2. Passive Transformations and Local Systems

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Overview
- Group properties of transformations
- Active and passive transformations
- Original/global and local systems

For this video see Förstner/Wrobel (2016), p. 261-266, 284-285

2.5 Transformation groups

Transformations are continuous groups

\( T_1 \circ T_2 \rightarrow T, \ M_1 \circ M_2 \rightarrow M, \ldots \)

- Group structure
  \( \rightarrow \) Properties remain after inversion and concatenation

- Differentiation w.r.t parameters
  \( \rightarrow \) allows uncertainty propagation
Transformation groups (1/7)

A group is a set \( G \) with an operation \( \circ \)
\[ \{ G, \circ \} \]

Here:
- Elements \( g \in G \) are transformations (matrices)
- Operation \( \circ \) is concatenation (multiplication)

Groups (2/7)

Properties
1. Closed
   \[ \text{if } g_1, g_2 \in G \text{ then } g_1 \circ g_2 \in G \]
2. Associative
   \[ g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \]
3. Unit element \( \delta \) (here: unit matrix)
   \[ \delta \circ g = g \circ \delta = g \]
4. For each \( g \) exists inverse element \( g^{-1} \)
   \[ g \circ g^{-1} = g^{-1} \circ g = \delta \]

Groups (3/7)

Here: non-abelian groups
(Norwegian mathematician Niels Henrik Abel, 1802-1829)
= non-commutative groups

Classical groups
- General linear group of regular \( n \times n \) matrices
  \[ \text{GL}(n) : \quad x' = Ax \]
- Affine general linear group (also \( \text{Aff}(n) \))
  \[ \text{AGL}(n) : \quad x' = Ax + a \]

Groups (4/7)

- Group of orthogonal transformations in 2D
  = rotations including mirroring
  \[ \text{O}(2) : \quad x' = Rx \quad \text{with} \quad RR^T = I \]
- Special group of orthogonal transformations in 2D
  = rotations without mirroring
  \[ \text{SO}(2) : \quad x' = Rx \quad \text{with} \quad RR^T = I \quad \text{and} \quad |R| = 1 \]
- Translations \( T \) and dilations \( D \)
Groups (5/7): in homogeneous coordinates

- Group $SE(2)$ motions $\mathcal{M}$ in the 2D plane
  \[ \mathcal{M}(R, t) : \quad M = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} \]

- Group $S$ of similarities
  \[ S(R, t, \lambda) : \quad S = \begin{bmatrix} \lambda R & t \\ 0^T & 1 \end{bmatrix} \]

Groups (6/7)

Geometric affine transformations

- Affine ($\in AGL(2)$) in non-homogeneous coordinates
  \[ \mathcal{A}(A, t) : \quad x' = Ax + t \]

- Linear ($\in GL(3)$) in homogeneous coordinates
  \[ \mathcal{A}(A, t) : \quad x' = Ax \quad \text{with} \quad A = \begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix} \]

Groups (7/7)

Perspective transformations

\[ \mathcal{P}(A, t, p) : \quad P = \begin{bmatrix} A & t \\ p^T & 1 \end{bmatrix} \]

Geometric transformation hierarchy

2.6 Passive transformations and local coordinate systems
Active and passive transformations

- Active: coordinate system $S(x, y)$ is fixed
  point/object $x_1$ moves to $x_2$
- Passive: point/object is fixed
  coordinate system $S_1(x', y')$ moves to $S_2(x'', y'')$

If active transformation is

$$M : \quad x_2 = M(p)x_1$$

then passive transformation with same parameters is

$$N : \quad ^2x = M^{-1}(p)^1x,$$

with the inverse transformation matrix

Remark: here dilations/scalings are included

Example: Translation

Active

$$x_2 = x_1 + t$$

$$x_2 = \begin{bmatrix} 1 & t \\ 0^T & 1 \end{bmatrix} x_1$$

Passive

$$^2x = ^1x - t$$

$$^2x = \begin{bmatrix} 1 \\ 0^T \\ 1 \end{bmatrix}^{-1} ^1x = \begin{bmatrix} 1 \\ 0^T \\ 1 \end{bmatrix} ^1x$$

Representation in original and local frame

Representation of transformation may relate to

- Original coordinate system
- Last/local/carried with coordinate system
**Ex.: active translation then rotation**

1. translation: \( t = [0,1]^T \), 2. rotation: \( \phi = -45^\circ \)

**Ex.: passive translation then rotation**

1. translation: \( t = [0,1]^T \), 2. rotation: \( \phi = -45^\circ \)

**Concatenation of transformations**

- Active and passive transformations
- Reference of transformation to original and local/carried with system

→ 4 cases A, B, C, and D (proofs below)

