Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

2. Passive Transformations and Local Systems

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The slides have been created by Wolfgang Förstner.

Overview

- Group properties of transformations
- Active and passive transformations
- Original/global and local systems

For this video see Förstner/Wrobel (2016), p. 261-266, 284-285

2.5 Transformation Groups

2.5 Transformation groups

Transformations are continuous groups

$$(\mathcal{T}_1 \circ \mathcal{T}_2 \to \mathcal{T} \, , \ \mathcal{M}_1 \circ \mathcal{M}_2 \to \mathcal{M} \, , \ldots)$$

- Group structure
  → Properties remain after inversion and concatenation
- Differentiation w.r.t parameters
  → allows uncertainty propagation

Transformation groups (1/7)

A group is a set $\mathcal{G}$ with an operation $\circ$

$$\{\mathcal{G}, \circ\}$$

Here:

- Elements $g \in \mathcal{G}$ are transformations (matrices)
- Operation $\circ$ is concatenation (multiplication)

Groups (2/7)

Properties

1. Closed
   if $g_1, g_2 \in \mathcal{G}$ then $g_1 \circ g_2 \in \mathcal{G}$
2. Associative
   $$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$$
3. Unit element δ (here: unit matrix)
   $$\delta \circ g = g \circ \delta = g$$
4. For each g exists inverse element g^{-1}
   $$g \circ g^{-1} = g^{-1} \circ g = \delta$$

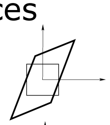
Groups (3/7)

Here: non-abelian groups
(Norwegian mathematician Niels Henrik Abel, 1802-1829)
= non-commutative groups

Classical groups

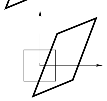
- General linear group of regular $n \times n$ matrices

$$\mathrm{GL}(n): \quad \boldsymbol{x}' = A\boldsymbol{x}$$



- Affine general linear group (also $\mathrm{Aff}(n)$)

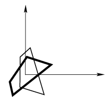
$$\mathrm{AGL}(n): \quad \boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{a}$$



Groups (4/7)

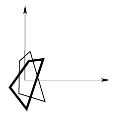
- Group of orthogonal transformations in 2D
  = rotations including mirroring

$$\mathrm{O}(2): \quad \boldsymbol{x}' = R\boldsymbol{x} \quad \text{with} \quad RR^{\mathsf{T}} = I$$



- Special group of orthogonal transformations in 2D
  = rotations without mirroring

$$\mathrm{SO}(2): \quad \boldsymbol{x}' = R\boldsymbol{x} \quad \text{with} \quad RR^{\mathsf{T}} = I \quad \text{and} \quad |R| = 1$$

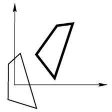


- Translations $\mathcal{T}$ and dilations $\mathcal{D}$

Groups (5/7): in homogeneous coordinates

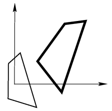
- Group $\mathrm{SE}(2)$ motions $\mathcal{M}$ in the 2D plane

$$\mathcal{M}(R, \boldsymbol{t}): \quad \mathsf{M} = \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$



- Group $\mathcal{S}$ of similarities

$$\mathcal{S}(R, \boldsymbol{t}, \lambda): \quad \mathsf{S} = \begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$



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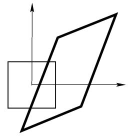
Groups (6/7)

Geometric affine transformations



- Affine ($\in \mathrm{AGL}(2)$) in non-homogeneous coordinates

$$\mathcal{A}(A, \boldsymbol{t}): \quad \boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{t}$$

- Linear ($\in \mathrm{GL}(3)$) in homogeneous coordinates

$$\mathcal{A}(A, \boldsymbol{t}): \quad \mathbf{x}' = \mathsf{A}\mathbf{x} \quad \text{with} \quad \mathsf{A} = \begin{bmatrix} A & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

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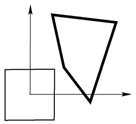
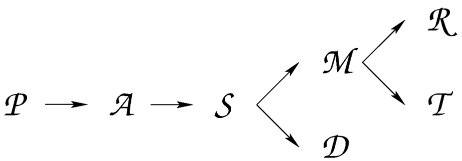
Groups (7/7)

Perspective transformations

$$\mathcal{P}(A, \boldsymbol{t}, \boldsymbol{p}): \quad \mathsf{P} = \begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{p}^\mathsf{T} & 1 \end{bmatrix}$$



Geometric transformation hierarchy



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2.6 Passive transformations and local coordinate systems

