Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

5. Rotations – Quaternions and Concatenation

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The slides have been created by Wolfgang Förstner.

3.6 Quaternions

Motivation

- Representations
  - Many parameters (R)
  - 3 or 4 parameters (Euler, axis angle)
    → contain trigonometric terms

→Representation with 4 parameters
  - Interpretable
  - No trigonometric terms
  - Unique except sign

Excellent visual tutorial on quaternions: https://eater.net/quaternions
For this video see Förstner/Wrobel (2016), p. 332-338

Representation of Quaternions

1. Pair of scalar and vector
   $\mathsf{q} = (q, \boldsymbol{q})$
2. 4-vector
   $$\mathsf{q} = \begin{bmatrix} q_0 \\ \hline q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q \\ \boldsymbol{q} \end{bmatrix}$$
3. Hyper-complex number
   $$\mathsf{q} = q_0 + iq_1 + jq_2 + kq_3 \quad \text{with} \quad i^2 = j^2 = k^2 = ijk = -1$$

Special quaternions

- $\mathbf{1}$-quaternion $(1, \mathbf{0})$ or $\boldsymbol{e}_1^{[4]}$ or 1
- Unit quaternion

$$q^2 + |\boldsymbol{q}|^2 = 1 \quad \text{or} \quad |\mathbf{q}| = 1 \quad \text{or} \quad q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

- $\overline{\mathbf{q}}$: conjugate quaternion (see complex numbers)

$$(q, -\boldsymbol{q}) \quad \text{or} \quad \begin{bmatrix} q \\ -\boldsymbol{q} \end{bmatrix} \quad \text{or} \quad q_0 - q_1 i - q_2 j - q_3 k$$

- Pure or vector quaternion, scalar part is 0

$$(0, \boldsymbol{q}) \quad \text{or} \quad \begin{bmatrix} 0 \\ \boldsymbol{q} \end{bmatrix} \quad \text{or} \quad q_1 i + q_2 j + q_3 k$$

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Algebra of quaternions

- Addition

$$\mathbf{p} = \mathbf{q} + \mathbf{r} \qquad (p, \boldsymbol{p}) = (q + r, \boldsymbol{q} + \boldsymbol{r})$$

$$\mathbf{p} = \mathbf{q} + \mathbf{r} \qquad \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0 + r_0 \\ q_1 + r_1 \\ q_2 + r_2 \\ q_3 + r_3 \end{bmatrix}$$

$$\mathbf{p} = \mathbf{q} + \mathbf{r} \qquad (p_0 + r_0) + (p_1 + r_1)i + (p_2 + r_2)j + (p_3 + r_3)k$$

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Algebra of quaternions

- Multiplication, not commutative

$$\mathbf{p} = \mathbf{qr} \qquad (p, \boldsymbol{p}) = (qr - \boldsymbol{q} \cdot \boldsymbol{r}, r\boldsymbol{q} + q\boldsymbol{r} + \boldsymbol{q} \times \boldsymbol{r})$$

$$\mathbf{p} = \mathbf{qr} \qquad \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 \\ q_1 r_0 + q_0 r_1 - q_3 r_2 + q_2 r_3 \\ q_2 r_0 + q_3 r_1 + q_0 r_2 - q_1 r_3 \\ q_3 r_0 - q_2 r_1 + q_1 r_2 + q_0 r_3 \end{bmatrix}$$

use $i^2 = j^2 = k^2 = ijk = -1$ → $ij = k = -ji$ etc.