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<td>( x'' = M_1^{-1}M_2^{-1}x )</td>
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**Rules**

A: active transformation, original system
→ multiplication with matrices from the left

B: active transformation, local/carried with system
→ multiplication with matrices from the right
### Rules

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C: passive transformation, original system  
→ multiplication with inverses from the right  
D: passive transformation, local/carried with system  
→ multiplication with inverses from the left  
→ Proofs

### A: active transformations, original system

(see above)

1. transformation  
   \( x_1 = M_1 x_0 \).  
2. transformation  
   \( x_2 = M_2 x_1 \).  
combined  
\[ x_2 = M_2 M_1 x_0. \]

### B: active transformations, local system

1. Transformation  
2. Transformation  
   a) Inverse 1. transformation  
   b) Second transformation  
   c) First transformation

1. Transformation  
2. Transformation in three steps  
   a) Inverse first transformation \( \rightarrow \) origin  
   b) Second transformation  
   c) First transformation  
combined  
\[ x_2 = M_1 M_2 M_1^{-1} x_1 = M_1 M_2 M_2^{-1} M_1 x_0 \]  
\[ x_2 = M_1 M_2 x_0. \]
### Passive transformations

Invert transformation matrices

**C: passive transformation in original system**

Inverse of case A: \(x'' = (M_2M_1)^{-1}x\)

\[x'' = M_1^{-1}M_2^{-1}x\]

**D: passive transformation in local system**

Inverse of case B: \(x'' = (M_1M_2)^{-1}x\)

\[x'' = M_2^{-1}M_1^{-1}x\]

### Summary of rules

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- A: multiply with matrices from the **left**
- B: multiply with matrices from the **right**
- C: multiply with **inverses** from the **right**
- D: multiply with **inverses** from the **left**

### Comments

- Transformations build continuous groups
- We need to distinguish
  - Active and passive transformations
    - Motions incl. dilations
    - Coordinate transformations
  - Representations in the original and the local system
- Default: active transformations
- Concatenation rules

### Notation for transformations (1/4)

**Problem:** given \(\mathcal{N}: \quad ^2x = M^{-1}(p)^1x\)

Is \(M^{-1}\) a passive or an inverse active transformation?

**Conventions**

- Coordinates of points are vectors \(x\) or \(x\)
- Name of **point** is attached as **lower right** index
- Name of **coordinate system** is **upper left** index
- Transformations have indices, such that matrix and point have same upper and lower index
Notation for transformations (2/4)
e.g. coordinates of point \( w_x \) in world system \( S_w \)

\[ w_x \rightarrow O_w \quad w^y \]

\[ \chi(w_x) = \chi(w_y) \]

\( \rightarrow \) space for hats and underlining 😊

Notation for transformations (2/4)
Active transformation (in non-named system)

\[ j M^i : \quad x_j = j M^i x_i \]

Inverse active transformation

\[ i M^j := (j M^i)^{-1} : \quad i M^j = (j M^i)^{-1} \]

Notation for transformations (3/4)

- Passive transformation of point \( \chi_i \)

\[ m M_i : \quad m x_i = m M_i \chi_i \]

- Inverse passive transformation

\[ i M_m = (m M_i)^{-1} : \quad i M_m = (m M_i)^{-1} \]

Notation for transformations (4/4)
Relation between active and passive transformations (enforcing the same indices, names of frames)

\[ j M_i = (j M^i)^{-1} : \quad j M_i = (j M^i)^{-1} \]

\( \rightarrow \) Generally

\[ j M_i = i M^j \]

Matrix of motion from \( j \) to \( i \) = matrix of coordinate transformation from \( i \) to \( j \)
Example: Car moving and observing

Car: at (4,1) looking in y-direction
Pole: at (1,5)
Q: in which direction is the pole seen from the car?

Car, viewing direction x-axis
Active motion: $S_O \rightarrow S_C$: $CM_O$
translation by (4,1),
rotation by $+90^\circ$ in local system
described in local system: multiplication from right

Example: Car moving and observing

Pole coordinates in car frame
Coordinate frame $S_O \rightarrow S_C$
passive transformation of pole

$C_{x_P} = CM_O \cdot o_{x_P} = (CM_O)^{-1} \cdot o_{x_P}$
$= \begin{bmatrix}
0 & -1 & 4 \\
+1 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}^{-1} \cdot \begin{bmatrix}
1 \\
5 \\
1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 4 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
5 \\
1
\end{bmatrix} = \begin{bmatrix}
4 \\
3 \\
1
\end{bmatrix}$
Direction $= (0.6,0.4) = 36.9^\circ$ (to the left)

Next lecture

3. Rotations - Overview, Rotation Matrices, Euler Angles
References