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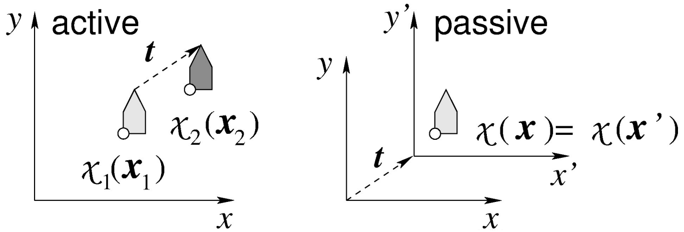
Active and passive transformations

- Active: coordinate system $S(x, y)$ is fixed
  point/object $\mathcal{X}_1$ moves to $\mathcal{X}_2$
- Passive: point/object is fixed
  coordinate system $S_1(x', y')$ moves to $S_2(x'', y'')$



Example: Translation

Active

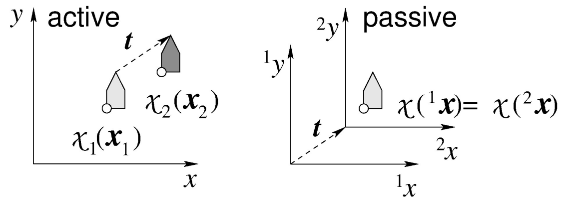
$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + \boldsymbol{t}$$

$$\mathbf{x}_2 = \begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \mathbf{x}_1$$

Passive

$$^2\boldsymbol{x} = {}^1\boldsymbol{x} - \boldsymbol{t}$$

$$^2\mathbf{x} = \begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}^{-1} {}^1\mathbf{x} = \begin{bmatrix} I & -\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} {}^1\mathbf{x}$$



Active and passive transformations

If active transformation is

$$\mathcal{M}: \quad \mathbf{x}_2 = \mathsf{M}(\boldsymbol{p})\mathbf{x}_1$$

then passive transformation with same parameters is

$$\mathcal{N}: \quad {}^2\mathbf{x} = \mathsf{M}^{-1}(\boldsymbol{p})\, {}^1\mathbf{x}\,,$$

with the inverse transformation matrix

Remark: here dilations/scalings are included

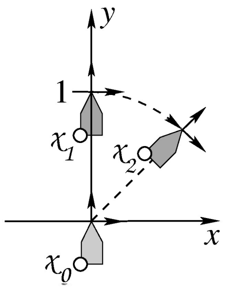
Representation in original and local frame

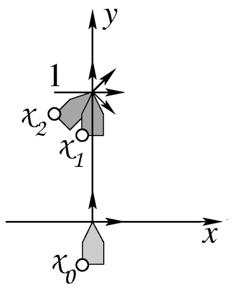
Representation of transformation may relate to

- Original coordinate system
- Last/local/carried with coordinate system

Ex.: *active* translation then rotation

1. translation: $\boldsymbol{t} = [0, 1]^{\mathsf{T}}$, 2. rotation: $\phi = -45°$

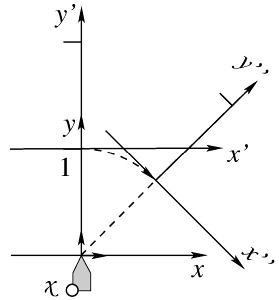


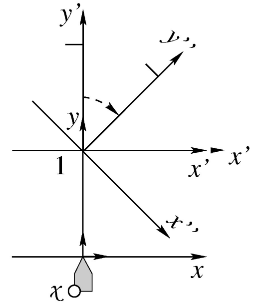


reference: original system reference: local system

Ex.: *passive* translation then rotation

1. translation: $\boldsymbol{t} = [0, 1]^{\mathsf{T}}$, 2. rotation: $\phi = -45°$





reference: original system reference: local system

Concatenation of transformations

- Active and passive transformations
- Reference of transformation to
  original and local/carried with system

→ 4 cases A, B, C, and D (proofs below)

	original	local/carried with
active	A: $\mathbf{x}_2 = \mathsf{M}_2\mathsf{M}_1\mathbf{x}_0$	B: $\mathbf{x}_2 = \mathsf{M}_1\mathsf{M}_2\mathbf{x}_0$
passive	C: $\mathbf{x}'' = \mathsf{M}_1^{-1}\mathsf{M}_2^{-1}\mathbf{x}$	D: $\mathbf{x}'' = \mathsf{M}_2^{-1}\mathsf{M}_1^{-1}\mathbf{x}$

Rules

	original	local/carried with
active	A: $\mathbf{x}_2 = \mathsf{M}_2\mathsf{M}_1\mathbf{x}_0$	B: $\mathbf{x}_2 = \mathsf{M}_1\mathsf{M}_2\mathbf{x}_0$
passive	C: $\mathbf{x}'' = \mathsf{M}_1^{-1}\mathsf{M}_2^{-1}\mathbf{x}$	D: $\mathbf{x}'' = \mathsf{M}_2^{-1}\mathsf{M}_1^{-1}\mathbf{x}$

