Photogrammetry & Robotics Lab

3D Coordinate Systems

(Bsc Geodesy & Geoinformation)

3. Rotations
– Overview, Rotation Matrices, Euler Angles

Wolfgang Förstner

The slides have been created by Wolfgang Förstner.

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3.1 Motivation and Overview

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Motivation

Roles of rotations

- Many applications
  - Instruments (total station, gyro, antenna)
  - Earth rotation
  - Attitude of object
- Nonlinear part of motion

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Overview

- Representations
- Concatenation
- Small rotations (differential equation, uncertainty)
- Rotations from pairs of vectors

For this video see Förstner/Wrobel (2016), p. 325-331

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Rotation: Definition

Linear transformations of n-dimensional space

$\mathcal{R}: \mathbb{R}^n \to \mathbb{R}^n \quad \boldsymbol{x}' = R\boldsymbol{x} \quad \text{with} \quad R^{\mathsf{T}} = \mathrm{R}^{-1} \quad \text{and} \quad |R| = 1$

Invariants of rotations

- Distances
- Point of origin

Properties of rotation matrices

- Orthogonal
- No mirror (special)

→Group $\mathrm{SO}(n)$ of special orthogonal linear mapping

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Representations of Rotations $\mathcal{R} \in \mathrm{SO}(3)$

We need 3 parameters at least, d.o.f. = 3

- 3×3 Rotation matrix R
- 3 Euler angles (ω, ϕ, κ)
- 3-vector of rotation axis and angle $(\boldsymbol{r}, \theta)$
- Exponential of skew matrix of 3-vector $\exp(S(\boldsymbol{\theta}))$
- Quaternions: 4-vector $\mathbf{q}$
  - Rodriguez: 3 vector $\boldsymbol{m}$
  - Cayley: 3-vector $\boldsymbol{u}$

All representations have advantages/disadvantages

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3.2 Matrix representation

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Matrix representation

Rotation matrix with columns and rows

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3] = \begin{bmatrix} \boldsymbol{r}_1^{\mathsf{T}} \\ \boldsymbol{r}_2^{\mathsf{T}} \\ \boldsymbol{r}_3^{\mathsf{T}} \end{bmatrix}$$

6 Constraints, e.g. $R^{\mathsf{T}} R = I_3$ → for columns

$$|\boldsymbol{c}_1|^2 = 1 \qquad |\boldsymbol{c}_2|^2 = 1 \qquad |\boldsymbol{c}_3|^2 = 1$$
$$\boldsymbol{c}_1^{\mathsf{T}}\ \boldsymbol{c}_2 = 0 \qquad \boldsymbol{c}_2^{\mathsf{T}}\ \boldsymbol{c}_3 = 0 \qquad \boldsymbol{c}_3^{\mathsf{T}}\ \boldsymbol{c}_1 = 0$$

→ Redundant representation with 9 numbers

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Interpretation of columns and rows

- Columns c_i are images of basis vectors $e_i^{[3]}$

$$c_i = R\, e_i = [c_1, c_2, c_3]\, e_i$$

- Basis vectors e_i are maps of the vectors r_i

$$e_i = R r_i \quad \text{since} \quad r_i = R^\mathsf{T} e_i \quad \text{or} \quad r_i^\mathsf{T} = e_i^\mathsf{T} R = e_i^\mathsf{T} \begin{bmatrix} r_1^\mathsf{T} \\ r_2^\mathsf{T} \\ r_3^\mathsf{T} \end{bmatrix}$$

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Interpretation of elements

- Elements are cosines of angles between old and new frame e_i and c_j

$$R = [r_{ij}] = \begin{bmatrix} e_1^\mathsf{T} \\ e_2^\mathsf{T} \\ e_3^\mathsf{T} \end{bmatrix} [c_1\ c_2\ c_3] = \begin{bmatrix} e_1 \cdot c_1 & e_1 \cdot c_2 & e_1 \cdot c_3 \\ e_2 \cdot c_1 & e_2 \cdot c_2 & e_2 \cdot c_3 \\ e_3 \cdot c_1 & e_3 \cdot c_2 & e_3 \cdot c_3 \end{bmatrix}$$

or

$$R = [r_{ij}] = [e_i \cdot c_j] = [\cos(e_i, c_j)]$$

Naming: "*direction cosine matrix* " (DCM)