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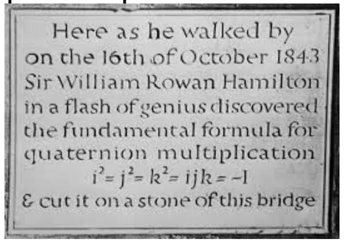
Hamilton's (1805-1865) goal

Integrate scalar and vector product

1. For pure quaternions $\mathbf{q} = (0, \boldsymbol{q})$ and $\mathbf{r} = (0, \boldsymbol{r})$

$$\mathbf{p} = \mathbf{qr} : \quad (p, \boldsymbol{p}) = (-\boldsymbol{q} \cdot \boldsymbol{r}, \boldsymbol{q} \times \boldsymbol{r}) \quad ✓$$

2. (1843) Hypercomplex numbers $\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$



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Multiplication is bilinear

Product

$$\mathbf{p} = \mathbf{qr} \qquad \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0r_0 - q_1r_1 - q_2r_2 - q_3r_3 \\ q_1r_0 + q_0r_1 - q_3r_2 + q_2r_3 \\ q_2r_0 + q_3r_1 + q_0r_2 - q_1r_3 \\ q_3r_0 - q_2r_1 + q_1r_2 + q_0r_3 \end{bmatrix} = \mathsf{T}_\mathrm{q}\mathbf{r}$$

- with

$$\mathsf{T}_\mathrm{q} = \left[\begin{array}{c|ccc} q_0 & -q_1 & -q_2 & -q_3 \\ \hline q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{array}\right] = \begin{bmatrix} q & -\boldsymbol{q}^\mathsf{T} \\ \boldsymbol{q} & qI_3 + S_q \end{bmatrix}$$

Multiplication is bilinear

Product in reverse order

$$\mathbf{p} = \mathbf{qr} \qquad \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0r_0 - q_1r_1 - q_2r_2 - q_3r_3 \\ q_1r_0 + q_0r_1 - q_3r_2 + q_2r_3 \\ q_2r_0 + q_3r_1 + q_0r_2 - q_1r_3 \\ q_3r_0 - q_2r_1 + q_1r_2 + q_0r_3 \end{bmatrix} = \overline{\mathsf{T}}_\mathrm{r}\mathbf{q}$$

- with

$$\overline{\mathsf{T}}_\mathrm{r} = \left[\begin{array}{c|ccc} r_0 & -r_1 & -r_2 & -r_3 \\ \hline r_1 & r_0 & r_3 & -r_2 \\ r_2 & -r_3 & r_0 & r_1 \\ r_3 & r_2 & -r_1 & r_0 \end{array}\right] = \begin{bmatrix} r & -\boldsymbol{r}^\mathsf{T} \\ \boldsymbol{r} & rI_3 - S_r \end{bmatrix}$$

Inverse quaternion

Inverse quaternion w.r.t. multiplication

$$\mathbf{q}^{-1} = \bar{\mathbf{q}}/|\mathbf{q}|^2$$

- conjugate quaternion

$$\bar{\mathbf{q}} = \begin{bmatrix} q \\ -\boldsymbol{q} \end{bmatrix} = \begin{bmatrix} q_0 \\ -q_1 \\ -q_2 \\ -q_3 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & -I_3 \end{bmatrix}\mathbf{q}$$

- quadratic norm

$$|\mathbf{q}|^2 = \bar{\mathbf{q}}\mathbf{q} = q^2 + \boldsymbol{q}\cdot\boldsymbol{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Properties of Multiplication Matrices

We have for quaternions and their matrix

- inverse quaternion ←→ inverse matrix

$$\mathsf{T}_{\mathrm{q}^{-1}} = \mathsf{T}_\mathrm{q}^{-1}$$

- conjugate quaternion ←→ transposed matrix

$$\mathsf{T}_{\bar{\mathrm{q}}} = \mathsf{T}_\mathrm{q}^\mathsf{T}$$

- for unit quaternions $\mathbf{e} = \mathbf{q}/|\mathbf{q}|$→ orthogonal matrices

$$\mathsf{T}_\mathrm{e}^\mathsf{T}\mathsf{T}_\mathrm{e} = I_4 \quad \text{and} \quad \overline{\mathsf{T}}_\mathrm{e}^\mathsf{T}\overline{\mathsf{T}}_\mathrm{e} = I_4$$

Rotations with Quaternions

Given: point $p(\boldsymbol{p})$, axis $\boldsymbol{r}$, angle θ
Sought: rotated point $p'(\boldsymbol{p}')$

