Photogrammetry & Robotics Lab

3D Coordinate Systems
(Bsc Geodesy & Geoinformation)

1. Motions and Similarities in the Plane

Wolfgang Förstner

The slides have been created by Wolfgang Förstner.

1

Notation

Objects and their representation

objects: calligraphic

representation: scalars, vectors, matrices

e.g. 2D point $\mathcal{X}$ with coordinates

$$\mathcal{X}(x, y) \quad \text{or} \quad \mathcal{X}(\boldsymbol{x})$$

or a 3D rotation $\mathcal{R}$ with angles or rotation matrix

$$\mathcal{R}(\alpha, \beta, \gamma) \quad \text{or} \quad \mathcal{R}(R)$$

2

Notation

- $|\cdot|$
  - absolute value $a|$
  - vector norm*, length $\boldsymbol{a}| = \sqrt{\sum_i a_i^2}$
  - determinant $A| = \det A$
- Frobenius norm $\|A\| = \sqrt{\mathrm{tr}(AA^\mathsf{T})} = \sqrt{\sum_{ij} a_{ij}^2}$
- $\underline{\mathcal{X}}, \underline{\boldsymbol{x}}$ stochastical entities (underscored)
- Alternative notation for vector norm $\|x\|$

For this video see Förstner/Wrobel (2016), p. 196-205, 250-253, 261

3

2.1 Basic 2D Transformations

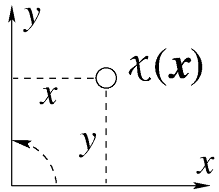
4

Representation of 2D points

Coordinates:

$$\chi = \chi(\boldsymbol{x}) : \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Right hand system: left rotation from x - to y - axis

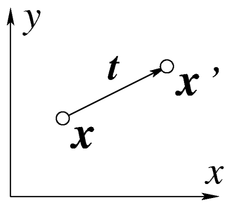


5

Translation

Shift of χ to χ'

$$\chi' = \mathcal{T}(\chi) : \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$



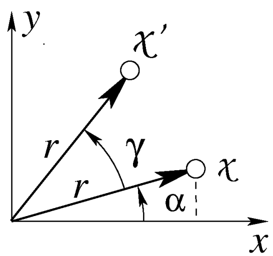
6

Rotation (1/4)

Rotation of χ to χ'

Polar coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$



7

Rotation (2/4)

... using the trigonometric addition formulas

$$\begin{aligned}
\begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} r\cos(\alpha+\gamma) \\ r\sin(\alpha+\gamma) \end{bmatrix} \\
&= \begin{bmatrix} r(\cos\alpha\cos\gamma - \sin\alpha\sin\gamma) \\ r(\sin\alpha\cos\gamma + \cos\alpha\sin\gamma) \end{bmatrix} \\
&= \begin{bmatrix} x\cos\gamma - y\sin\gamma \\ x\sin\gamma + y\cos\gamma \end{bmatrix}
\end{aligned}$$

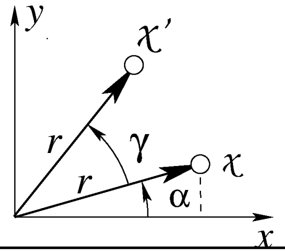
$$\chi' = \mathcal{R}(\chi) : \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

8

Rotation (3/4)

Rotation of χ to χ'

$$\chi' = \mathcal{R}(\chi) : \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotation (4/4)

Sign of $\sin\gamma$: small angle

$$\chi' = \mathcal{R}(\chi) : \begin{bmatrix} x' \\ y' \end{bmatrix} \approx \begin{bmatrix} 1 & -\gamma \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - \gamma y \\ \gamma x + y \end{bmatrix}$$

x -coordinate in 1. quadrant diminishes by γy

Dilation/Scaling

Dilation of χ to χ'

$$\chi' = \mathcal{D}(\chi) : \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

- $\lambda > 1$ magnification
- $0 < \lambda < 1$ diminution
- $\lambda = -1$ mirroring
- $\lambda < 0$ mirroring and magnification or diminution

2.2 Homogeneous Coordinates

Motivation – Problems (1/4)

1. How do we join translations and rotations?

Translation is additive

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t}$$

Rotaton and dilations are multiplicative

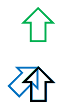
$$\boldsymbol{x}' = \lambda R \boldsymbol{x}$$

Motivation – Problems (2/4)

2. Concatenation is not commutative

$$\chi''_1 \equiv \mathcal{M}_1(\chi) \equiv \mathcal{R}(\mathcal{T}(\chi)) \;\not\equiv\; \chi''_2 \equiv \mathcal{M}_2(\chi) \equiv \mathcal{T}(\mathcal{R}(\chi))$$

First translation then rotation
vs
First rotation then translation.