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Matrix representation

Advantages

- Useful for rotating points
- **Best exchange format (add example)**
- Some direct solutions of estimation problem

Disadvantages

- Redundant representation
  → not good for least squares solution
- No immediate interpretation wrt instruments

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3.3 Euler angles

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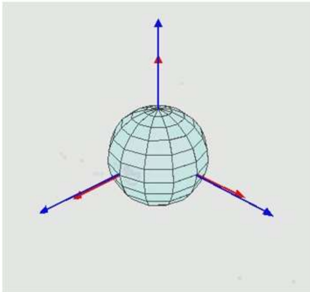
Euler angles

Leonard Euler (1707-1783)

3 rotations around 3 axes
(from Wikipedia)

1. Rotation around z
2. Rotation around local x
3. Rotation around local z



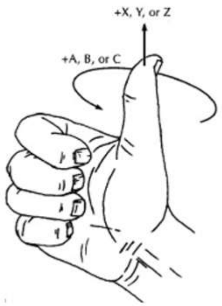
Euler angles

- 3 Rotations
  - Around three axes
  - Within three planes (*)
- Many options for sequence
  - Euler: (z, x, z)
  - Tait-Bryan: (x, y, z)
  - Others (how many?)

* *Q: Which explanation is more general?*
Rotation around axes or rotation within planes?

Right hand rule

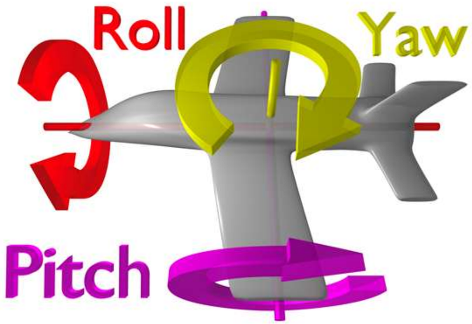
- Positive rotation following right hand rule



(https://www.machsupport.com/forum/index.php?topic=15220.20)

Application dependent naming

- Roll
  G: Rollen, Wanken
- Pitch
  G: Nicken, Stampfen
- Yaw
  G: Gier



[Image Courtesy: Wikipedia Commons, User: ZeroOne]

Basic rotations

Rotation around axis $i \in \{1,2,3\}$ with angle α

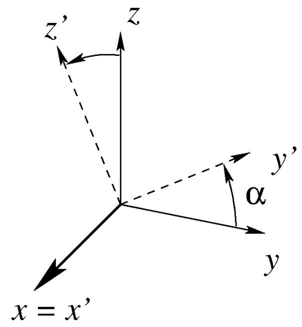
$$R_i(\alpha)$$

$$\begin{array}{ll} i=1: & (y \rightarrow z) \\ i=2: & (z \rightarrow x) \\ i=3: & (x \rightarrow y) \end{array}$$

Basic rotation 1

$$\begin{aligned} x' &= x \\ y' &= y\cos\alpha - z\sin\alpha \\ z' &= y\sin\alpha + z\cos\alpha \end{aligned}$$

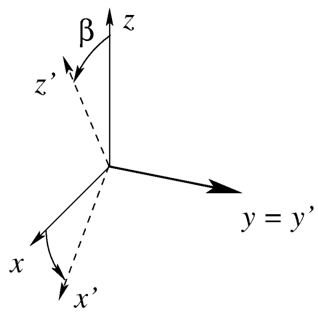
$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$



Basic rotation 2

$$\begin{aligned} x' &= x\cos\beta + z\sin\beta \\ y' &= y \\ z' &= -x\sin\beta + z\cos\beta \end{aligned}$$

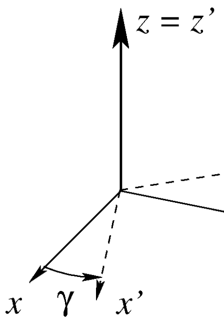
$$R_2(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$



Basic rotation 3

$$\begin{aligned} x' &= x\cos\gamma - y\sin\gamma \\ y' &= x\sin\gamma + y\cos\gamma \\ z' &= z \end{aligned}$$

$$R_3(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



→ rotation in the (x,y)-plane

Concatenation

A: **Active rotation** w.r.t **original system**
multiplication from the left

$$R_A(\alpha,\beta,\gamma) = R_3(\gamma)R_2(\beta)R_1(\alpha)$$

$$\begin{bmatrix} \cos\gamma\cos\beta & -\sin\gamma\cos\alpha+\cos\gamma\sin\beta\sin\alpha & \sin\gamma\sin\alpha+\cos\gamma\sin\beta\cos\alpha \\ \sin\gamma\cos\beta & \cos\gamma\cos\alpha+\sin\gamma\sin\beta\sin\alpha & -\cos\gamma\sin\alpha+\sin\gamma\sin\beta\cos\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}$$

