Photogrammetry & Robotics Lab

Homogeneous Coordinates

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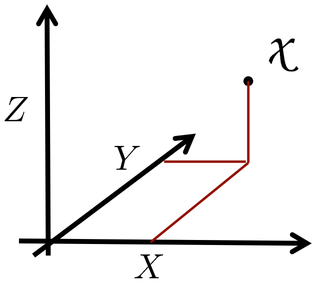
The slides have been created by Cyrill Stachniss.

A Point in the 3D Euclidean World



$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Is this always the best representation of geometric object in the 3D world?

Pinhole Camera

- Popular model to approximate the imaging process of a perspective camera

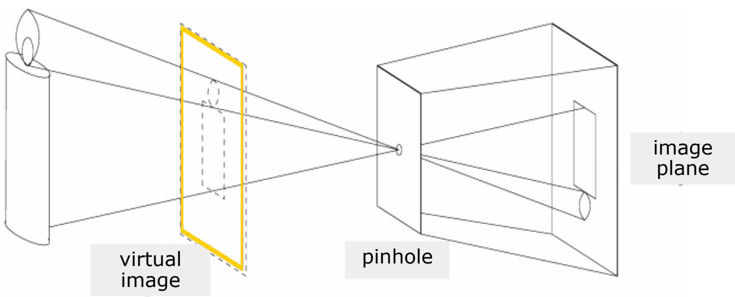


Image Courtesy: Forsyth and Ponce

Pinhole Camera Model

- A box with an **infinitesimal small hole**
- **Camera center** is the intersection point of the rays
- The back wall is the **image plane**
- The distance between the camera center and image plane is the **camera constant**

Geometry and Images

What can we say about the geometry?

Image courtesy: Förstner

Pinhole Camera Properties

- **Line-preserving:** straight lines are mapped to straight lines
- **Not length-preserving:** size of objects is inverse proportional to the distance
- **Not angle-preserving:** Angles between lines change

Perspective Projection

- Straight lines stay straight
- Parallel lines may not remain parallel

Image courtesy: Förstner

Vanishing Point
(DE: Fluchtpunkt)

Image Courtesy: J. Jannene

Vanishing Points

- Parallel lines are not parallel anymore
- All mapped parallel lines intersect in a vanishing point
- The vanishing point is the "point at infinity" for the parallel lines
- Every direction has exactly one vanishing point

How to describe "points at infinity"?

Projective Geometry Motivation

- **Euclidian geometry is suboptimal to describe the central projection**
- In Euclidian geometry, the math can get difficult
- **Projective geometry** is an alternative algebraic representation of geometric objects and transformations

Homogeneous Coordinates

- H.C. are a system of coordinates used in projective geometry
- Formulas involving H.C. are often simpler than in the Cartesian world
- Points at infinity can be represented using finite coordinates
- A single matrix can represent affine and projective transformations

Notation

Point $\mathcal{X}$ (or $\mathcal{Y}$ or $\mathcal{P}$)

- in homogeneous coordinates $\mathbf{x}$
- in Euclidian coordinates $\boldsymbol{x}$

Line $\mathcal{L}$ (or $\mathcal{M}$)

- in homogeneous coordinates $\mathbf{l}$

Plane $\mathcal{A}$

- in homogeneous coordinates $\mathbf{A}$

2D vs. **3D** space

- lowercase = 2D; capitalized = 3D

Homogeneous Coordinates

Definition

The representation $\mathbf{x}$ of a geometric object is **homogeneous** if $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same object for $\lambda \neq 0$

Example

$$\mathbf{x} = \lambda\,\mathbf{x} \qquad\qquad \boldsymbol{x} \neq \lambda\,\boldsymbol{x}$$

homogeneous — Euclidian

Homogeneous Coordinates

- H.C. use a n+1 dimensional vector to represent the n-dimensional Euclidian point
- Set dimension n+1 to the value 1
- Example for $\mathbb{R}^2/\mathbb{P}^2$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Definition

The representation $\mathbf{x}$ of a geometric object is **homogeneous** if $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same object for $\lambda \neq 0$

Example

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2 \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix}$$

