Earth Ellipsoid, ellipsoidal coordinates, satellite coordinate systems

(Part II)

MSc Geodetic Engineering

Module Coordinate Systems
1st Semester, 2020/21

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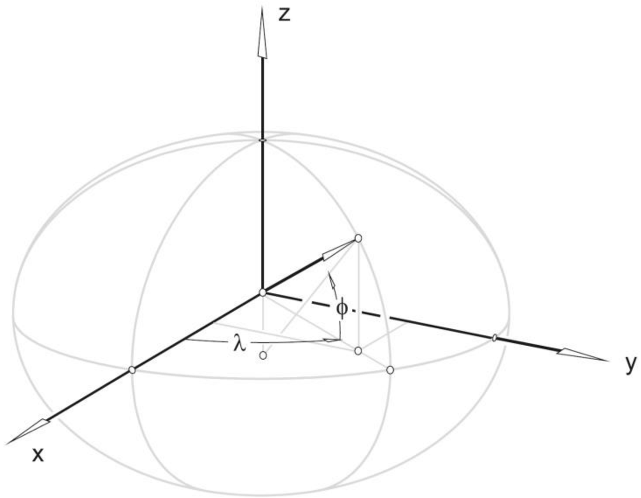
Earth's rotation and inertial coordinate system

Inertial reference system:

- Satellite motion (Kepler orbit) defined in inertial space
- Required for satellite orbit computations, geodetic data analysis, Earth rotation measurements, interplanetary trajectory planning, …
- Practical realization: ICRF (Int. Celestial Reference Frame)



Terrestrial reference system:

- Where all terrestrial geodetic markers are located
- Where the reference ellipsoid refers to (z-axis = rotation axis)
- Practical realization: ITRF (Int. Terr. Reference Frame)

(See lecture MGE-05)

Earth rotation matrix R(t) → Rotation of the ITRF w.r.t. ICRF

Coordinate Systems

Orientation of ITRF in inertial space (ICRF) is described through

- a sequence of rotation matrices
- containing Earth Orientation Parameters (EOPs)

Rotation matrix inertial vs. terrestrial as a function of time t

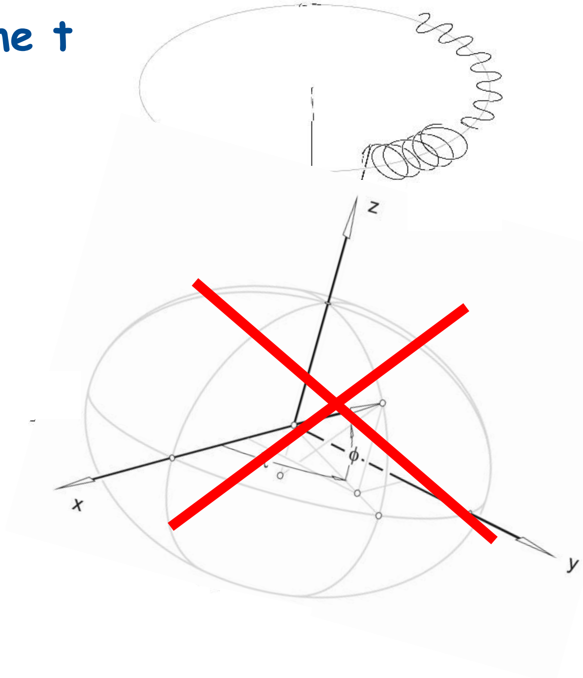
Earth Rotation angle (360° per siderial day)

$$\mathbf{R} \sim \mathbf{R}_3(\Phi^g) = \mathbf{R}_3(\omega_e(t - t_0)) = \mathbf{R}_3(\omega_e t - \Phi_0^g) \qquad \omega_e = \frac{2\pi}{d}$$

(if the Earth would uniformly rotate about its z-axis)

$$= \mathbf{S}(\Phi^g, x_p, y_p)\mathbf{N}(\epsilon_s, \Delta\epsilon, \Delta\psi)\mathbf{P}(-z, \theta, -\zeta)$$