Concatenation

B: **Active rotation** w.r.t **local system**
multiplication from the right

$$R_B(\alpha,\beta,\gamma) = R_1(\alpha)R_2(\beta)R_3(\gamma)$$

$$\begin{bmatrix} \cos\gamma\cos\beta & -\sin\gamma\cos\beta & \sin\beta \\ \cos\gamma\sin\beta\sin\alpha+\sin\gamma\cos\alpha & -\sin\gamma\sin\beta\sin\alpha+\cos\gamma\cos\alpha & -\cos\beta\sin\alpha \\ -\cos\gamma\sin\beta\cos\alpha+\sin\gamma\sin\alpha & \sin\gamma\sin\beta\cos\alpha+\cos\gamma\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}$$

Concatenation

Passive rotations → use of inverse/transposed

C: **Passive rotation** w.r.t **original system**
multiplication of inverses from the right

$$R_C(\alpha,\beta,\gamma) = R_1^{\mathsf{T}}(\alpha)R_2^{\mathsf{T}}(\beta)R_3^{\mathsf{T}}(\gamma) = R_A^{\mathsf{T}}$$

D: **Passive rotation** w.r.t **local system**
multiplication of inverses from the left

$$R_D(\alpha,\beta,\gamma) = R_3^{\mathsf{T}}(\gamma)R_2^{\mathsf{T}}(\beta)R_1^{\mathsf{T}}(\alpha) = R_B^{\mathsf{T}}$$

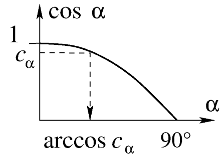
Angles from rotation matrix

Given: rotation matrix
Known: algebraic form of rotation matrix
Sought: rotation angles

In all cases
- One term only referring to one angle (e.g. $\sin\beta$)
  → useful for that angle?
- Corresponding row/column → all angles

Numerical consideration

Do not use $\arccos(\cdot)$ or $\arcsin(\cdot)$!



Example: given angle $\alpha = 0.01$

10 digits (using MAPLE)

$c_\alpha = \cos\alpha = .9999500004\,, \qquad s_\alpha = \sin\alpha = .009999833334.$

Then

$$\hat{\alpha}_{\text{arccos}} = \arccos(c_\alpha) = .01000000167$$
$$\hat{\alpha}_{\text{arctan}} = \text{atan2}\,(s_\alpha, c_\alpha) = .01000000000.$$

→ **Use two-valued function** $\text{atan2}\,(\cdot,\cdot)$ **!**

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General solution

Given matrix $R_A = (r_{ij})$

$$\begin{bmatrix} \cos\gamma\cos\beta & -\sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha & \sin\gamma\sin\alpha + \cos\gamma\sin\beta\cos\alpha \\ \sin\gamma\cos\beta & \cos\gamma\cos\alpha + \sin\gamma\sin\beta\sin\alpha & -\cos\gamma\sin\alpha + \sin\gamma\sin\beta\cos\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}$$

→
$$\alpha = \text{atan2}\,(r_{32}, r_{33})$$
$$\beta = \text{atan2}\,(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})$$
$$\gamma = \text{atan2}\,(r_{21}, r_{11})$$

If $\cos\beta \neq 0$ or $\beta \neq \pm 90°$

Independent of sign of $\cos\beta$

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Singularity: Gimbal lock

If $\cos\beta = 0$ or $\beta = \pm 90°$, hence $\sin\beta = 1$

$$R_A(\beta = 90°) = \begin{bmatrix} 0 & -\sin(\gamma-\alpha) & \cos(\gamma-\alpha) \\ 0 & \cos(\gamma-\alpha) & \sin(\gamma-\alpha) \\ -1 & 0 & 0 \end{bmatrix}$$

→

- Angle β determinable
- Only difference $\gamma - \alpha$ determinable

  axes for α and γ parallel

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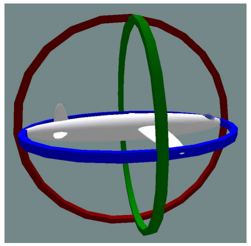
Gimbal lock

