Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

5a. Rotations
- as two Reflections using Quaternions

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The slides have been created by Wolfgang Förstner.

Motivation

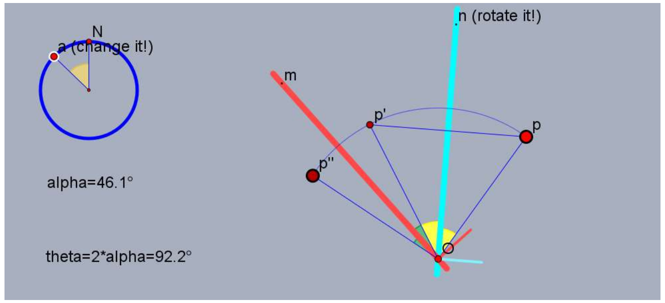
Previous lectures proved form of rotation matrix
- Axis angle representation
- Quaternion representation

Here: We construct both
- Using geometric insight
- Using basic rules of quaternions

Derivation following H. S. M. Coxeter (1946):
Quaternions and Reflections,
The American Mathematical Monthly, 53(3), 136-146

Rotation as two reflections

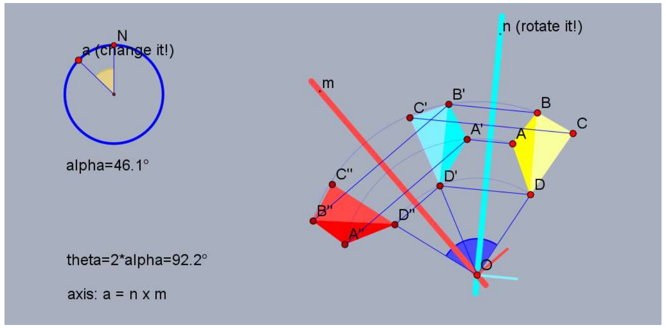
A rotation can be represented by two reflections



Graphics using Cinderella

Rotation as two reflections

A rotation can be represented by two reflections



Graphics using Cinderella

Rotation as two reflections

... in 2D

- Rotation angle $\theta = 2\,\alpha$, angle between lines
- Angle between lines may be restricted to $[-90°, +90°]$
- Jointly rotating the lines → fixed rotation

... in 3D

- Rotation angle $\theta = 2\,\alpha$, angle between planes
- Angle between lines may be restricted to $[-90°, +90°]$
- Rotation axis = intersection line of planes
- Jointly rotating planes around axis → fixed rotation

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Quaternions

Here, quaternions are pairs with scalar and vector

$$\mathbf{q} = (q, \boldsymbol{q})$$

Addition and multiplication (not commutative)

$$\mathbf{r} + \mathbf{p} = (r + p, \boldsymbol{r} + \boldsymbol{p}) \quad \text{and} \quad \mathbf{r}\mathbf{p} = (rp - \boldsymbol{r}\cdot\boldsymbol{p}, r\boldsymbol{p} + p\boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{p})$$

Conjugation and norm

$$\overline{\mathbf{q}} = (q, -\boldsymbol{q}) \quad \text{and} \quad |\mathbf{q}| = \sqrt{\mathbf{q}\overline{\mathbf{q}}} = \sqrt{q^2 + \boldsymbol{q}\cdot\boldsymbol{q}}$$

Inverse

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}$$

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Special quaternions

- One
$$\mathbf{1} = (1, \mathbf{0}) = 1$$
- Unit quaternion
$$\mathbf{q} \quad \text{with} \quad |\mathbf{q}| = 1$$
- Pure quaternion (used regularly in the following)
$$\mathbf{q} = (0, \boldsymbol{q})$$

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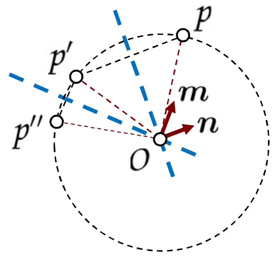
Rotation as two reflections

