in matrix $S(\boldsymbol{u})$

3.10 Rotations from Pairs of Points

Rotations from pairs of points

Given: sets of corresponding points/unit vectors

$$\{\boldsymbol{a}', \boldsymbol{b}', ...\} \quad \text{and} \quad \{\boldsymbol{a}'', \boldsymbol{b}'', ...\}$$

Known: relations, e.g.

$$\boldsymbol{a}'' = R\boldsymbol{a}'$$

Sought: rotation matrix R

Q: How many point pairs necessary? (1, 2, or 3?)

Rotations from pairs of points

Observation:

- Each point pair yields 2 constraints
- → at least two point pairs are necessary

Outline

1. 3 vector pairs (orthogonal, non-orthogonal)
2. 2 vector pairs
3. 1 vector pair (great circle)
4. Best rotation in case of perturbated data

Rotation from three orthogonal pairs

Given: two sets of corresponding orthogonal vectors

$$\{e'_1, e'_2, e'_3\} \quad \text{and} \quad \{e''_1, e''_2, e''_3\}$$

The two rotation matrices

$$R' = [e'_1, e'_2, e'_3] \quad \text{and} \quad R'' = [e''_1, e''_2, e''_3]$$

contain axis of tripods as columns

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Rotation from three orthogonal pairs

From

$$R'' = [e''_1, e''_2, e''_3] = R\,[e'_1, e'_2, e'_3] = R\,R'$$

Follows

$$\boxed{R = R''R'^{\mathsf{T}} = [e''_1, e''_2, e''_3]\begin{bmatrix} {e_1}'^{\mathsf{T}} \\ {e_2}'^{\mathsf{T}} \\ {e_3}'^{\mathsf{T}} \end{bmatrix}}$$

Remark: may include a mirroring

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Rotation from three general pairs

Given: two sets of corresponding general vectors

$$\{a', b', c'\} \quad \text{and} \quad \{a'', b'', c''\}$$

then from

$$A'' = [a'', b'', c''] = R\,[a', b', c'] = R\,A'\,.$$

we have

$$\boxed{R = A''(A')^{-1} = [a'', b'', c''][a', b', c']^{-1}}$$

Matrix A' not singular, if vectors $\{a', b', c'\}$ not coplanar
May include a mirroring

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Rotation from two vector pairs

Given: two corresponding vectors $a \nparallel b$

$$\{a', b'\} \quad \text{and} \quad \{a'', b''\}$$

We generate third vectors

$$c' = a' \times b' \quad \text{and} \quad c'' = a'' \times b''$$

and use previous result with three general vectors
Contains no mirroring!

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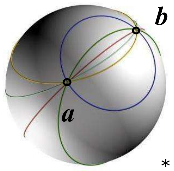
Minimal rotation from one vector pair

Given: one point pair $\boldsymbol{a}$ and $\boldsymbol{b}$

There are many rotations (circular paths)

Rotation **on great circle**

(valid in all dimensions) Weber (2003)



$$R_{ab}(\boldsymbol{a},\boldsymbol{b}) = I + 2\tilde{\boldsymbol{b}}\tilde{\boldsymbol{a}}^{\mathsf{T}} - \frac{1}{1+\tilde{\boldsymbol{a}}\cdot\tilde{\boldsymbol{b}}}(\tilde{\boldsymbol{a}}+\tilde{\boldsymbol{b}})(\tilde{\boldsymbol{a}}+\tilde{\boldsymbol{b}})^{\mathsf{T}}$$

with normalized vectors

$$\tilde{\boldsymbol{a}} = \mathrm{N}(\boldsymbol{a}) \qquad \tilde{\boldsymbol{b}} = \mathrm{N}(\boldsymbol{b})$$

Rotation axis $\boldsymbol{a}\times\boldsymbol{b}$, since $R_{ab}\,(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\times\boldsymbol{b}$

* https://www.quora.com/On-the-surface-of-the-sphere-from-one-point-to-the-other-is-it-different-in-different-directions

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Best rotation matrix

Problem:

Observations are noisy

→ estimated matrix only approximates rotation matrix

→ derive best rotation matrix R from matrix Q

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Best rotation matrix

Theorem: Best approximating orthogonal matrix

For an arbitrary regular matrix Q with the singular value decomposition (SVD)

$$Q = USV^{\mathsf{T}}$$

the orthogonal matrix

$$R = UV^{\mathsf{T}}$$

minimizes the Frobenius norm of the difference

$$||R - Q||_F = \sqrt{\textstyle\sum_{ij}(r_{ij} - q_{ij})^2}$$

Arun (1987)

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Best rotation matrix

Comments:

- The SVD $Q = USV^{\mathsf{T}}$ is the product of
  - two orthogonal matrices U and V and
  - a diagonal matrix S with positive entries s_i
- We have for the determinants

$$\mathrm{sign}|Q| = \mathrm{sign}(|U|\,|S|\,|V^{\mathsf{T}}|) \overset{|S|>0}{=} \mathrm{sign}(|U||V^{\mathsf{T}}|) = \mathrm{sign}(|R|)$$

- If $|Q| > 0$ then R is a proper rotation matrix otherwise it contains a mirroring

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Rotation from > 3 pairs

See the example: Cayley representation in last lecture

Faster algorithm in context of motion estimation

Rotations in 3D

Summary

- Multiple representations → choose an adequate one
- Uncertainty representation = CovM of 3 parameters
- Estimation of rotation matrix from pairs of directions

Next lecture

4. Spatial motions and similarities

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