Euclidian — homogeneous

Definition

- Homogeneous Coordinates of a point χ in the plane $\mathbb{R}^2$ is a 3-dim. vector

$$\chi: \quad \mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ with } |\mathbf{x}|^2 = u^2 + v^2 + w^2 \neq 0$$

- it corresponds to Euclidian coordinates

$$\chi: \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \end{bmatrix} \text{ with } w \neq 0$$

Example: Projective Plane

The projective plane $\mathbb{P}^2(\mathbb{R})$ or $\mathbb{P}^2$ contains

- All points $\mathcal{X}$ of the Euclidian plane $\mathbb{R}^2$ with $\boldsymbol{x} = [x, y]^\top$ expressed through the 3-valued vector (e.g., $\mathbf{x} = [x, y, 1]^\top$)
- and all points at infinity, i.e.,

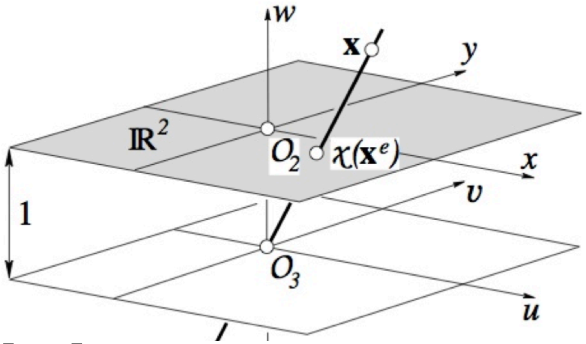
  $\mathbf{x} = [x, y, 0]^\top$
- except $[0, 0, 0]^\top$

From Homogeneous to Euclidian Coordinates

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

homogeneous Euclidian

From Homogeneous to Euclidian Coordinates



$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Image courtesy: Förstner

3D Points

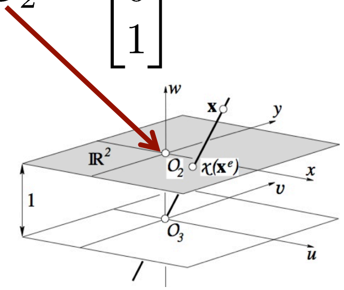
Analogous for points in 3D Euclidian space $\mathbb{R}^3$

homogeneous Euclidian

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

Origin of the Euclidian Coordinate System in H.C.

$$\mathbf{O}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{O}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



Transformations

Transformations

- A projective transformation is an invertible linear mapping

$$\underbrace{\mathbf{X}'}_{\text{homogeneous vector}} = \underbrace{\mathsf{H}}_{\text{homogeneous matrix}} \underbrace{\mathbf{X}}_{\text{homogeneous vector}}$$

Fundamental Theorem of Projective Geometry

- Every one-to-one, straight-line preserving mapping of a projective space $\mathbb{P}^n$ onto itself is a homography (projectivity) for $2 \leq n < \infty$
- Implies that all one-to-one, straight-line preserving transformations are linear if we use projective coordinates

3D Transformations

- General projective mapping

$$\mathbf{X}' = \mathsf{H}\mathbf{X}$$

- Question: How should H look like to realize relevant transformation?
- Eg, translation, rotation, scale change, rigid-body, similarity, affine, projective

Important 3D Transformations

- General projective mapping $\mathbf{X}' = \mathsf{H}\mathbf{X}$
- Translation: 3 parameters
  (3 translations)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{H} = \lambda \begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

$$\boldsymbol{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

homogeneous property

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Important 3D Transformations

- Rotation: 3 parameters
  (3 rotation)

rotation matrix

$$\mathsf{H} = \lambda \begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

Recap – Rotation Matrices

- 2D:

$$R^{2D}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- 3D:

$$R_x^{3D}(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) \\ 0 & \sin(\omega) & \cos(\omega) \end{bmatrix} \quad R_y^{3D}(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$$R_z^{3D}(\kappa) = \begin{bmatrix} \cos(\kappa) & -\sin(\kappa) & 0 \\ \sin(\kappa) & \cos(\kappa) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^{3D}(\omega, \phi, \kappa) = R_z^{3D}(\kappa) R_y^{3D}(\phi) R_x^{3D}(\omega)$$