Changing orientation of Earth's rotation axis (see lecture by Prof. Schindelegger)



Orientation of ITRF in inertial space (ICRF) is described through

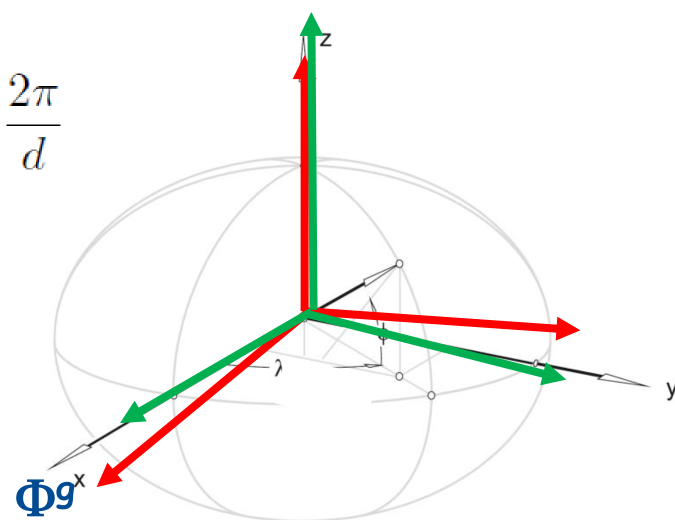
Here: simple rotation about z-axis

Rotation matrix inertial vs. terrestrial as a function of time t

Earth Rotation angle (360° per siderial day)

$$\mathbf{R} \sim \mathbf{R}_3(\Phi^g) = \mathbf{R}_3(\omega_e(t - t_0)) = \mathbf{R}_3(\omega_e t - \Phi^g_0) \qquad \omega_e = \frac{2\pi}{d}$$

(if the Earth would uniformly rotate about its z-axis)