1. Choose adequate quaternion $\mathbf{p}(\boldsymbol{p})$
2. Choose adequate quaternion $\mathbf{q}(\boldsymbol{r}, \theta)$
3. Perform double multiplication
$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$
4. Derive $\boldsymbol{p}'$ from $\mathbf{p}'$

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Double Multiplication

Expand $\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$

$$\begin{aligned}\mathbf{p}' &= \frac{1}{|\mathbf{q}|^2}\mathbf{q}\,(\mathbf{p}\bar{\mathbf{q}}) = \frac{1}{|\mathbf{q}|^2}\mathsf{T}_{\mathrm{q}}(\overline{\mathsf{T}}_{\bar{\mathrm{q}}}\mathbf{p}) \\ &= \frac{1}{|\mathbf{q}|^2}\begin{bmatrix} q & -\boldsymbol{q}^\mathsf{T} \\ \boldsymbol{q} & q I_3 + S_q \end{bmatrix}\begin{bmatrix} q & \boldsymbol{q}^\mathsf{T} \\ -\boldsymbol{q} & q I_3 + S_q \end{bmatrix}\mathbf{p}\end{aligned}$$

Yields

$$\begin{bmatrix} p' \\ \boldsymbol{p}' \end{bmatrix} = \frac{1}{|\mathbf{q}|^2}\begin{bmatrix} q^2 + \boldsymbol{q}^\mathsf{T}\boldsymbol{q} & \overbrace{q\boldsymbol{q}^\mathsf{T} - q\boldsymbol{q}^\mathsf{T} I_3}^{=0^\mathsf{T}} \\ q\boldsymbol{q} - q I_3 \boldsymbol{q} & \boldsymbol{q}\boldsymbol{q}^\mathsf{T} + (q I_3 + S_q)^2 \end{bmatrix}\begin{bmatrix} p \\ \boldsymbol{p} \end{bmatrix}$$

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Double Multiplication

or $\quad p' = p \qquad \boldsymbol{p}' = \underbrace{\frac{1}{|\mathbf{q}|^2}(\boldsymbol{q}\boldsymbol{q}^\mathsf{T} + (q I_3 + S_q)^2)}_{R}\ \boldsymbol{p}$

Using $S_q^2 = D_q - \boldsymbol{q}^\mathsf{T}\boldsymbol{q}\, I_3$ yields **rotation matrix**

$$R_Q(\mathbf{q}) = \frac{1}{|\mathbf{q}|^2}\left((q^2 - \boldsymbol{q}^\mathsf{T}\boldsymbol{q})\, I_3 + 2\, D_q + 2\, q\, S_q\right)$$

$$R_Q = \frac{1}{|\mathbf{q}|^2}\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\,(q_1 q_2 - q_0 q_3) & 2\,(q_1 q_3 + q_0 q_2) \\ 2\,(q_2 q_1 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\,(q_2 q_3 - q_0 q_1) \\ 2\,(q_3 q_1 - q_0 q_2) & 2\,(q_3 q_2 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

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Rotation Matrix $R_Q(\mathbf{q})$ with Quaternions

Proof: We compare

$$R_Q(\mathbf{q}) = \frac{1}{|\mathbf{q}|^2}\left((q^2 - \boldsymbol{q}^\mathsf{T}\boldsymbol{q})\, I_3 + 2\, D_q + 2\, q\, S_q\right)$$

with

$$R_{r,\theta}(\boldsymbol{r}, \theta) = \cos\theta\ I_3 + (1 - \cos\theta)\ D_r + \sin\theta\ S_r$$

Assume unit quaternion $\mathbf{q}$, $||\mathbf{q}|| = 1$
→ three constraints between $(\boldsymbol{r}, \theta)$ and $(q, \boldsymbol{q})$

$$\begin{aligned} \cos\theta\ I_3 &= (q^2 - |\boldsymbol{q}|^2)\ I_3 && (1) \\ (1 - \cos\theta)\ \boldsymbol{r}\boldsymbol{r}^\mathsf{T} &= 2\boldsymbol{q}\boldsymbol{q}^\mathsf{T} && (2) \\ \sin\theta\ S_r &= 2\, q\, S_q && (3) \end{aligned}$$