Important 3D Transformations

- Rotation: 3 parameters
  (3 rotation)

$$H = \lambda \begin{bmatrix} \underline{R} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

- Rigid body transformation: 6 params
  (3 translation + 3 rotation)

$$H = \lambda \begin{bmatrix} \underline{R} & \underline{\boldsymbol{t}} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

Important 3D Transformations

- Similarity transformation: 7 params
  (3 trans + 3 rot + 1 scale)

$$H = \lambda \begin{bmatrix} \underline{m}R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

(angle-preserving)

- Affine transformation: 12 parameters
  (3 trans + 3 rot + 3 scale + 3 sheer)

$$H = \lambda \begin{bmatrix} \underline{A} & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

(not angle-preserving but parallel lines remain parallel)

Important 3D Transformations

- Projective transformation: 15 params.

$$H = \lambda \begin{bmatrix} A & \boldsymbol{t} \\ \underline{\boldsymbol{a}^\mathsf{T}} & 1 \end{bmatrix}$$

affine transformation + 3 parameters

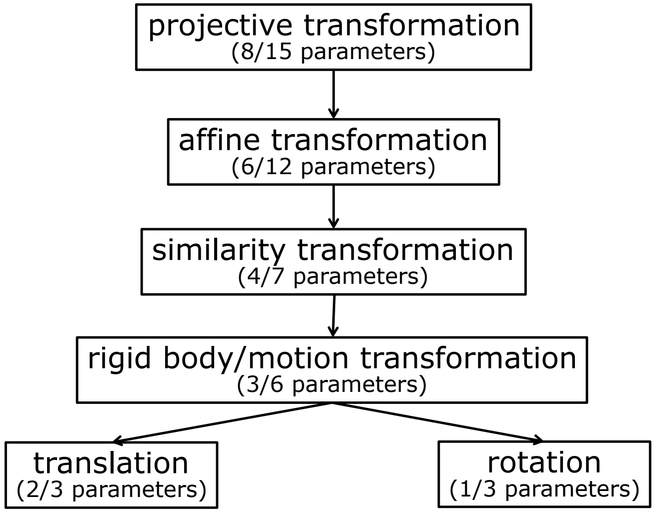
- These 3 parameters are the projective part and they are the reason that **parallel lines may not stay parallel**

Transformations for 2D

2D Transformation	Figure	d. o. f.	H	H
Translation		2	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Mirroring at y-axis		1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} Z & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Rotation		1	$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Motion		3	$\begin{bmatrix} \cos\varphi & -\sin\varphi & t_x \\ \sin\varphi & \cos\varphi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Similarity		4	$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Scale difference		1	$\begin{bmatrix} 1+m/2 & 0 & 0 \\ 0 & 1-m/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Shear		1	$\begin{bmatrix} 1 & s/2 & 0 \\ s/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Asym. shear		1	$\begin{bmatrix} 1 & s' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S' & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Affinity		6	$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$
Projectivity		8	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{p}^\mathsf{T} & 1/\lambda \end{bmatrix}$

Image courtesy: Schindler

Transformations Hierarchy



Inverting and Chaining

- Inverting a transformation

$$\begin{aligned} \mathbf{X}' &= \mathsf{H}\mathbf{X} \\ \mathbf{X} &= \mathsf{H}^{-1}\mathbf{X}' \end{aligned}$$

- Chaining transformations via matrix products (not commutative)

$$\begin{aligned} \mathbf{X}' &= \mathsf{H}_1\mathsf{H}_2\mathbf{X} \\ &\neq \mathsf{H}_2\mathsf{H}_1\mathbf{X} \end{aligned}$$

Chaining and Inverting Transformations

- Chaining transformations via matrix products (not commutative)

$$\begin{aligned} \mathbf{X}' &= \mathsf{H}_1\mathsf{H}_2\mathbf{X} \\ &\neq \mathsf{H}_2\mathsf{H}_1\mathbf{X} \end{aligned}$$