Quaternions to represent reflections and rotations

1. Reflection of $p(0, \boldsymbol{p})$ at plane with normal $\mathbf{n} = (0, \boldsymbol{n})$
$$p \mapsto p' : \quad \mathbf{p}' = \mathbf{n}\,\mathbf{p}\,\mathbf{n}$$
2. Concatenation of reflections $\{\boldsymbol{n}, \boldsymbol{m}\}$
   → yields rotation
$$\begin{aligned} p \mapsto p'' : \quad \mathbf{p}'' &= \mathbf{q}\,\mathbf{p}\,\overline{\mathbf{q}} \\ \text{with} \quad \mathbf{q} &= -\mathbf{m}\,\mathbf{n} \end{aligned}$$



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Some rules for *pure* quaternions

Product of quaternions	$\mathbf{x}\,\mathbf{y} = (-\boldsymbol{x}\cdot\boldsymbol{y}, \boldsymbol{x}\times\boldsymbol{y})$
Square $\mathbf{x}^2 = \mathbf{x}\,\mathbf{x}$ of quaternion	$\mathbf{x}^2 = -\lvert\boldsymbol{x}\rvert^2$
Cube of quaternion	$\mathbf{x}^3 = -\lvert\boldsymbol{x}\rvert^2\,\mathbf{x}$
Product of conjugates	$\overline{\mathbf{x}}\,\overline{\mathbf{y}} = \mathbf{x}\,\mathbf{y}$
Reverse product	$\mathbf{y}\,\mathbf{x} = \overline{\mathbf{x}\,\mathbf{y}}$
Product of orthogonal quaternions	$0 = \mathbf{x}\,\mathbf{z} + \mathbf{z}\,\mathbf{x}$
Square, cube of unit quaternion	$\mathbf{x}^2 = -1 \quad \mathbf{x}^3 = -\mathbf{x}$

Product of two unit quaternions with angle α

$$\mathbf{x}\,\mathbf{y} = (-\cos\alpha,\, \sin\alpha\,\mathsf{N}(\boldsymbol{x}\times\boldsymbol{y})) \quad \text{with} \quad \lvert\mathbf{x}\,\mathbf{y}\rvert = 1$$

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Reflection at a plane

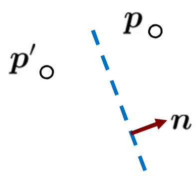
Given:

1. Point $p(\boldsymbol{p})$ with pure quaternion

$$\mathbf{p} = (0, \boldsymbol{p}) \quad \text{with} \quad \boldsymbol{p} = [x, y, z]^{\mathrm{T}}$$

2. Plane through origin
   with normalized normal $\boldsymbol{n}$
   pure unit quaternion

$$\mathbf{n} = (0, \boldsymbol{n}) \quad \text{with} \quad \mathbf{n}\overline{\mathbf{n}} = 1$$



Task: Determine reflected point $p'(\boldsymbol{p}')$

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Point f in a plane

Constraint for $\mathbf{f} = (0, \boldsymbol{f})$

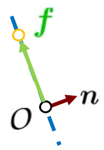
$$\mathbf{f}\mathbf{n} + \mathbf{n}\mathbf{f} = 0$$

Constraint for unit quaternion

$$\mathbf{n}^2 = -1$$

Multiplication with $\mathbf{n}$ → constraint

$$\mathbf{f} = \mathbf{n}\,\mathbf{f}\,\mathbf{n}$$



Interprete constraint as mapping

$$\mathbf{f} \mapsto \mathbf{n}\,\mathbf{f}\,\mathbf{n}$$

All points f on plane are **fixed points** of this mapping

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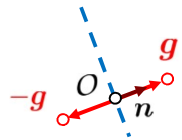
Point g on line of normal

Point $\mathbf{g} = (0, \boldsymbol{g}) = \mu\,\mathbf{n}$ maps to opposite point

$$\mathbf{g} \mapsto -\mathbf{g}$$



Since

$$\mu\mathbf{n} \mapsto \mathbf{n}\;\mu\mathbf{n}\;\mathbf{n} = \mu\mathbf{n}^3 = -\mu\mathbf{n}$$

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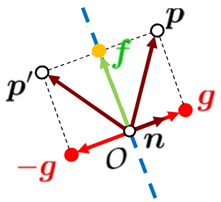
Reflection in a plane

