Motivation

Previous lectures proved form of rotation matrix
- Axis angle representation
- Quaternion representation

Here: We construct both
- Using geometric insight
- Using basic rules of quaternions


Rotation as two reflections

A rotation can be represented by two reflections

Graphics using Cinderella
Rotation as two reflections

... in 2D
- Rotation angle $\alpha = 2\alpha$, angle between lines
- Angle between lines may be restricted to $[-90^\circ, +90^\circ]$
- Jointly rotating the lines $\rightarrow$ fixed rotation

... in 3D
- Rotation angle $\alpha = 2\alpha$, angle between planes
- Angle between lines may be restricted to $[-90^\circ, +90^\circ]$
- Rotation axis = intersection line of planes
- Jointly rotating planes around axis $\rightarrow$ fixed rotation

Quaternions

Here, quaternions are pairs with scalar and vector $\mathbf{q} = (q, \mathbf{q})$

Addition and multiplication (not commutative)
\[ r + p = (r + p, r + p) \quad \text{and} \quad rp = (rp - r \cdot p, rp + pr + r \times p) \]

Conjugation and norm
\[ \mathbf{q} \bar{\mathbf{q}} = q^2 + \mathbf{q} \cdot \mathbf{q} \]

Inverse
\[ q^{-1} = \frac{\bar{q}}{|q|^2} \]

Special quaternions

- One $1 = (1, 0) = 1$
- Unit quaternion $\mathbf{q}$ with $|\mathbf{q}| = 1$
- Pure quaternion (used regularly in the following) $\mathbf{q} = (0, \mathbf{q})$

Rotation as two reflections

Quaternions to represent reflections and rotations
1. Reflection of $p(0, p)$ at plane with normal $\mathbf{n} = (0, \mathbf{n})$
\[ p \mapsto p' : \quad p' = n \cdot p \]

2. Concatenation of reflections $\{n, m\}$ $\rightarrow$ yields rotation
\[ p \mapsto p'' : \quad p'' = q \cdot p \cdot \mathbf{q} \quad \text{with} \quad q = -m \cdot n \]
Some rules for pure quaternions

Product of quaternions \( x y = (-x \cdot y, x \times y) \)
Square \( x^2 = xx \) of quaternion \( x^2 = -|x|^2 \)
Cube of quaternion \( x^3 = -|x|^2 x \)
Product of conjugates \( \overline{xy} = xy \)
Reverse product \( yx = \overline{xy} \)
Product of orthogonal quaternions \( 0 = xz + zx \)
Square, cube of unit quaternion \( x^2 = -1 \quad x^3 = -x \)
Product of two unit quaternions with angle \( \alpha \)
\( xy = (-\cos \alpha, \sin \alpha N(x \times y)) \) with \( |xy| = 1 \)

Reflection at a plane

Given:
1. Point \( p(p) \) with pure quaternion \( p = (0, p) \) with \( p = [x, y, z]^T \)
2. Plane through origin with normalized normal \( n \)
   pure unit quaternion \( n = (0, n) \) with \( n\overline{n} = 1 \)

Task: Determine reflected point \( p'(p') \)

Point \( f \) in a plane

Constraint for \( f = (0, f) \)
\( fn + nf = 0 \)
Constraint for unit quaternion \( n^2 = -1 \)
Multiplication with \( n \rightarrow \) constraint \( f = nf n \)

Interpret constraint as mapping \( f \mapsto nf n \)
All points \( f \) on plane are fixed points of this mapping

Point \( g \) on line of normal

Point \( g = (0, g) = \mu n \) maps to opposite point \( g \mapsto -g \)
Since \( \mu n \leftrightarrow n \mu n n = \mu n^3 = -\mu n \)
**Reflection in a plane**

Reflection

\[ p \mapsto p' : \quad p' = n p n \]

since **general point** \( p = f + g \) maps to

\[ p = f + g \mapsto f - g \]

reflected point

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**Rotation as two reflections**

