Photogrammetry & Robotics Lab

3D Coordinate Systems

(Bsc Geodesy & Geoinformation)

4. Rotations
– Axis Angle and Exponential Representation

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The slides have been created by Wolfgang Förstner.

3.4 Axis Angle Representation

Axis-Angle Representation

Due to Euler's theorem
a rotation can be represented as
rotation around some axis with some angle.

Result: Given axis $\boldsymbol{r}$ with $||\boldsymbol{r}|| = 1$ and angle θ

$$R_{r,\theta} = I_3 + \sin\theta \; S_r + (1 - \cos\theta) \; S_r^2$$

With skew S_r matrix of direction vector $\boldsymbol{r}$ of axis

For this video see Förstner/Wrobel (2016), p. 331-332

Euler's theorem on 3D Rotations

Every rotation of a 2D sphere has a fixed point.

Proof: (see Palais & Palais (2007))
Assume skew symmetric matrix

$$A = \tfrac{1}{2}(R - R^\mathsf{T}) \quad \text{with} \quad A^\mathsf{T} = -A$$

then

$$RA = AR \quad \text{or} \quad RAR^\mathsf{T} = A$$

since

$$R \, \tfrac{1}{2}(R - R^\mathsf{T}) \, R^\mathsf{T} = \tfrac{1}{2}(R \, RR^\mathsf{T} - RR^\mathsf{T} \, R^\mathsf{T}) = \tfrac{1}{2}(R - R^\mathsf{T})$$

Lemma: Skew and rotation matrix

Generally $R(\boldsymbol{x} \times \boldsymbol{y}) = R\boldsymbol{x} \times R\boldsymbol{y}$

We use skew symmetric matrix

$$S(\boldsymbol{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \quad \text{with} \quad S(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}$$

since

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - y_2x_3 \\ x_3y_1 - y_3x_1 \\ x_1y_2 - y_1x_2 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

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Lemma: Skew and rotation matrix

Generally

$$R(\boldsymbol{x} \times \boldsymbol{y}) = R\boldsymbol{x} \times R\boldsymbol{y}$$

We have for all $\boldsymbol{x}$ and $\boldsymbol{y}$

$$RS(\boldsymbol{x})\boldsymbol{y} = S(R\boldsymbol{x})R\boldsymbol{y}$$

Hence

$$\boxed{RS(\boldsymbol{x}) = S(R\boldsymbol{x})R \quad \text{or} \quad RS(\boldsymbol{x})R^{\mathsf{T}} = S(R\boldsymbol{x})}$$

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Euler's theorem on 3D Rotations

Using $A = S(\boldsymbol{a})$

From $RA = AR$ we have

$$RS(\boldsymbol{a}) = S(\boldsymbol{a})R$$

Due to

$$RS(\boldsymbol{a}) = S(R\boldsymbol{a})R$$

we follow

$$\boldsymbol{a} = R\boldsymbol{a}$$

→ The point $\boldsymbol{a}$ is a fixed point

Case $A = 0$ needs special discussion

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Axis-Angle Representation

Result: Given axis $\boldsymbol{r}$ with $||\boldsymbol{r}|| = 1$ and angle θ then

$$\boxed{R_{r,\theta} = I_3 + \sin\theta \; S_r + (1 - \cos\theta) \; S_r^2}$$

Observe non uniqueness: $R(\boldsymbol{r}, \theta) = R(-\boldsymbol{r}, -\theta)$

→ Proof

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Proof for Axis-Angle Representation

1. Rotation axis is $\boldsymbol{r}$
2. Matrix $R_{r,\theta}$ is a rotation matrix
3. Angle of rotation is θ

We use skew and dyadic product matrix of unit vector

$$S_r = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}, \quad D_r = \boldsymbol{r}\boldsymbol{r}^\mathsf{T} = \begin{bmatrix} r_1^2 & r_1 r_2 & r_1 r_3 \\ r_2 r_1 & r_2^2 & r_2 r_3 \\ r_3 r_1 & r_3 r_2 & r_3^2 \end{bmatrix}$$