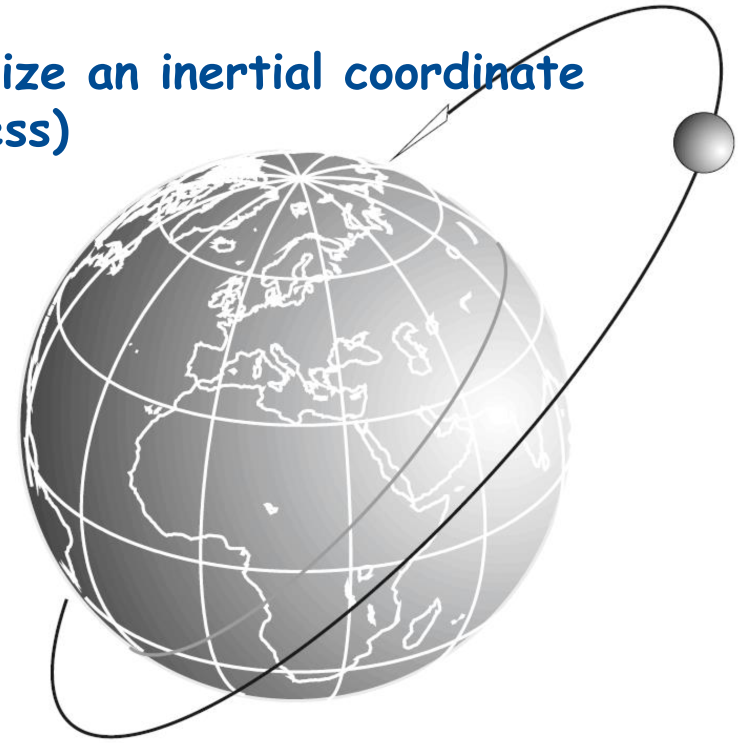


Kepler angles and satellite coordinate systems

Kepler's laws state that

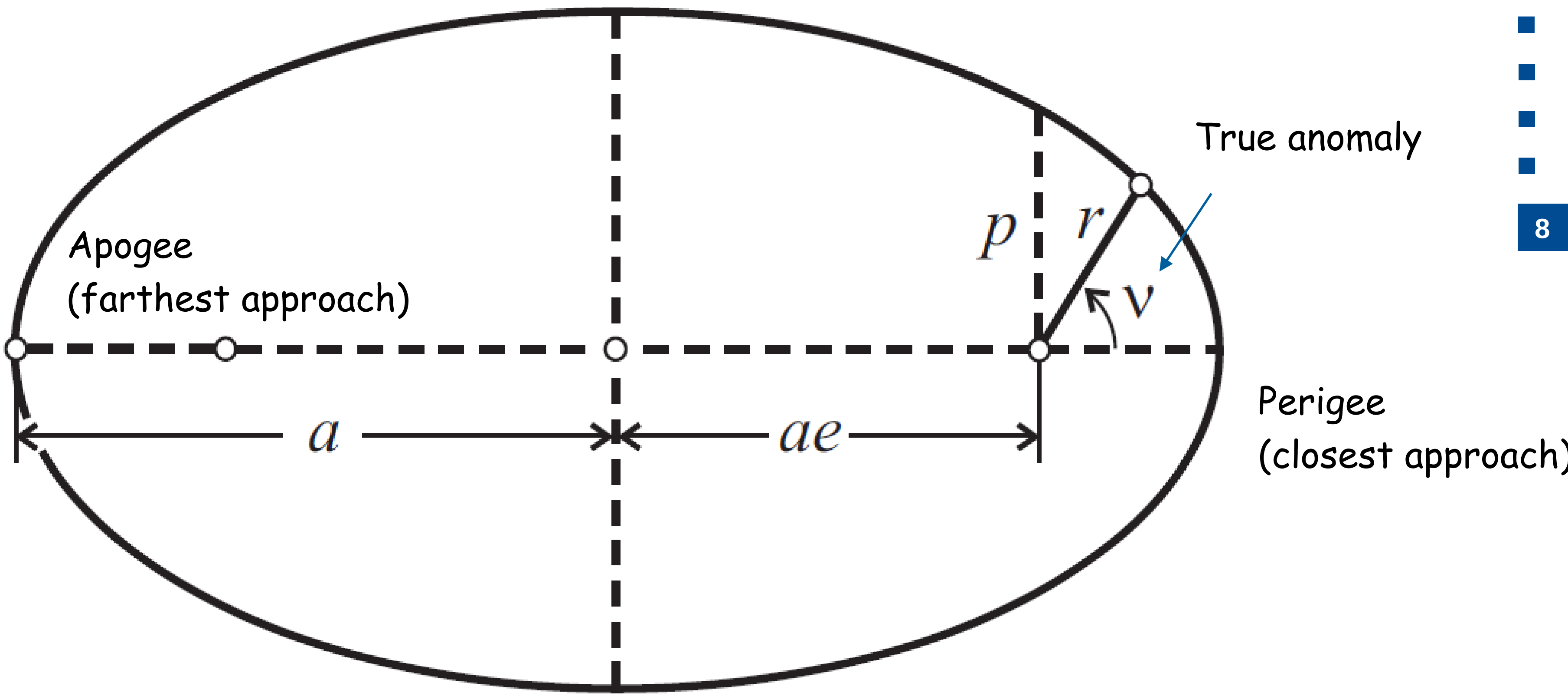
- Planet/satellite orbiting a body with a central gravitational field will be an ellipse
- Orientation of orbital plane of a planet remains fixed in inertial space
- Orientation of orbital plane of a satellite remains fixed in space

→ satellite orbit can be used to realize an inertial coordinate system (which is much easier to access)