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Rotation Matrix $R_Q(\mathbf{q})$ with Quaternions

Vectors $\boldsymbol{q}$ and $\boldsymbol{r}$ need to be parallel

$$\boldsymbol{q} = f\boldsymbol{r}$$

Thus $$||\boldsymbol{q}||^2 = f^2$$

Hence (1) and (3) $$\cos\theta = q^2 - f^2$$ $$\sin\theta = 2\,q\,f$$

Therefore with trigonometric relations $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, $\sin 2\alpha = 2\cos\alpha\sin\alpha$

$$q = \cos\frac{\theta}{2} \quad \text{and} \quad f = \sin\frac{\theta}{2}$$

Fulfil condition (2) → constraints consistent

$$(1 - \cos\theta) = 2f^2 = 2\sin^2\frac{\theta}{2}$$

Rotation with quaternion

Choose unit quaternion

$$\mathbf{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\boldsymbol{r} \end{bmatrix} = \cos\frac{\theta}{2}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \sin\frac{\theta}{2}\begin{bmatrix}0\\r_1\\r_2\\r_3\end{bmatrix}$$

with unit vector $\boldsymbol{r}$ of rotation axis and angle θ

- Elements of quaternion can be interpreted!
- Quaternions for rotations: homogeneous entities!

$$R_Q(\mathbf{q}) = R_Q(\mu\mathbf{q}) \quad \text{with} \quad \mu \neq 0$$

Rotation with unit quaternion

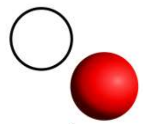
If $||\mathbf{q}|| = 1$ then the rotation matrix is

$$R_{\overline{Q}} = I_3 + 2(q\,S_q + S_q^2)$$
$$= \begin{bmatrix} 1 - 2\,(q_2^2 + q_3^2) & 2\,(q_1q_2 - q_0q_3) & 2\,(q_1q_3 + q_0q_2) \\ 2\,(q_2q_1 + q_0q_3) & 1 - 2\,(q_1^2 + q_3^2) & 2\,(q_2q_3 - q_0q_1) \\ 2\,(q_3q_1 - q_0q_2) & 2\,(q_3q_2 + q_0q_1) & 1 - 2\,(q_1^2 + q_2^2) \end{bmatrix}$$

- Bilinear in the elements of $\mathbf{q}$, quadratic constraint
- No trigonometric terms

Rotations are points on the 3-sphere

- Unit quaternions are unit 4-vectors
- They build the 3-sphere S^3 in $\mathbb{R}^4$ similar to
  - the circle S^1 in $\mathbb{R}^2$ with all unit 2-vectors and
  - the sphere S^2 in $\mathbb{R}^3$ with all unit 3-vectors



- The sphere has no border and is smooth everywhere
- No singularity
- Only sign ambiguity (for unit quaternions)

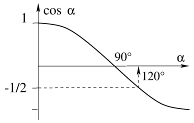
$$R_Q(\mathbf{q}) = R_Q(-\mathbf{q})$$

Examples (1,2/3)

1. Rotation matrix

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{corresponds to} \quad \mathbf{q} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

2. Rotation matrix



$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{corresponds to} \quad \mathbf{q} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Examples (3/3)

3. Rotation in the plane depending on 2-vector

$$\boldsymbol{a} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$R(\boldsymbol{a}) = \frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$= \frac{1}{|\boldsymbol{a}|^2}(aI_2 + bS_1) \quad \text{with} \quad S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

corresponds to rotation $R_Q([q_0, 0, 0, q_3])$ around z-axis

Questions to 2D Rotation

- What is the relation between (a, b) and (q_0, q_3) ?
- What is S_1^2 ?
- Compare with the rotation with complex numbers

$$p' = r\,p \quad \text{with} \quad p = x + iy \quad \text{and} \quad r = \frac{a+ib}{\sqrt{a^2+b^2}}$$