Q: Do we need general proof or just an example?

Motivation – Problems (3/4)

3. How to determine the inverse transformation?

Given $\mathcal{M}_1 \; : \; \chi''_1 = \mathcal{M}_1(\chi)$

What is $\mathcal{M}_1^{-1} : \; \chi = \mathcal{M}_1^{-1}(\chi''_1)$?

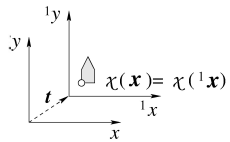
Q: Do we need to concatenate the inverse transformations or is there a simpler path?

Motivation – Problems (4/4)

4. How does the situation change if not the point is moved, but the coordinate system?

E.g. shifting the coordinate system changes the coordinates from $\boldsymbol{x}$ to ${}^1\boldsymbol{x}$ leaving the point χ fixed

$$\chi({}^1\boldsymbol{x}) = {}^1\mathcal{T}(\chi(\boldsymbol{x}))$$



Q: How does this situation change the concatenation?

Homogeneous coordinates

- Easy concatenation and inversion
- Preserves collinearity of points
- Easy join of two points or intersection of two lines
- Classical tool of projective geometry

Here:

- Focus: angular preserving transformations, no perspective mappings

→ Simplified representation

17

Definition

Homogeneous element:

An algebraic element $\mathbf{e}$ is called homogeneous, if for $\lambda \neq 0$ the algebraic Element $\lambda\mathbf{e}$ represents the same geometric element e :

$$e(\mathbf{e}) \equiv e(\lambda\mathbf{e}), \quad \lambda \neq 0\,.$$

Remark: In our context

the element e may be a point or a transformation,

the element $\mathbf{e}$ may be a vector or a matrix.

18

Homogeneous point coordinates

A point χ with (non-homogeneous) coordinates $\boldsymbol{x}$ has homogeneous coordinates

$$\mathbf{x} \cong \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cong w \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{for} \quad w \neq 0$$

The factor w may be chosen arbitrarily.

In our context we (can) use $w = 1$.

19

Non-homogeneous from homogenous c.

For given homogeneous coordinates $\mathbf{x}$

if $w \neq 0$ the non-homogeneous coordinates are

$$x = \frac{u}{w} \qquad y = \frac{v}{w}$$

Remarks: A homogeneous vector $[u, v, 0]^\mathsf{T}$ with $w = 0$ represents a point at infinity in the direction $[u, v]^\mathsf{T}$.

We assume $w = 1$. This assumption is removed in Photogrammetry or computer vision.

20

Basic homogeneous transformations

- Translation

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

or compactly

$$\mathbf{x}' = \mathsf{T}\mathbf{x} \quad \text{with} \quad \mathsf{T} = \begin{bmatrix} I_2 & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix T is homogeneous. Q: Why?

21

Basic homogeneous transformations

- Rotation around origin by φ

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

or compactly

$$\mathbf{x}' = \mathsf{R}\mathbf{x} \quad \text{with} \quad \mathsf{R} = \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

22

Basic homogeneous transformations

- Dilation or scaling

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

or compactly

$$\mathbf{x}' = \mathsf{D}\mathbf{x} \quad \text{with} \quad \mathsf{D} = \begin{bmatrix} \lambda I_2 & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

23

Basic homogeneous transformations

- Summary of transformations

$$\begin{aligned} \mathcal{T}(\mathsf{T}): \quad & \mathcal{x}' = \mathcal{T}(\mathcal{x}) & \mathbf{x}' &= \mathsf{T}\mathbf{x} \\ \mathcal{R}(\mathsf{R}): \quad & \mathcal{x}' = \mathcal{R}(\mathcal{x}) & \mathbf{x}' &= \mathsf{R}\mathbf{x} \\ \mathcal{D}(\mathsf{D}): \quad & \mathcal{x}' = \mathcal{D}(\mathcal{x}) & \mathbf{x}' &= \mathsf{D}\mathbf{x} \end{aligned}$$