A: active transformation, original system
→ multiplication with matrices from the left

B: active transformation, local/carried with system
→ multiplication with matrices from the right

Rules

	original	local/carried with
active	A: $\mathbf{x}_2 = \mathsf{M}_2\mathsf{M}_1\mathbf{x}_0$	B: $\mathbf{x}_2 = \mathsf{M}_1\mathsf{M}_2\mathbf{x}_0$
passive	C: $\mathbf{x}'' = \mathsf{M}_1^{-1}\mathsf{M}_2^{-1}\mathbf{x}$	D: $\mathbf{x}'' = \mathsf{M}_2^{-1}\mathsf{M}_1^{-1}\mathbf{x}$

C: passive transformation, original system

→ multiplication with inverses from the right

D: passive transformation, local/carried with system

→ multiplication with inverses from the left

→ Proofs

A: active transformations, original system

(see above)

1. transformation

$$\mathbf{x}_1 = \mathsf{M}_1\mathbf{x}_0.$$

2. transformation

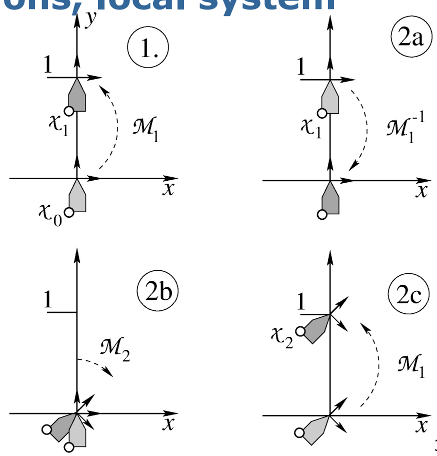
$$\mathbf{x}_2 = \mathsf{M}_2\mathbf{x}_1.$$

combined

$$\boxed{\mathbf{x}_2 = \mathsf{M}_2\mathsf{M}_1\mathbf{x}_0.}$$

B: active transformations, local system

1. Transformation
2. Transformation
   a) Inverse 1. transformation
   b) Second transformation
   c) First transformation



B: active transformations, local system

1. Transformation

$$\mathbf{x}_1 = \mathsf{M}_1\mathbf{x}_0.$$

2. Transformation in three steps
   a) Inverse first transformation → origin
   b) Second transformation
   c) First transformation

$$\mathbf{x}_2 = \mathsf{M}_1\mathsf{M}_2\mathsf{M}_1^{-1}\,\mathbf{x}_1 = \mathsf{M}_1\mathsf{M}_2\,\underbrace{\mathsf{M}_1^{-1}\,\mathsf{M}_1}_{I}\,\mathbf{x}_0$$

combined

$$\boxed{\mathbf{x}_2 = \mathsf{M}_1\mathsf{M}_2\mathbf{x}_0.}$$

Passive transformations

Inverse transformation matrices

C: passive transformation in original system

Inverse of case A: $\mathbf{x}'' = (\mathsf{M}_2\mathsf{M}_1)^{-1}\mathbf{x}$

$$\boxed{\mathbf{x}'' = \mathsf{M}_1^{-1}\mathsf{M}_2^{-1}\mathbf{x}}$$

D: passive transformation in local system

Inverse of case B: $\mathbf{x}'' = (\mathsf{M}_1\mathsf{M}_2)^{-1}\,\mathbf{x}$

$$\boxed{\mathbf{x}'' = \mathsf{M}_2^{-1}\mathsf{M}_1^{-1}\mathbf{x}}$$

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Summary of rules

	original	local/carried with
active	A: $\mathbf{x}_2 = \mathsf{M}_2\mathsf{M}_1\mathbf{x}_0$	B: $\mathbf{x}_2 = \mathsf{M}_1\mathsf{M}_2\mathbf{x}_0$
passive	C: $\mathbf{x}'' = \mathsf{M}_1^{-1}\mathsf{M}_2^{-1}\mathbf{x}$	D: $\mathbf{x}'' = \mathsf{M}_2^{-1}\mathsf{M}_1^{-1}\mathbf{x}$

A: multiply with **matrices** from the **left**

B: multiply with **matrices** from the **right**

C: multiply with **inverses** from the **right**

D: multiply with **inverses** from the **left**

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Comments

- Transformations build continuous groups
- We need to distinguish
  - Active and passive transformations
    - Motions incl. dilations
    - Coordinate transformations
  - Representations in the original and the local system
- Default: active transformations
- Concatenation rules

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Notation for transformations (1/4)

Problem: given $\mathcal{N}:\quad {}^2\mathbf{x} = \mathsf{M}^{-1}(\boldsymbol{p})\ {}^1\mathbf{x}$

Is M^{-1}a passive or an inverse active transformation?