Satellite orbit and groundtrack

Orbital ellipse in the Kepler problem



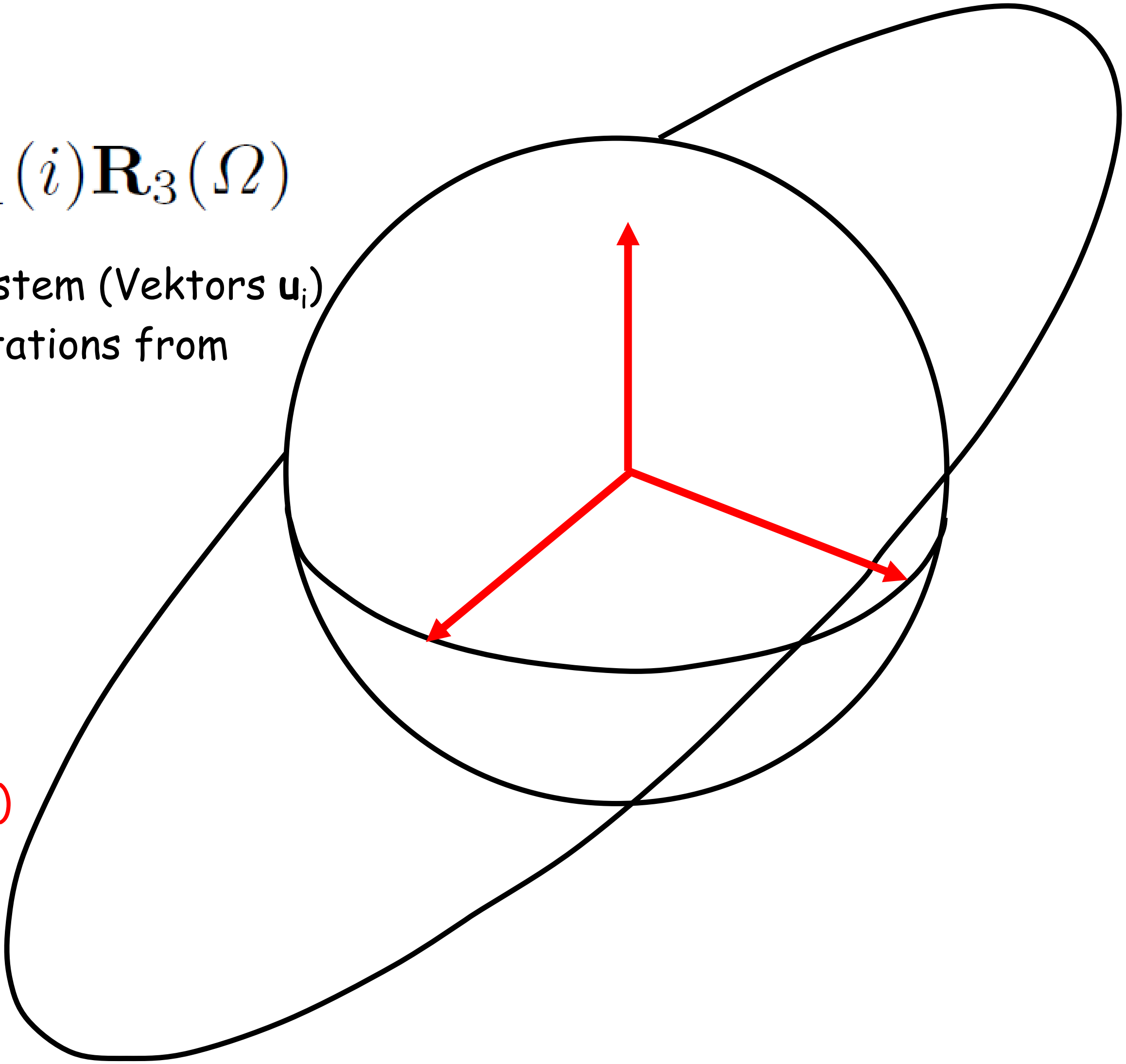
a semi-major axis of orbit, e eccentricity

Coordinate Systems

$$\mathbf{u}_i = \mathbf{R}\mathbf{e}_i$$

$$\mathbf{R} = \mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{R}_3(\Omega)$$

Orbital coordinate system (Vektors $\mathbf{u}_i$) created through 3 rotations from the inertial system
→ Kepler angles