which are closely related

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Properties of S_r and D_r

$$\begin{aligned}
D_r^\mathsf{T} &= D_r \text{ (Symmetry)} \\
D_r^2 &= D_r \text{ (Idempotence, due to } \boldsymbol{r}\boldsymbol{r}^\mathsf{T}\boldsymbol{r}\boldsymbol{r}^\mathsf{T} = \boldsymbol{r}\boldsymbol{r}^\mathsf{T}) \\
S_r^\mathsf{T} &= -S_r \text{ (Antisymmetry)} \\
S_r \boldsymbol{s} &= \boldsymbol{r} \times \boldsymbol{s} = -S_s \boldsymbol{r} = -\boldsymbol{s} \times \boldsymbol{r} \text{ (cross product)} \\
S_r \boldsymbol{r} &= \boldsymbol{0} \\
S_r^2 &= -(I_3 - D_r) \qquad \text{(s. Ex.)} \\
S_r^3 &= -S_r \\
S_r^4 &= I_3 - D_r \\
S_r D_r &= 0
\end{aligned}$$

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Alternative representations

1. With skew symmetric matrices only

$$R(\boldsymbol{r}, \theta) = I_3 + \sin\theta\, S(\boldsymbol{r}) + (1 - \cos\theta)\, S^2(\boldsymbol{r})$$

2. Without squares of matrices, using $S_r^2 = -(I_3 - D_r)$

$$R(\boldsymbol{r}, \theta) = \cos\theta\, I_3 + (1 - \cos\theta)\, D(\boldsymbol{r}) + \sin\theta\, S(\boldsymbol{r})$$

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Alternative representations

3. With rotation vector (M. O. Rodriguez, 1795-1851)

$$\boldsymbol{\theta} = \boldsymbol{r}\,\theta \quad \text{and} \quad \theta = |\boldsymbol{\theta}|, \quad \boldsymbol{r} = \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}$$

$$R(\boldsymbol{\theta}) = I_3 + \frac{\sin|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|}\, S(\boldsymbol{\theta}) + \frac{1 - \cos|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|^2}\, S^2(\boldsymbol{\theta}).$$

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Properties of $R_{r,\theta}$: axis, orthogonality

Given $R_{r,\theta} = I_3 + \sin\theta\ S_r + (1-\cos\theta)\ S_r^2$

1. For $\boldsymbol{d} = \mu \boldsymbol{r}$ on rotation axis we have

$$R\boldsymbol{d} = \boldsymbol{d}$$

due to $S_r \boldsymbol{r} = \boldsymbol{r} \times \boldsymbol{r} = \boldsymbol{0}$ → all points on axes are fixed

2. The transposed is

$$R_{r,\theta}^{\mathsf{T}} = I_3 - \sin\theta\ S_r + (1-\cos\theta)\ S_r^2$$

using relations for S_r yields $RR^{\mathsf{T}} = I_3$ (s. Ex.)

Properties of $R_{r,\theta}$: angle

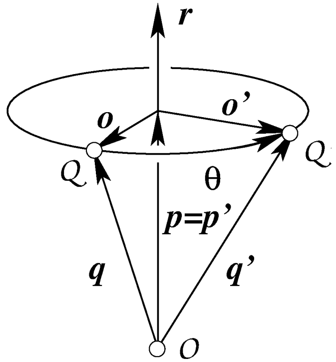
3. Rotation angle for general point

$$Q: \quad \boldsymbol{q} = \boldsymbol{p} + \boldsymbol{o}$$

parallel component $\boldsymbol{p} = \lambda \boldsymbol{r}$
orthogonal component $\boldsymbol{o}$

→We show, if $\boldsymbol{q}' = R_{r,\theta}\ \boldsymbol{q}$
then

$$\angle(\boldsymbol{o}, \boldsymbol{o}') = \theta$$



Properties of $R_{r,\theta}$: angle

We analyse

$$\begin{aligned}
\boldsymbol{q}' &= R\boldsymbol{q} = (\cos\theta\, I_3 + (1-\cos\theta)\, \boldsymbol{r}\boldsymbol{r}^{\mathsf{T}} + \sin\theta\, S(\boldsymbol{r}))(\boldsymbol{p}+\boldsymbol{o}) \\
&= \cos(\theta)\ (\boldsymbol{p}+\boldsymbol{o}) + (1-\cos\theta)\ \boldsymbol{r}\boldsymbol{r}^{\mathsf{T}}(\boldsymbol{p}+\boldsymbol{o}) + \sin(\theta)\ \boldsymbol{r}\times(\boldsymbol{p}+\boldsymbol{o}) \\
&= \cos(\theta)\ (\lambda\boldsymbol{r}+\boldsymbol{o}) + \lambda(1-\cos\theta)\boldsymbol{r} + \sin(\theta)\ \boldsymbol{r}\times\boldsymbol{o} \\
&= \underbrace{\lambda\boldsymbol{r}}_{\boldsymbol{p}'} + \underbrace{(\cos(\theta)\ \boldsymbol{o} + \sin(\theta)\ (\boldsymbol{r}\times\boldsymbol{o}))}_{\boldsymbol{o}'}
\end{aligned}$$