Two reflections at planes with normals \( n \) and \( m \)

1. Reflection

\[ p' = n p n \]

2. Reflection

\[ p'' = m p' m = m n p n m \]

= Rotation (by geometric insight, see demo)

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**Rotation with unit quaternion**

General rotation \( p \mapsto p' \)

\[ R(q) : \quad p' = q p \tilde{q} \quad \text{with} \quad |q| = 1 \]

Pure quaternions \( p = (0, p) \) and \( p' = (0, p') \)

Rotation angle \( \theta = 2\alpha \) and direction of axis

\[ \cos \frac{\theta}{2} = q \quad \text{and} \quad r = N(q) \]

Rotation with unit quaternion

\[ q = \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} r \right) \]
Rotation with quaternions

Rotation with **unit quaternion**
\[ p' = q p q^{-1} \text{ with } |q| = 1 \]

Rotation with **general quaternion** (not length 1)
\[ \mathcal{R}(q) : p' = q p q^{-1} \]

since
\[ q^{-1} = \frac{q}{|q|^2} \]

Arbitrary scaling of quaternion (homogeneous wrt \( \mathcal{R} \))
\[ \mathcal{R}(q) = \mathcal{R}(\lambda q) \text{ especially } \mathcal{R}(q) = \mathcal{R}(-q) \]

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Rotation matrix from quaternions

We rearrange and use* \[ \mathbf{x} \mathbf{x}^T = S_x^2 + |\mathbf{x}|^2 \mathbf{l}_3, \ |\mathbf{q}| = 1 \]

\[ q p q^{-1} = (q, q) (0, p) (q, -q) \]

\[ = (q, q) (p \cdot q, q p - p \times q) \]

\[ = (q \cdot p - q \cdot (q p - p \times q), \]

\[ q(p - q) + p \cdot q q + q \times (q p - p \times q)) \]

\[ = (0, (q^2 \mathbf{l}_3 + 2q \mathbf{S}_q + qq^T + q \mathbf{S}_q^2) \mathbf{p}) \]

\[ = (0, (q^2 \mathbf{l}_3 + 2q \mathbf{S}_q + qq^T + q \mathbf{S}_q^2) \mathbf{p}) \]

\[ * \]

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Rotation matrix from quaternions

Rotation matrix, assuming general quaternion
\[ R_Q(q) = \frac{1}{|q|^2} (I_3 + 2q \mathbf{S}_q + 2q^2) \]

or
\[ R_Q(q) = \frac{1}{|q|^2} \left( (q^2 - q^T q) I_3 + 2 q q^T + 2 q \mathbf{S}_q \right) \]

Explicitly, using \( q_0 = q, \mathbf{q} = [q_1, q_2, q_3]^T \)

\[ R_Q = \frac{1}{|q|^2} \left[ \begin{array}{ccc}
q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\
2(q_2 q_1 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\
2(q_0 q_1 - q_2 q_3) & 2(q_3 q_2 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2
\end{array} \right] \]
Rotation matrix from quaternions

Rotation matrix, assuming unit quaternion

\[ R_Q(q) = I_3 + 2qS_q + 2S_q^2 \quad \text{with} \quad |q| = 1 \]

With

\[ q = \cos \frac{\theta}{2}, \quad q = \sin \frac{\theta}{2} r \]

and

\[ 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} = \sin \theta, \quad 1 - 2 \sin^2 \frac{\theta}{2} = \cos \theta \]

we obtain **axis angle form of rotation matrix**

\[ R_{r,\theta}(r, \theta) = I_3 + \sin \theta \ S_r + (1 - \cos \theta) \ S_r^2 \]

Conclusions

- Rotation as two reflections
- Reflection with pure quaternions \( p' = npn \)
- Rotation with unit quaternions \( p' = qpq \)
- Rotation with mirroring \( p' = -qpq \)
- Unit quaternion: axis \( r \), angle \( \theta \)
- \( \rightarrow \) axis-angle representation for rotation by construction