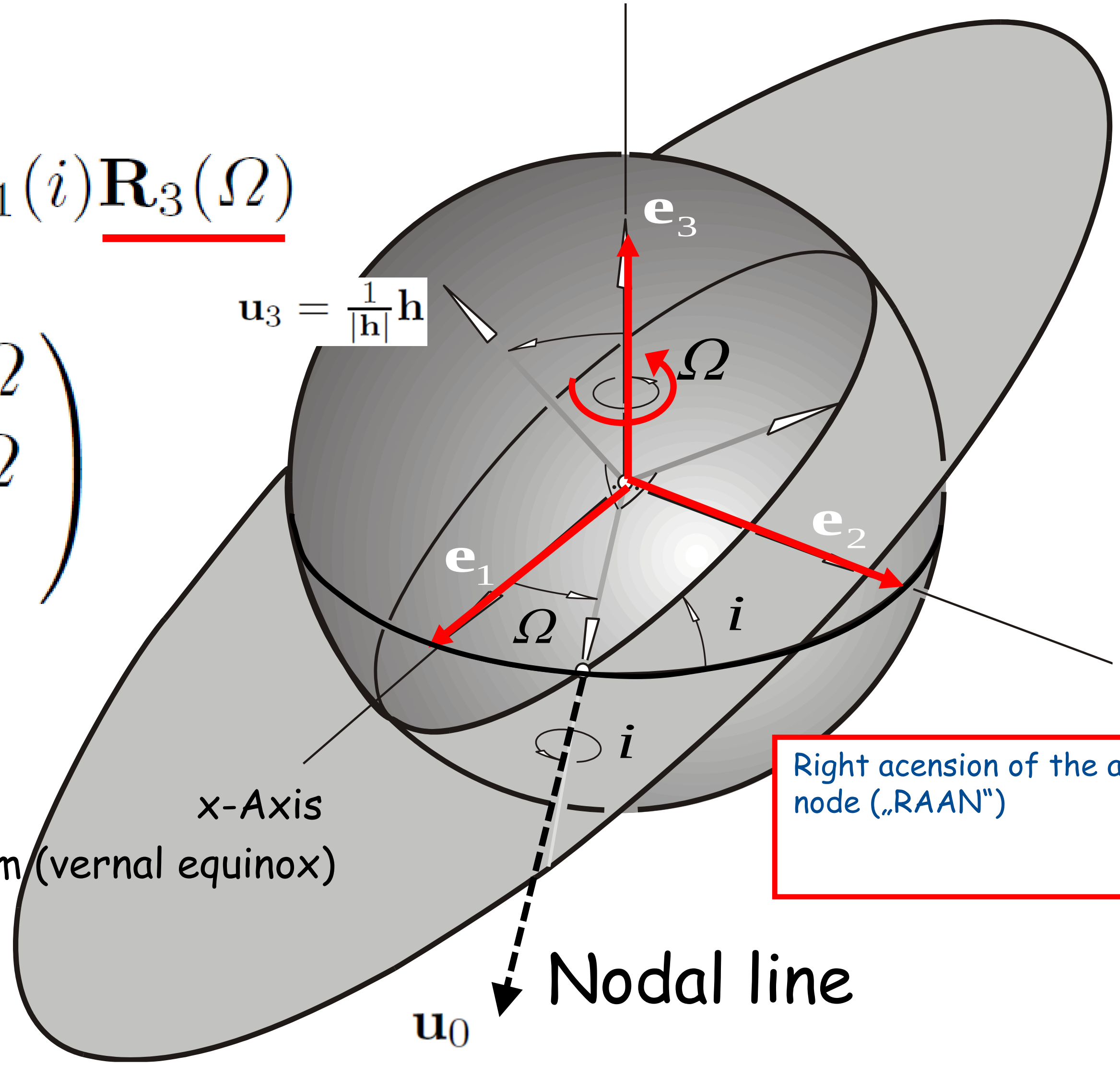


Inertial system (ICRF)

$$\mathbf{u}_i = \mathbf{R}\mathbf{e}_i$$

$$\mathbf{R} = \mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{R}_3(\Omega)$$