Hence scalar product

$$\boldsymbol{o}'\boldsymbol{\cdot}\boldsymbol{o} = (\cos(\theta)\ \boldsymbol{o} + \sin(\theta)\ (\boldsymbol{r}\times\boldsymbol{o}))\boldsymbol{\cdot}\boldsymbol{o} = \cos(\theta)\ \underbrace{\boldsymbol{o}\boldsymbol{\cdot}\boldsymbol{o}}_{|o|^2} + 0$$

→ Angle $\angle(\boldsymbol{o}, \boldsymbol{o}') = \theta$

Axis and Angle from Rotation Matrix

Given: rotation matrix R
Sought: axis $\boldsymbol{r}$ and angle $\theta \in [0, 2\pi)$

1. Angle from: $R = (r_{ij}) = \underbrace{\cos\theta\ I_3}_{\mathrm{tr}=3\cos\theta} + \underbrace{(1-\cos\theta) D_r}_{\mathrm{tr}=1-\cos\theta} + \sin\theta\ S_r$

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = 3\cos\theta + (1-\cos\theta) = 1 + 2\cos\theta$$

$$\theta = \arccos\left(\tfrac{1}{2}(\mathrm{tr}R - 1)\right)$$

… but we want to use $\mathrm{atan2}(\cdot,\cdot)$

Angle from Rotation Matrix

Skew vector $\boldsymbol{a}$ from

$$S_a = 2\sin\theta\, S_r = R - R^\mathsf{T}$$

parallel to axis

$$\boldsymbol{a} = -\begin{bmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{bmatrix} = 2\,\sin\theta\,\boldsymbol{r}$$

Angle from

$$\boxed{\theta = \text{atan2}(|\boldsymbol{a}|, \text{tr}\, R - 1)}$$

Since $|\boldsymbol{a}| = 2\sin\theta$ and $\text{tr}\, R - 1 = 2\cos\theta$

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Axis from Rotation Matrix

Cases

1. If $\theta = 0$ → axis is undetermined
2. If $\theta = \pi = 180°$ → $\sin\theta = 0$, symmetric rotation matrix

$$R = -I_3 + 2D_r \quad \text{or} \quad D_r = \frac{1}{2}(R + I_3) = \boldsymbol{r}\,\boldsymbol{r}^\mathsf{T}$$

rank $\text{rk}(D_r) = 1$,
rotation axis $\boldsymbol{r}$ is any normalized column $\neq \boldsymbol{0}$
best with largest entry
sign of $\boldsymbol{r}$ irrelevant, since rotation around 180°

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Example

Given $\boldsymbol{r} = [1, 0, 0]^\mathsf{T}$ and $\theta = 180°$

$$R = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

→ rotation axis $\boldsymbol{r}$ from 1. column of D_r

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Axis from Rotation Matrix

3. General case $\theta \notin \{0, \pi\}$

$$\boldsymbol{r} = \frac{\boldsymbol{a}}{||\boldsymbol{a}||}$$

Tests $\theta \notin \{0, \pi\}$ numerically stable

$$|\theta| < t_\theta \quad \text{and} \quad |\theta - \pi| < t_\theta$$

with threshold t_θ depending on numerical accuracy

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Examples

1. Rotation matrix

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

*

→ $\boldsymbol{a} = [-1, -1, -1]^\mathsf{T}$

→ $\theta = \text{atan2}\,(\sqrt{3}, -1) = +120°$, $\boldsymbol{r}^\mathsf{T} = -\sqrt{3}/3\ [1, 1, 1]$

2. Rotation matrix

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

→ $\boldsymbol{a} = \boldsymbol{0}$, $R + I_3 = \text{Diag}(2, 0, 0)$, 180° around x-axis