Normal situation

3 independent gimbals

(Kardanring zur Messung von Winkeln)



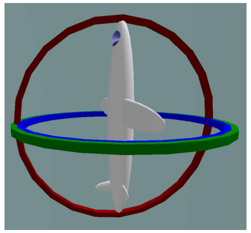
Gimbal lock

after upward pitch of airplane by 90°

2 gimbals are in the same plane

→one rotation not observable

(https://en.wikipedia.org/wiki/Gimbal_lock)



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Singularities, discontinuities, uniqueness

- Singularity: Gimbal lock
- Discontinuity: angles in $[0, 2\pi)$ or in $[-\pi, +\pi)$
- Uniqueness

$$R_3(\gamma)R_2(\beta)R_1(\alpha) = R_3(\gamma+\pi)R_2(\pi-\beta)R_1(\alpha+\pi)$$

→ sign of $\cos\beta$ changes sign of α

We chose positive square root, i.e. $\beta \in [-90°, +90°]$

- All representations with 3 parameters show defect

→Not useful for estimation

- Angles **not useful for exchanging 3D rotations**

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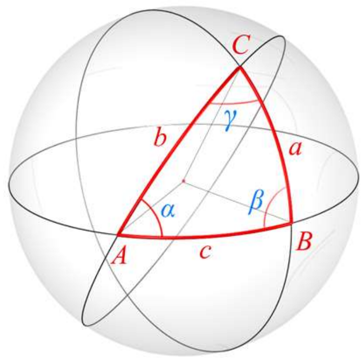
Example: Coordinates on the sphere

Calculations on the sphere
→spherical trigonometry



Here:
use of coordinates

- Geographical
- Cartesian
- Polar

Assumption: radius = 1

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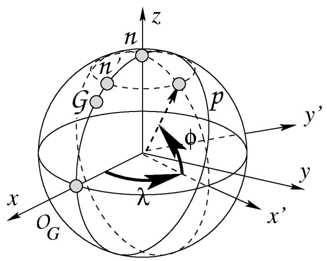
Cartesian and geographic coordinates

- Cartesian coordinates $p([x, y, z]^\mathsf{T})$
  - Equator $= (x, y)$-plane
  - North pole n = z-axis
  - Meridian plane Greenwich $\mathcal{G}$ (1851) $= (x, z)$-plane



- Geographic coordinates $p([\lambda, \phi])$ rectangular spherical system at O_G
  - Longitude λ : Angle of p to Greenwich meridian
  - Latitude ϕ: Angle of p to equator

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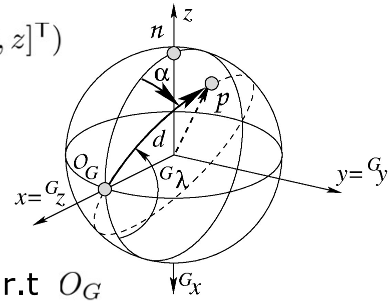
Cartesian and spherical polar coordinates

- Cartesian coordinates $p([x, y, z]^\mathsf{T})$
  - Equator $= (x, y)$-plane
  - North pole = z-axis
  - Merdian plane Greenwich $= (x, z)$-plane



- Polar coordinates $p([d, \alpha])$ w.r.t O_G
  - Distance d of p to Greenwich origin O_G
  - Azimuth α of p w.r.t. north

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From geographic to cartesian coordinates

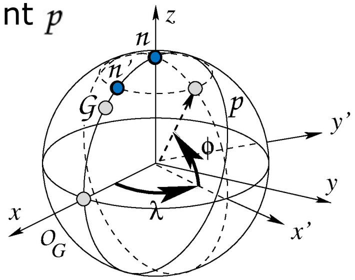
Given: (λ, ϕ) Sought: (x, y, z)

Use rule: columns of R are images of base vectors

→ rotate north pole $n(\boldsymbol{e}_3)$ into point p
coordinates are 3. column

Two rotations:

1. around y -axis by $90° - \phi$
   → $n \rightarrow n'$
2. around original z -axis by λ
   → $n' \rightarrow p$



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From geographic to cartesian coordinates