- Inverting a transformation

$$\begin{aligned} \mathbf{X}' &= \mathsf{H}\mathbf{X} \\ \mathbf{X} &= \mathsf{H}^{-1}\mathbf{X}' \end{aligned}$$

Homogeneous Lines (Images, 2D)

Representations of Lines

- Hesse normal form (angle ϕ, distance d)

  $$x \cos \phi + y \sin \phi - d = 0$$

- Intercept form

  $$\frac{x}{x_0} + \frac{y}{y_0} = 1 \quad \text{or} \quad \frac{x}{x_0} + \frac{y}{y_0} - 1 = 0$$

- Standard form

  $$ax + by + c = 0$$

Representations of Lines

- Hesse normal form

  $$x \cos \phi + y \sin \phi - d = 0 \Rightarrow (\cos \phi)x + (\sin \phi)y - d = 0$$

- Intercept form

  $$\frac{x}{x_0} + \frac{y}{y_0} - 1 = 0 \Rightarrow \left(\frac{1}{x_0}\right)x + \left(\frac{1}{y_0}\right)y - 1 = 0$$

- Standard form

  $$ax + by + c = 0 \Rightarrow ax + by + c = 0$$

All form linear equations that are equal to zero

Representations of Lines

$$\underset{\text{standard}}{\mathbf{l}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \underset{\text{Hesse}}{\mathbf{l}} = \begin{bmatrix} \cos \phi \\ \sin \phi \\ -d \end{bmatrix}$$

$$\underset{\text{intercept}}{\mathbf{l}} = \begin{bmatrix} \frac{1}{x_0} \\ \frac{1}{y_0} \\ -1 \end{bmatrix}$$

Line Equation Can be Expressed by the Dot-Product

$$\underset{\text{standard}}{\mathbf{l}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \underset{\text{Hesse}}{\mathbf{l}} = \begin{bmatrix} \cos \phi \\ \sin \phi \\ -d \end{bmatrix}$$

$$\underset{\text{intercept}}{\mathbf{l}} = \begin{bmatrix} \frac{1}{x_0} \\ \frac{1}{y_0} \\ -1 \end{bmatrix} \qquad \underset{\text{point}}{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x} \cdot \mathbf{l} = 0$$

Definition

- Homogeneous Coordinates of a line ℓ in the plane is a 3-dim. vector

$$\ell: \quad \mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \text{ with } |\mathbf{l}|^2 = l_1^2 + l_2^2 + l_3^2 \neq 0$$

- Corresponds to the Euclidian representation

$$l_1 x + l_2 y + l_3 = 0$$

Test If a Point Lies on a Line

- A point

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- lies on a line

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$$

- if $\mathbf{x} \cdot \mathbf{l} = 0$

Intersecting Lines

- Given two lines ℓ, m expressed in H.C., we look for the intersection $\chi = \ell \cap m$

How to find the intersection of two lines?

Intersecting Lines

- Given two lines ℓ, m expressed in H.C., we look for the intersection $\chi = \ell \cap m$
- Find the point $\boldsymbol{x} = [x, y]^T$ through the following system linear equations

$$\begin{bmatrix} \mathbf{l} \cdot \mathbf{x} \\ \mathbf{m} \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}.$$

Reminder: Cramer's rule

- A system of linear equations can be solved via Cramer's rule

$$A\boldsymbol{x} = \boldsymbol{b} \qquad x_i = \frac{\det(A_i)}{\det(A)}$$

- with A_i being the matrix in which the i^{th} column is replaced by $\boldsymbol{b}$
- Easily applicable for 2 by 2 systems

Intersecting Lines

- Solution of

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}$$

- through Cramer's rule

$$x = \frac{D_1}{D_3} \qquad y = \frac{D_2}{D_3} \qquad \begin{aligned} D_1 &= \det(A_1) = l_2 m_3 - l_3 m_2 \\ D_2 &= \det(A_2) = l_3 m_1 - l_1 m_3 \\ D_3 &= \det(A) = l_1 m_2 - l_2 m_1 \end{aligned}$$