Rodriguez parameters $\boldsymbol{m}$

Rodriguez (1840) proposed rotation with 3-vector $\boldsymbol{m}$

$$\mathbf{q} = \left[1, \frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right]^\mathsf{T} = \left[1, \frac{1}{2}\boldsymbol{m}^\mathsf{T}\right]^\mathsf{T}$$

Scalar part of quaternion = 1

Equivalent to quaternion

$$\mathbf{q} = \left(1, \tan\frac{\theta}{2}\,\boldsymbol{r}\right).$$

→ Only useful if $\theta \neq 180°$, for factor ½ see later

Rodriguez parameters

Rotation matrix with

$$\boldsymbol{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$R_R(a,b,c) = \frac{1}{4+a^2+b^2+c^2} \begin{bmatrix} 4+a^2-b^2-c^2 & 2ab-4c & 2ac+4b \\ 2ab+4c & 4-a^2+b^2-c^2 & 2bc-4a \\ 2ac-4b & 2bc+4a & 4-a^2-b^2+c^2 \end{bmatrix}$$

Rotation axis and angle

$$\boldsymbol{r} = \frac{\boldsymbol{m}}{|\boldsymbol{m}|} \quad \text{and} \quad \theta = 2\arctan(\tfrac{1}{2}|\boldsymbol{m}|)$$

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Cayley Representation

With the quaternion

$$\mathbf{q} = (1, \boldsymbol{u}) = \left(1, \tan\tfrac{\theta}{2}\boldsymbol{r}\right)$$

we can show

$$I_3 + S_u = R_Q(\mathbf{q})\,(I_3 - S_u) \quad \text{and} \quad I_3 + S_u = (I_3 - S_u)\,R_Q(\mathbf{q})$$

Hence we have the Cayley transformation (of $-S_u$)

$$R_C(\boldsymbol{u}) = (I_3 + S_u)(I_3 - S_u)^{-1}$$

or

$$R_C(\boldsymbol{u}) = (I_3 - S_u)^{-1}(I_3 + S_u)$$

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Cayley transformation (optional)

Given an $n \times n$ matrix A with no eigenvalue =-1
then a Cayley transformation* is

$$B = (I - A)(I + A)^{-1} = (I + A)^{-1}(I - A)$$

with its inverse transformation

$$A = (I - B)(I + B)^{-1} = (I + B)^{-1}(I - B)$$

If A is skew, then B is a rotation matrix

* Alternatively: $B_a = (I + A)(I - A)^{-1} = B^{-1}$, $A = (B_a - I)(B_a + I)^{-1}$
(Golub/van Loan 1996, p. 73; Wu et al. (2009), Sect. 3)

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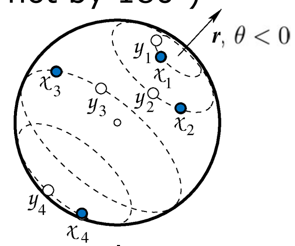
Application: Rotation from Point Pairs

Given: pairs $(x_i, y_i), i = 1, ..., I$ of corresponding points
Constraint: $\boldsymbol{y}_i = R\boldsymbol{x}_i$ (rotation not by 180°)
Sought: rotation matrix R



→Direct, non-iterative least squares solution
(more on estimating rotations from point pairs later)

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Application: Rotation from Point Pairs

Solution: use Cayley representation, starting from

$$\boldsymbol{y}_i = (I_3 - S_u)^{-1}(I_3 + S_u)\boldsymbol{x}_i$$

$$(I_3 - S_u)\boldsymbol{y}_i = (I_3 + S_u)\boldsymbol{x}_i$$

$$(S_{x_i} + S_{y_i})\boldsymbol{u} = \boldsymbol{x}_i - \boldsymbol{y}_i \quad \text{or} \quad A\boldsymbol{u} = \boldsymbol{b}$$

with $3I \times 3$ matrix $A = [S_{x_i} + S_{y_i}]$, $3I$-vector $\boldsymbol{b} = [\boldsymbol{x}_i - \boldsymbol{y}_i]$