Reflection

$$\boxed{p \mapsto p' : \quad \mathbf{p}' = \mathbf{n}\,\mathbf{p}\,\mathbf{n}}$$



since **general point** $\mathbf{p} = \mathbf{f} + \mathbf{g}$ maps to

$$\mathbf{p} = \mathbf{f} + \mathbf{g} \;\mapsto\; \mathbf{f} - \mathbf{g}$$

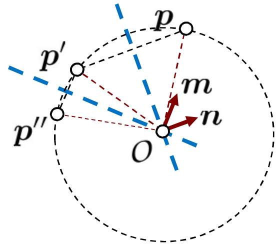
reflected point

Rotation as two reflections

Two reflections at planes with normals $\boldsymbol{n}$ and $\boldsymbol{m}$

1. Reflection

$$\mathbf{p}' = \mathbf{n}\,\mathbf{p}\,\mathbf{n}$$

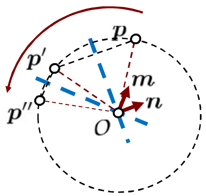


2. Reflection

$$\mathbf{p}'' = \mathbf{m}\,\mathbf{p}'\,\mathbf{m} = \mathbf{m}\,\mathbf{n}\;\mathbf{p}\;\mathbf{n}\,\mathbf{m}$$

= Rotation (by geometric insight, see demo)

Rotation as two reflections



Rotation with two reflections

$$\mathbf{p}'' = \mathbf{m}\,\mathbf{n}\;\mathbf{p}\;\mathbf{n}\,\mathbf{m}$$

Only depending on product $\mathbf{m}\,\mathbf{n}$

Define (such that scalar part is **positive** dot product)

$$\mathbf{q} = -\mathbf{m}\,\mathbf{n} = (q, \boldsymbol{q}) = (\boldsymbol{n}\cdot\boldsymbol{m}, \boldsymbol{n} \times \boldsymbol{m})$$
$$\overline{\mathbf{q}} = -\overline{\mathbf{m}\,\mathbf{n}} = -\mathbf{n}\,\mathbf{m}$$

Axis parallel to $\boldsymbol{n} \times \boldsymbol{m}$

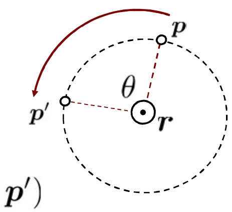
Unit quaternion

$$\mathbf{q} = (q, \boldsymbol{q}) = -\mathbf{m}\,\mathbf{n} = (\cos\alpha,\ \sin\alpha\, \mathsf{N}(\mathbf{n} \times \mathbf{m}))$$

Rotation with unit quaternion

General rotation $p \mapsto p'$

$$\boxed{\mathcal{R}(\mathbf{q}) : \qquad \mathbf{p}' = \mathbf{q}\,\mathbf{p}\,\overline{\mathbf{q}} \quad \text{with} \quad |\mathbf{q}| = 1}$$



Pure quaternions $\mathbf{p} = (0, \boldsymbol{p})$ and $\mathbf{p}' = (0, \boldsymbol{p}')$

Rotation angle $\theta = 2\alpha$ and direction of axis

$$\cos\tfrac{\theta}{2} = q \quad \text{and} \quad \boldsymbol{r} = \mathsf{N}(\boldsymbol{q})$$

Rotation with unit quaternion

$$\boxed{\mathbf{q} = \left(\cos\tfrac{\theta}{2},\ \sin\tfrac{\theta}{2}\boldsymbol{r}\right)}$$

Rotation with quaternions

Rotation with **unit quaternion**

$$\mathbf{p}' = \mathbf{q}\,\mathbf{p}\,\overline{\mathbf{q}} \quad \text{with} \quad |\mathbf{q}| = 1$$

Rotation with **general quaternion** (not length 1)

$$\boxed{\mathcal{R}(\mathbf{q}): \qquad \mathbf{p}' = \mathbf{q}\,\mathbf{p}\,\mathbf{q}^{-1}}$$

since

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}$$

Arbitrary scaling of quaternion (homogeneous wrt $\mathcal{R}$)

$$\mathcal{R}(\mathbf{q}) = \mathcal{R}(\lambda\mathbf{q}) \quad \text{especially} \quad \mathcal{R}(\mathbf{q}) = \mathcal{R}(-\mathbf{q})$$

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Rotation matrix from quaternions

Rotation

$$\mathbf{p}' = \mathbf{q}\,\mathbf{p}\,\mathbf{q}^{-1}$$

written as matrix vector product

$$\boldsymbol{p}' = R_Q(\mathbf{q})\,\boldsymbol{p}$$

What is $R_Q(\mathbf{q})$?