* https://hbfs.files.wordpress.com/2018/06/cube-and-hsv.gif

Summary

Rotation with axis angle representation

$$R(\boldsymbol{r}, \theta) = I_3 + \sin\theta\ S(\boldsymbol{r}) + (1 - \cos\theta)\ S^2(\boldsymbol{r})$$

$$R(\boldsymbol{\theta}) = I_3 + \frac{\sin|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|}\ S(\boldsymbol{\theta}) + \frac{1 - \cos|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|^2}\ S^2(\boldsymbol{\theta})\,.$$

Non-unique

- change of sign of $(\boldsymbol{r}, \theta)$ does not change R
- Adding $k\,2\pi$ to the angle does not change R

→ Restriction $\theta \in [0, 2\pi)$ → difficulties in estimation

3.5 Exponential Form of Rotation Matrix

Definition

Given a rotation vector

$$\boldsymbol{\theta} = \theta\boldsymbol{r} \quad \text{or} \quad \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \theta \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

the rotation matrix is

$$\boxed{R(\boldsymbol{\theta}) = \mathrm{e}^{\mathsf{S}_\theta}}$$

With

$$\mathrm{e}^{\mathsf{S}_\theta} = I_3 + S_\theta + \frac{1}{2!}S_\theta^2 + \frac{1}{3!}S_\theta^3 + \frac{1}{4!}S_\theta^4 + \ldots$$

see Förstner/Wrobel (2016), p. 236-237

Exponential Form

Proof

1. points $\boldsymbol{d} = \mu\boldsymbol{\theta}$ are fixed points → on **rotation axis**
2. Angle = ?

Collect odd and even terms using $\boldsymbol{r} = \boldsymbol{\theta}/|\boldsymbol{\theta}|$

$$S_\theta = \theta S_r \qquad S_r^3 = -S_r \qquad S_r^4 = -S_r^2$$

$$R = I_3 + \left(\theta S_r - \frac{1}{3!}\theta^3 S_r + ... \right) + \left(\frac{1}{2!}\theta^2 S_r^2 - \frac{1}{4!}\theta^4 S_r^2 + ... \right)$$

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Angle in Exponential Form

Simplifying

$$R = I_3 + \left(\theta - \frac{1}{3!}\theta^3 + ... \right) S_r + \left(\frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + ... \right) S_r^2$$

with

$$\sin x = x - \frac{1}{3!}x^3 + ... \quad \text{and} \quad \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - ...$$

→ rotation matrix

$$R = I_3 + \sin\theta\, S_r + (1 - \cos\theta) S_r^2$$

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Exponential Form

Rotation matrix:

exponential of skew matrix of rotation vector $\boldsymbol{\theta} = \theta\boldsymbol{r}$

$$\boxed{R(\boldsymbol{\theta}) = \mathrm{e}^{S(\theta)} = I_3 + \frac{\sin|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|} S(\boldsymbol{\theta}) + \frac{1 - \cos|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|^2} S^2(\boldsymbol{\theta})}$$

→Infinite sum shortened to 3 terms!

Exponential form: $R(\boldsymbol{\theta}) = \mathrm{e}^{S(\theta)}$

shortest definition of rotation matrix

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Exponential Form

Skew matrix: logarithm of rotation matrix, $\theta \neq 180°$

$$\boxed{S(\boldsymbol{\theta}) = \log R(\boldsymbol{\theta})} \qquad \boldsymbol{\theta} = \theta\boldsymbol{r}$$

see algorithm for determining axis $\boldsymbol{r}$ and angle θ

MATLAB routines `expm()` and `logm()` for matrices

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Axis angle and exponential representation

- Axis angle representation
  - Useful if rotation axis is given by hardware
  - Easy to visualize
  - Ambiguities
    - Sign: $R_{r,\theta}(\boldsymbol{r},\theta) = R_{r,\theta}(-\boldsymbol{r},-\theta)$
    - Periodicity: $R_{r,\theta}(\boldsymbol{r},\theta) = R_{r,\theta}(\boldsymbol{r},\theta + 2\pi\, k)\,,\ k \in \mathbb{Z}$
- Exponential representation
  - Shortest interpretable representation
  - Defines axis angle representation $R_{\exp}(\theta \boldsymbol{r}) = R_{r,\theta}(\boldsymbol{r},\theta)$
  - Ambiguity: $R_{\exp}(\boldsymbol{\theta}) = R_{\exp}\left((1 + 2\pi\, k/|\boldsymbol{\theta}|)\ \boldsymbol{\theta}\right)$

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Next lecture

3.6 Rotations –
Quaternions and Concatenation

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