Intersecting Lines

- Solution from Cramer's rule

$$x = \frac{D_1}{D_3} \qquad y = \frac{D_2}{D_3} \qquad \begin{aligned} D_1 &= \det(A_1) = l_2 m_3 - l_3 m_2 \\ D_2 &= \det(A_2) = l_3 m_1 - l_1 m_3 \\ D_3 &= \det(A) = l_1 m_2 - l_2 m_1 \end{aligned}$$

- can be homogenously rewritten as

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} D_1/D_3 \\ D_2/D_3 \\ 1 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

Intersecting Lines

- Thus, the solution of

$$\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -l_3 \\ -m_3 \end{bmatrix}$$

- can be expressed in vector form as

$$\mathbf{x} = \frac{1}{D_3}\boldsymbol{D} = \mathbf{l} \times \mathbf{m}$$

- This is the cross product of the lines!

Intersecting Lines

- The intersection of two lines in H.C. is

$$\boxed{\chi = \ell \cap m : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m}}$$

- **Simple way for computing the intersection of two lines using H.C.**

Line Between Two Points

- H.C. also offer a simple way for computing a line through two points
- Given two points $\chi = [x_i], y = [y_i]$, find the line $\ell = [l_i]$ connecting both points

How to find a line that connects two given points?

Line Between Two Points

- H.C. also offer a simple way for computing a line through two points
- Given two points $\chi = [x_i], y = [y_i]$, find the line $\ell = [l_i]$ connecting both points
- We write that as $\ell = \chi \wedge y$ ("wedge")
- Solution via a system of linear eqns.

$$\begin{bmatrix} \mathbf{x} \cdot \mathbf{l} \\ \mathbf{y} \cdot \mathbf{l} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -x_3\, l_3 \\ -y_3\, l_3 \end{bmatrix}$$

Line Between Two Points

- Cramer's rule again solves

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -x_3\, l_3 \\ -y_3\, l_3 \end{bmatrix}$$

- by

$$l_1 = \frac{D_1}{D_3} \qquad l_2 = \frac{D_2}{D_3}$$

- with

$$\begin{aligned} D_1 &= \det(A_1) = l_3(x_2 y_3 - y_2 x_3) \\ D_2 &= \det(A_2) = l_3(x_3 y_1 - y_3 x_1) \\ D_3 &= \det(A) = x_1 y_2 - x_2 y_1 \end{aligned}$$

Line Between Two Points

- Cramer's leads to

$$l_1 = \frac{D_1}{D_3} \qquad l_2 = \frac{D_2}{D_3}$$

$$\begin{aligned} D_1 &= l_3(x_2y_3 - y_2x_3) \\ D_2 &= l_3(x_3y_1 - y_3x_1) \\ D_3 &= x_1y_2 - x_2y_1 \end{aligned}$$

- and we use

$$l_3 = l_3 \frac{D_3}{D_3}$$

- which results in

$$\mathbf{l} = \left[\frac{D_1}{D_3}, \frac{D_2}{D_3}, l_3\frac{D_3}{D_3}\right]^\top \longrightarrow \mathbf{l} = \frac{l_3}{D_3}\begin{bmatrix} x_2y_3 - y_2x_3 \\ x_3y_1 - y_3x_1 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

Line Between Two Points

- We again exploit the cross product and the homogeneous property

$$\mathbf{l} = \frac{l_3}{D_3}\begin{bmatrix} x_2y_3 - y_2x_3 \\ x_3y_1 - y_3x_1 \\ x_1y_2 - x_2y_1 \end{bmatrix} = \begin{bmatrix} x_2y_3 - y_2x_3 \\ x_3y_1 - y_3x_1 \\ x_1y_2 - x_2y_1 \end{bmatrix} = \mathbf{x} \times \mathbf{y}$$

- Thus we obtain

$$\boxed{\mathcal{l} = \mathcal{x} \wedge \mathcal{y} : \quad \mathbf{l} = \mathbf{x} \times \mathbf{y}}$$

Typical Line Operations

- A point lies on a line if

$$\mathbf{x} \cdot \mathbf{l} = 0$$

- Intersection of two lines

$$\mathcal{x} = \mathcal{l} \cap \mathcal{m} : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m}$$