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Rotation matrix from quaternions

We rearrange and use* $\boldsymbol{x}\boldsymbol{x}^\mathsf{T} = S_x^2 + |\boldsymbol{x}|^2 I_3,\ |\mathbf{q}| = 1$

$$\begin{aligned}
\mathbf{q}\mathbf{p}\overline{\mathbf{q}} &= (q, \boldsymbol{q})\,(0, \boldsymbol{p})\,(q, -\boldsymbol{q}) \\
&= (q, \boldsymbol{q})\,(\boldsymbol{p}\cdot\boldsymbol{q}, q\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{q}) \\
&= (q\boldsymbol{p}\cdot\boldsymbol{q} - \boldsymbol{q}\cdot(q\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{q}), \\
&\qquad q(q\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{q}) + \boldsymbol{p}\cdot\boldsymbol{q}\boldsymbol{q} + \boldsymbol{q}\times(q\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{q})) \\
&= (0, (q^2 I_3 + qS_q + \boldsymbol{q}\boldsymbol{q}^\mathsf{T} + qS_q + S_q^2)\,\boldsymbol{p}) \\
&= (0, (q^2 I_3 + 2qS_q + \boldsymbol{q}\boldsymbol{q}^\mathsf{T} + S_q^2)\,\boldsymbol{p}) \\
&\overset{*}{=} (0, (I_3 + 2qS_q + 2S_q^2)\,\boldsymbol{p})
\end{aligned}$$

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Rotation matrix from quaternions

Rotation matrix, assuming general quaternion

$$R_Q(\mathbf{q}) = \frac{1}{|\mathbf{q}|^2}(I_3 + 2qS_q + 2S_q^2)$$

or

$$R_Q(\mathbf{q}) = \frac{1}{|\mathbf{q}|^2}\left((q^2 - \boldsymbol{q}^\mathsf{T}\boldsymbol{q})\,I_3 + 2\,\boldsymbol{q}\boldsymbol{q}^\mathsf{T} + 2\,q\,S_q\right)$$

Explicitly, using $q_0 = q,\ \boldsymbol{q} = [q_1, q_2, q_3]^\mathsf{T}$

$$\boxed{R_Q = \frac{1}{|\mathbf{q}|^2}\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\,(q_1q_2 - q_0q_3) & 2\,(q_1q_3 + q_0q_2) \\ 2\,(q_2q_1 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\,(q_2q_3 - q_0q_1) \\ 2\,(q_3q_1 - q_0q_2) & 2\,(q_3q_2 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}}$$

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Rotation matrix from quaternions

Rotation matrix, assuming unit quaternion

$$R_Q(\boldsymbol{q}) = I_3 + 2qS_q + 2S_q^2 \quad \text{with} \quad |\mathbf{q}| = 1$$

With

$$q = \cos\tfrac{\theta}{2}\,, \quad \boldsymbol{q} = \sin\tfrac{\theta}{2}\,\boldsymbol{r}$$

and

$$2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2} = \sin\theta\,, \quad 1 - 2\sin^2\tfrac{\theta}{2} = \cos\theta$$

we obtain **axis angle form of rotation matrix**

$$R_{r,\theta}(\boldsymbol{r},\theta) = I_3 + \sin\theta\; S_r + (1-\cos\theta)\; S_r^2$$

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Conclusions

- Rotation as two reflections
- Reflection with pure quaternions $\mathbf{p}' = \mathbf{n}\,\mathbf{p}\,\mathbf{n}$
- Rotation with unit quaternions $\mathbf{p}' = \mathbf{q}\,\mathbf{p}\,\overline{\mathbf{q}}$
- Rotation with mirroring $\mathbf{p}' = -\mathbf{q}\,\mathbf{p}\,\overline{\mathbf{q}}$
- Unit quaternion: axis $\boldsymbol{r}$, angle θ $\left(\cos\tfrac{\theta}{2},\ \sin\tfrac{\theta}{2}\boldsymbol{r}\right)$
- → axis-angle representation for rotation

by construction

Derivation following H. S. M. Coxeter (1946):
Quaternions and Reflections, The American Mathematical Monthly, 53(3), 136-146

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