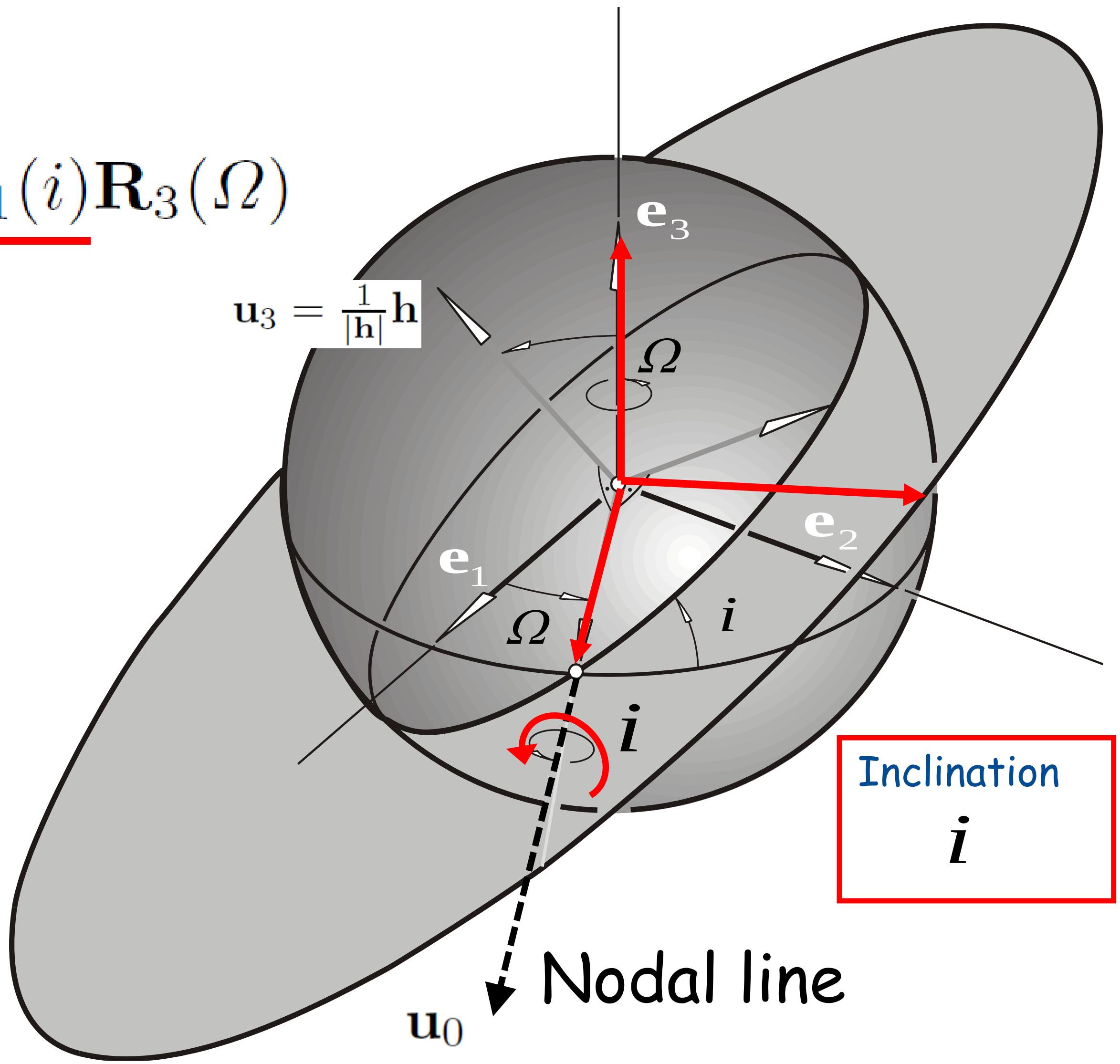
$$\mathbf{u}_0 = \begin{pmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{pmatrix}$$