- A line through two given points

$$\mathcal{l} = \mathcal{x} \wedge \mathcal{y} : \quad \mathbf{l} = \mathbf{x} \times \mathbf{y}$$

Points and Lines at Infinity

Points at Infinity

- It is possible to **explicitly** model infinitively distant points **with finite coordinates**

$$\mathcal{X}_\infty : \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- We can **maintain the direction** to that infinitively distant point
- Great tool when working with cameras as they are bearing-only sensors

Intersection at Infinity

- All lines ℓ with $\ell \cdot \mathcal{X}_\infty = 0$ pass through $\mathcal{X}_\infty$
- We can interpret ℓ as a line in Hesse form

$$\underset{\text{Hesse}}{\mathbf{l}} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ -d \end{bmatrix}$$

- First two dimensions determine the direction of the line ℓ
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$

Intersection at Infinity

- All lines ℓ with $\ell \cdot \mathcal{X}_\infty = 0$ pass through $\mathcal{X}_\infty$
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$
- This hold for any line $\mathbf{m} = [\cos\phi, \sin\phi, *]^T$ i.e. for any line that is parallel to ℓ

All parallel lines meet at one point at infinity!

Intersection at Infinity

- All lines ℓ with $\ell \cdot \mathcal{X}_\infty = 0$ pass through $\mathcal{X}_\infty$
- This means $[u, v] \cdot [\cos\phi, \sin\phi] = 0$
- This hold for any line $\mathbf{m} = [\cos\phi, \sin\phi, *]^T$ i.e. for any line that is parallel to ℓ
- This can also be seen by

$$\mathbf{l} \times \mathbf{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ 0 \end{bmatrix}$$

All parallel lines meet at one point at infinity!

Parallel Lines Meet at Infinity

Image Courtesy: J. Jannene

Infinitively Distant Objects

- Infinitively distant point

$$\chi_\infty : \quad \mathbf{x}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

- The infinitively distant line is the **ideal line**

$$\ell_\infty : \quad \mathbf{l}_\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- ℓ_∞ can be interpreted as the horizon

Infinitively Distant Objects

- All points at infinity lie on the line at infinity called the **ideal line** given by

$$\mathbf{x}_\infty \cdot \mathbf{l}_\infty = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

- The ideal line can be seen as the horizon

Analogous for 3D Objects

- 3D point

$$\mathbf{X} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = \begin{bmatrix} U/T \\ V/T \\ W/T \\ 1 \end{bmatrix} \to \begin{bmatrix} U/T \\ V/T \\ W/T \end{bmatrix}$$

- Plane

$$\mathbf{A} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Point on a Plane

- Via the scalar product, we can again test if a point lies on a plane

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{A}^{\mathsf{T}}\mathbf{X} = \mathbf{X}^{\mathsf{T}}\mathbf{A} = 0$$

- which is based on

$AX + BY + CZ + D = 0$ or $N \cdot X - S = 0$

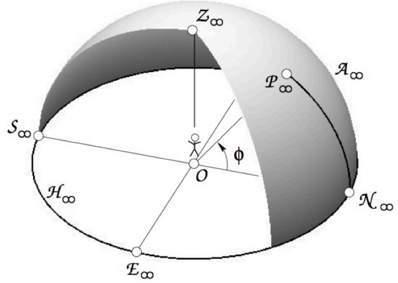
3D Objects at Infinity

- 3D point

$$\mathbf{P}_\infty = \begin{bmatrix} U \\ V \\ W \\ 0 \end{bmatrix}$$

- Plane

$$\mathbf{A}_\infty = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$



Conclusion

- Homogeneous coordinates are an alternative representation for geometric objects
- They can simplify mathematical expressions
- They can model points at infinity
- Easy chaining and inversion of transformations
- Uses an extra dimension (n+1)
- Equivalence up to scale

Being Familiar with Homogeneous Coordinates is Key for the Remaining Course

Literature

- Förstner & Wrobel: Photogrammetric Computer Vision, Springer, 2016
  - Chapter 5.1 – 5.3: H.C., points & lines
  - Chapter 6.1 – 6.4: transformations

Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- **I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.**
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

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