!! linear in $\boldsymbol{u}$ → least squares solution

$$\widehat{\boldsymbol{u}} = \min_u |A\boldsymbol{u} - \boldsymbol{b}|^2 = (A^\mathsf{T} A)^{-1} A^\mathsf{T} \boldsymbol{b} \quad \text{and} \quad \widehat{R} = R_C(\widehat{\boldsymbol{u}})$$

Approximation (no use of uncertainty information)

3.7 Concatenation of Rotations

Concatenation

Concatenation of two rotations R' and R'' → rotation

$$R = R'' R'$$

If rotations are parametrized by $\boldsymbol{p}'$ and $\boldsymbol{p}''$

→how do parameters $\boldsymbol{p}$ depend on $\boldsymbol{p}'$ and $\boldsymbol{p}''$?

Euler angles: $R(\alpha, \beta, \gamma) = R(\alpha'', \beta'', \gamma'') R(\alpha', \beta', \gamma')$

No simple expression $(\alpha, \beta, \gamma) = f(\alpha', \beta', \gamma', \alpha'', \beta'', \gamma'')$

But for quaternions and Rodriguez and Cayley

Concatenation of rotations with quaternions

First rotation with $\mathbf{q}'$

$$\mathbf{p}' = \mathbf{q}' \mathbf{p} \mathbf{q}'^{-1}$$

Second rotation with $\mathbf{q}''$

$$\mathbf{p}'' = \mathbf{q}'' \mathbf{p}' \mathbf{q}''^{-1} = \mathbf{q}'' \mathbf{q}' \mathbf{p} \mathbf{q}'^{-1} \mathbf{q}''^{-1} = (\mathbf{q}'' \mathbf{q}')\, \mathbf{p}\, (\mathbf{q}'' \mathbf{q}')^{-1}$$

Thus

$$\begin{aligned} R(\mathbf{q}) &= R(\mathbf{q}'')\, R(\mathbf{q}') \\ \mathbf{q} &= \mathbf{q}'' \mathbf{q}' \\ \begin{bmatrix} q \\ \boldsymbol{q} \end{bmatrix} &= \begin{bmatrix} q'' q' - \boldsymbol{q}'' \cdot \boldsymbol{q}' \\ q' \boldsymbol{q}'' + q'' \boldsymbol{q}' + \boldsymbol{q}'' \times \boldsymbol{q}' \end{bmatrix} \end{aligned}$$

Observe: same order of multiplication

Concatenation with Rodriguez form

Rodriguez representation uses special quaternion

If

$$\boldsymbol{m}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} \quad \text{and} \quad \boldsymbol{m}'' = \begin{bmatrix} a'' \\ b'' \\ c'' \end{bmatrix}$$

then

$$\boldsymbol{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{2}{4 - a'a'' - b'b'' - c'c''} \begin{bmatrix} 2\,(a' + a'') + b'c'' - b''c' \\ 2\,(b' + b'') + c'a'' - c''a' \\ 2\,(c' + c'') + a'b'' - a''b' \end{bmatrix}$$

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Concatenation with Cayley form

Cayley representation uses special quaternion

If

$$\boldsymbol{u}' = \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}'' = \begin{bmatrix} u'' \\ v'' \\ w'' \end{bmatrix}$$

then

$$\boldsymbol{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1 - u'u'' - v'v'' - w'w''} \begin{bmatrix} u' + u'' + v'w'' - v''w' \\ v' + v'' + w'u'' - w''u' \\ w' + w'' + u'v'' - u''v' \end{bmatrix}$$

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Summary

- Quaternions are 4-vectors
- Quaternions build an algebra
- Quaternions can represent rotations
- Purely rational, fraction of bilinear forms
- Space of rotations: unit quaternions = 3-sphere
- Close relation to axis and angle representation
- Special forms: Rodriguez and Cayley representation
- Easy concatenation

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Next lecture

3.8 Small and uncertain rotations
3.9 Relations
3.10 Rotations from point pairs

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