Right acension of the ascensing node („RAAN") Ω

$$\mathbf{u}_i = \mathbf{R}\mathbf{e}_i$$

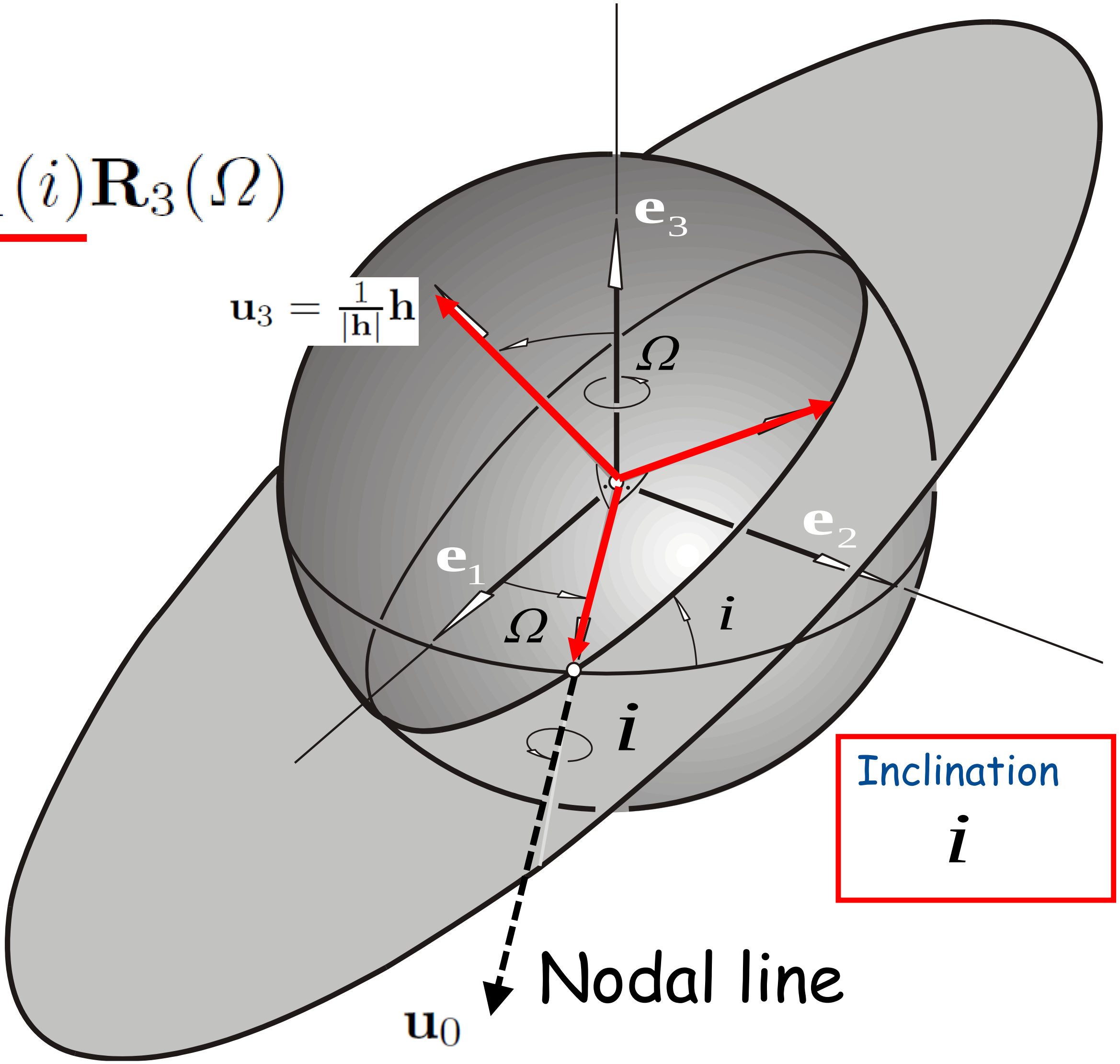
$$\mathbf{R} = \mathbf{R}_3(\omega)\underline{\mathbf{R}_1(i)}\mathbf{R}_3(\Omega)$$