!! All transformations are
- matrix-vector product
- linear transformations

→ Easy inversion and concatenation

24

2.3 Inversion

Inverting homogeneous transformations

Given transformation $\mathcal{A}: \ \mathbf{x}' = \mathsf{A}\mathbf{x}$
its inverse is

$$\mathcal{B} = \mathcal{A}^{-1}: \quad \mathbf{x} = \mathsf{B}\mathbf{x}' \quad \text{with} \quad \mathsf{B} = \mathsf{A}^{-1}$$

Inverse transformations

- Translation

$$\mathsf{T}^{-1} = \begin{bmatrix} I_2 & -\boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

- Rotation

$$\mathsf{R}^{-1} = \begin{bmatrix} R^{-1} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} R^\mathsf{T} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

- Dilation

$$\mathsf{D}^{-1} = \begin{bmatrix} \lambda^{-1} I_2 & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \cong \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & \lambda \end{bmatrix}$$

2.4 Concatenation

Concatenation

Given two transformations

$$\mathcal{M}_1(\mathsf{M}_1) \quad \text{and} \quad \mathcal{M}_2(\mathsf{M}_2)$$

First applying $\mathcal{M}_1$ then $\mathcal{M}_2$ yields

$$\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1 : \quad \mathcal{X}' = \mathcal{M}_2(\mathcal{M}_1(\mathcal{X}))$$

Or

$$\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1 : \quad \mathbf{x}' = \mathsf{M}_2\left(\mathsf{M}_1\mathbf{x}\right) = \mathsf{M}_2\mathsf{M}_1\mathbf{x}$$

Concatenation = matrix multiplication

Example 1: Similarity

Similarity transformation

$$\mathcal{S}(\mathsf{S}) = \mathcal{T}(t_x, t_y) \circ \mathcal{R}(\phi) \circ \mathcal{D}(\lambda)$$

or

$$\mathsf{S}(\lambda, \phi, t_x, t_y) = \begin{bmatrix} \lambda\cos\phi & -\lambda\sin\phi & t_x \\ \lambda\sin\phi & \lambda\cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

or linearly, without trigonometric expressions

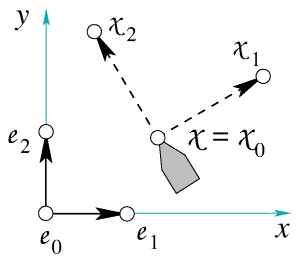
$$\mathsf{S}(a, b, c, d) = \begin{bmatrix} a & -b & c \\ b & a & d \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2: Motion of a frame

Frame = coordinate system

- for describing poses
- attached to point

Possibly including scale



Example 2: Representation of a frame

- Rotation and translation (point, origin)

$$\mathcal{F}: \quad \{R, \boldsymbol{t}\} \quad \text{e.g.} \quad \{I_2, \mathbf{0}_{2\times 1}\}$$

- Transformation matrix

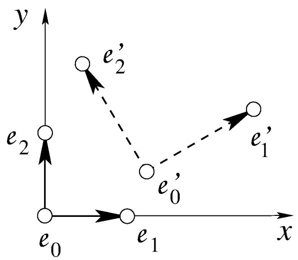
$$\mathcal{F}: \quad \mathsf{F} = \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix} \quad \text{e.g.} \quad \mathsf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Tripod: origin, unit points

$$\mathcal{F}: \quad F = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_0] \quad \text{e.g.} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Motion of a frame

Similarity transformation of frame
Scaling, rotation, translation



Remark: interpretation = motion of e_0
as part of a moved object

33

Motion of a frame

First similarity

$$\mathsf{S}_1(\lambda, \phi, t_x, t_y) = \begin{bmatrix} \lambda\cos\phi & -\lambda\sin\phi & t_x \\ \lambda\sin\phi & \lambda\cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Frame after first similarity

$$F' = [\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_0] = S_1 F = [S_1\mathbf{e}_1, S_1\mathbf{e}_2, S_1\mathbf{e}_0]\, F$$

$$F' = \begin{bmatrix} \lambda\cos\phi + t_x & -\lambda\sin\phi + t_x & t_x \\ \lambda\sin\phi + t_y & \lambda\cos\phi + t_y & t_y \\ 1 & 1 & 1 \end{bmatrix}$$

34

Motion of a frame

Second similarity

$$\mathsf{S}_2(\mu, \psi, u_x, u_y) = \begin{bmatrix} \mu\cos\psi & -\mu\sin\psi & u_x \\ \mu\sin\psi & \mu\cos\psi & u_y \\ 0 & 0 & 1 \end{bmatrix}$$