Conventions

- Coordinates of points are vectors $\boldsymbol{x}$ or $\mathbf{x}$
- Name of **point** is attached as **lower right** index
- Name of **coordinate system** is **upper left** index
- Transformations have indices, such that matrix and point have same upper and lower index

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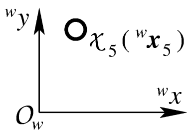
Notation for transformations (2/4)

e.g. coordinates of point 5 in world system S_w

$${}^{w}\boldsymbol{x}_5$$

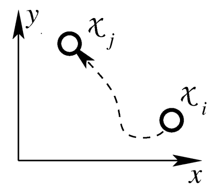


→ space for hats and underlining ☺

Notation for transformations (2/4)

Active transformation (in non-named system)

$${}_{j}\mathcal{M}^{i}: \qquad \mathbf{x}_j = {}_{j}\mathsf{M}^{i}\,\mathbf{x}_i\,.$$



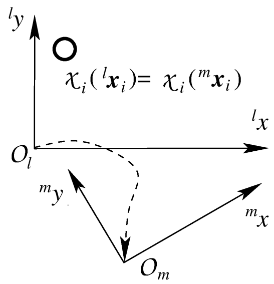
Inverse active transformation

$${}_{i}\mathcal{M}^{j} := \left({}_{j}\mathcal{M}^{i}\right)^{-1}: \qquad {}_{i}\mathsf{M}^{j} = \left({}_{j}\mathsf{M}^{i}\right)^{-1}.$$

Notation for transformations (3/4)

- Passive transformation of point $\mathcal{X}_i$

$${}^{m}\mathcal{M}_{l}: \qquad {}^{m}\mathbf{x}_i = {}^{m}\mathsf{M}_{l}\,{}^{l}\mathbf{x}_i\,.$$



- Inverse passive transformation

$${}^{l}\mathcal{M}_{m} = ({}^{m}\mathcal{M}_{l})^{-1}: \qquad {}^{l}\mathsf{M}_{m} = ({}^{m}\mathsf{M}_{l})^{-1}$$

Notation for transformations (4/4)

Relation between active and passive transformations (enforcing the same indices, names of frames)

$${}^{j}\mathcal{M}_{i} = ({}_{j}\mathcal{M}^{i})^{-1}: \qquad {}^{j}\mathsf{M}_{i} = ({}_{j}\mathsf{M}^{i})^{-1}.$$

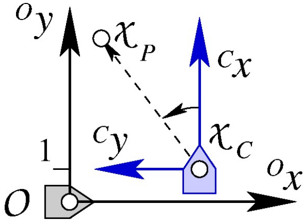
→Generally

$${}^{j}\mathsf{M}_{i} = {}_{i}\mathsf{M}^{j}$$

Matrix of motion from j to i =
matrix of coordinate transformation from i to j

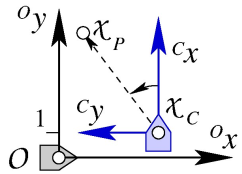
Example: Car moving and observing

Car : at (4,1) looking in y-direction
Pole: at (1,5)
Q : in which direction is the pole seen from the car?



Example: Car moving and observing

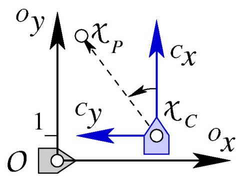
Car $\mathcal{X}_C$, viewing direction x-axis
Active motion: $S_O \rightarrow S_C$: $_C\mathsf{M}^O$
translation by (4,1),
rotation by +90° in local system
described in local system: multiplication from right



$$_C\mathsf{M}^O = \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{T}} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathsf{R}} = \begin{bmatrix} 0 & -1 & 4 \\ +1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Car moving and observing

Pole coordinates in car frame
Coordinate frame $S_O \rightarrow S_C$
passive transformation of pole



$$^{C}\mathbf{x}_P = {}^{C}\mathsf{M}_O \, {}^{O}\mathbf{x}_P = \left({}_C\mathsf{M}^O\right)^{-1} {}^{O}\mathbf{x}_P$$

$$= \begin{bmatrix} 0 & -1 & 4 \\ +1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Direction = (0.6,0.4) = 36.9° (to the left)

Next lecture

3. Rotations -
Overview, Rotation Matrices, Euler Angles

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