We obtain the rotation matrix

$$R(\lambda, \phi) = R_3(\lambda) R_2(90^o - \phi) = \begin{bmatrix} \cos\lambda \sin\phi & -\sin\lambda & \cos\lambda\cos\phi \\ \sin\lambda\sin\phi & \cos\lambda & \sin\lambda\cos\phi \\ -\cos\phi & 0 & \sin\phi \end{bmatrix}$$

Sought cartesian coordinates are (3. column)

$$\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \cos\lambda\cos\phi \\ \sin\lambda\cos\phi \\ \sin\phi \end{bmatrix}$$

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From cartesian to geographic coordinates

Given

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \cos\lambda\cos\phi \\ \sin\lambda\cos\phi \\ \sin\phi \end{bmatrix}$$

Geographic coordinates (using $\operatorname{atan2}(\cdot,\cdot)$)

$$\lambda = \operatorname{atan2}(p_2, p_1) \qquad \phi = \operatorname{atan2}\left(p_3, \sqrt{p_1^2 + p_2^2}\right)$$

Singularity: $\cos\phi = 0$ or $\phi = 90°$

→ At north pole longitude λ is not defined

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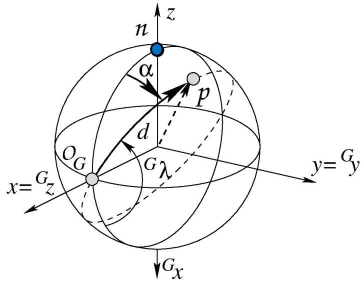
From geographic to polar coordinates

Rotation of north pole n to O_G

→Azimuth

= 180° – longitude in S_G

= $180° - {}^G\lambda$



Passive transformation

$${}^G\boldsymbol{p} = \begin{bmatrix} {}^Gx \\ {}^Gy \\ {}^Gz \end{bmatrix} = R_2^{\mathsf{T}}(90°)\boldsymbol{p} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\lambda\cos\phi \\ \sin\lambda\cos\phi \\ \sin\phi \end{bmatrix}$$

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From geographic to polar coordinates

Point in cartesian system S_G

$$
{}^G\boldsymbol{p} \;=\; \begin{bmatrix} -\sin\phi \\ \sin\lambda\cos\phi \\ \cos\lambda\cos\phi \end{bmatrix} = \begin{bmatrix} \cos{}^G\lambda\cos{}^G\phi \\ \sin{}^G\lambda\cos{}^G\phi \\ \sin{}^G\phi \end{bmatrix}
$$

Azimuth w.r.t.

$$
\begin{aligned}
\alpha &= 180^\circ - {}^G\lambda = 180^\circ - \operatorname{atan2}\left({}^Gy, {}^Gx\right) \\
&= 180^\circ - \operatorname{atan2}\left(\sin\lambda\cos\phi, -\sin\phi\right)
\end{aligned}
$$

or

$$
\boxed{\alpha = \operatorname{atan2}\left(\sin\lambda\cos\phi, \sin\phi\right)}
$$

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From geographic to polar coordinates

Distance (assuming radius R)

Two numerically instable solutions

1. Using $\sin{}^G\phi = \cos\lambda\cos\phi$ (instable for distances $\approx 90^\circ$)

$$
d = R(90^\circ - \arcsin(\cos\lambda\cos\phi))
$$

2. Using $\cos(\boldsymbol{e}_1^\mathsf{T}\boldsymbol{p}) = d/R$ (instable for small distances)

$$
d = R\arccos(\cos\lambda\cos\phi)
$$

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From geographic to polar coordinates

Numerically stable distance from

$$
d = R(\pi/2 - {}^G\phi) = R(\pi/2 - \operatorname{atan2}\left({}^Gz, \sqrt{{}^Gx^2 + {}^Gy^2}\right))
$$

Using $\tan(\pi/2 - \delta) = 1/\tan\delta$

$$
\boxed{d = R\operatorname{atan2}\left(\sqrt{\sin^2\phi + \sin^2\lambda\cos^2\phi}, \cos\lambda\cos\phi\right)}
$$

→ Useful for determining azimuth and distance of shortest flight (to be generalized)

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Comments

- Relations between
  - different coordinate systems on the sphere
  - using basic rotations
- Relations could be derived in many other ways
- Generally: use spherical trigonometry

https://en.wikipedia.org/wiki/Spherical_trigonometry

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Next lecture

3.4 Rotations -
Axis Angle and Exponential Representation

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