Frame after second similarity

$$F'' = \mathsf{S}_2 F' = \begin{bmatrix} \mu\cos\psi & -\mu\sin\psi & u_x \\ \mu\sin\psi & \mu\cos\psi & u_y \\ 0 & 0 & 1 \end{bmatrix} F'$$

35

Motion of a frame

Frame after second similarity

$$F'' = S_2 F' = S_2 S_1 F$$

or

$$= \begin{bmatrix} \lambda\mu(\cos(\phi+\psi)) + v_x & -\lambda\mu(\sin(\phi+\psi)) + v_x & v_x \\ \lambda\mu(\sin(\phi+\psi)) + v_y & \lambda\mu(\cos(\phi+\psi)) + v_y & v_y \\ 1 & 1 & 1 \end{bmatrix}$$

with the origin after the second similarity

$$\mathbf{e}''_0 = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \mu(t_x\cos\psi - t_y\sin\psi) + u_x \\ \mu(t_x\sin\psi + t_y\cos\psi) + u_y \end{bmatrix}$$

36

Comments

Concatenation is

- **interpretable**
  - Scales multiply
  - Rotations add
  - First translation vector is transformed
- **not very transparent in general**
  - Three or more transformations?
  - Inversion of concatenated transformations?

Next lecture

2.6 Passive Transformations and Local Systems

References of the video series

Arun, K. S., T. S. Huang, and S. B. Blostein (1987). Least-Squares Fitting of Two 3D Point Sets. IEEE T-PAMI 9 (5), 698-700.

Bishop, C. (2006). Pattern Recognition and Machine Learning. Springer.

Boyd, S. and L. Vandenberghe (2004). Convex optimization. Cambridge University Press.

Cayley, A. (1846). Sur quelques proprétés des déterminants gauches. Journal für die reine und angewandte Mathematik 32, 119-123

Cover, T. and J. A. Thomas (1991). Elements of Information Theory. John Wiley & Sons.

H. S. M. Coxeter (1946): Quaternions and Reflections, The American Mathematical Monthly, 53(3), 136-146

Fallat, S. M. and M. J. Tsatsomeros (2002). On the Cayley Transform of Positivity Classes of Matrices. Electronic Journal of Linear Algebra 9, 190-196.

Fischler, M. A. and R. C. Bolles (1981). Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. Communications of the ACM 24 (6), 381-395.

Förstner, W. and B. P. Wrobel (2016). Photogrammetric Computer Vision Statistics, Geometry, Orientation and Reconstruction. Springer.

Horn, B. K. B. (1987). Closed-form Solution of Absolute Orientation Using Unit Quaternions. Optical Soc. of America.

Howell, T. D. and J.-C. Lafon (1975). The Complexity of the Quaternion Product. Technical Report TR75-245, Cornell University.

Kanatani, K. (1990). Group Theoretical Methods in Image Understanding. New York: Springer.

Koch, K.-R. (1999). Parameter Estimation and Hypothesis Testing in Linear Models (2nd ed.). Springer.

Li, S. Z. (2000). Markov random field modeling in computer vision. Springer.

McGlone, C. J., E. M. Mikhail, and J. S. Bethel (2004). Manual of Photogrammetry (5th ed.). Maryland, USA: American Society of Photogrammetry and Remote Sensing.

Mikhail, E. M. and F. Ackermann (1976). Observations and Least Squares. University Press of America
Mikhail, E. M., J. S. Bethel, and J. C. McGlone (2001). Introduction to Modern Photogrammetry. Wiley.
Palais, B. and R. Palais (2007). Euler's fixed point theorem: The axis of a rotation. J. fixed point theory appl. 2, 215-220.
Raguram, R., O. Chum, M. Pollefeys, J. Matas, and J.-M. Frahm (2013). USAC: A Universal Framework for Random Sample Consensus. IEEE Transactions on Pattern Analysis and Machine Intelligence 35 (8), 2022-2038.
Rao, R. C. (1973). Linear Statistical Inference and Its Applications. New York: Wiley.
Rodriguez, O. (1840). Des lois géometriques qui regissent les deplacements d'un système solide independament des causes qui peuvent les produire. Journal de mathématiques pures et appliquées 1 (5), 380440.
Sansò, F. (1973). An Exact Solution of the Roto-Translation Problem. Photogrammetria 29 (6), 203-206.
Vaseghi, S. V. (2000). Advanced Digital Signal Processing and Noise Reduction. Wiley.
Weber, M. (2003). Rotation between two vectors. Personal communication, Bonn.