Inclination
i

$$\mathbf{u}_i = \mathbf{R}\mathbf{e}_i$$

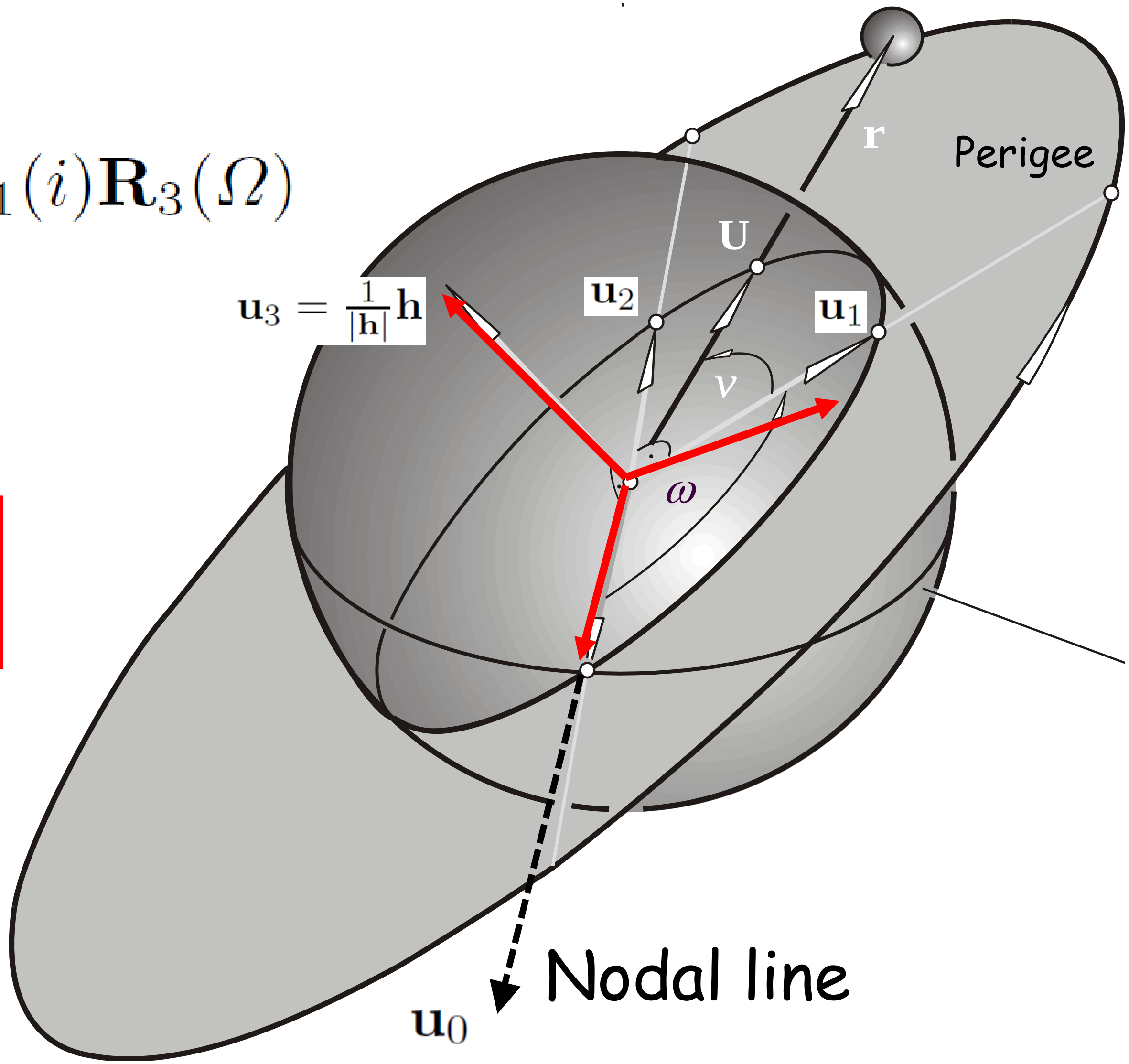
$$\mathbf{R} = \mathbf{R}_3(\omega)\underline{\mathbf{R}_1(i)}\mathbf{R}_3(\Omega)$$



$$\mathbf{u}_i = \mathbf{R}\mathbf{e}_i$$

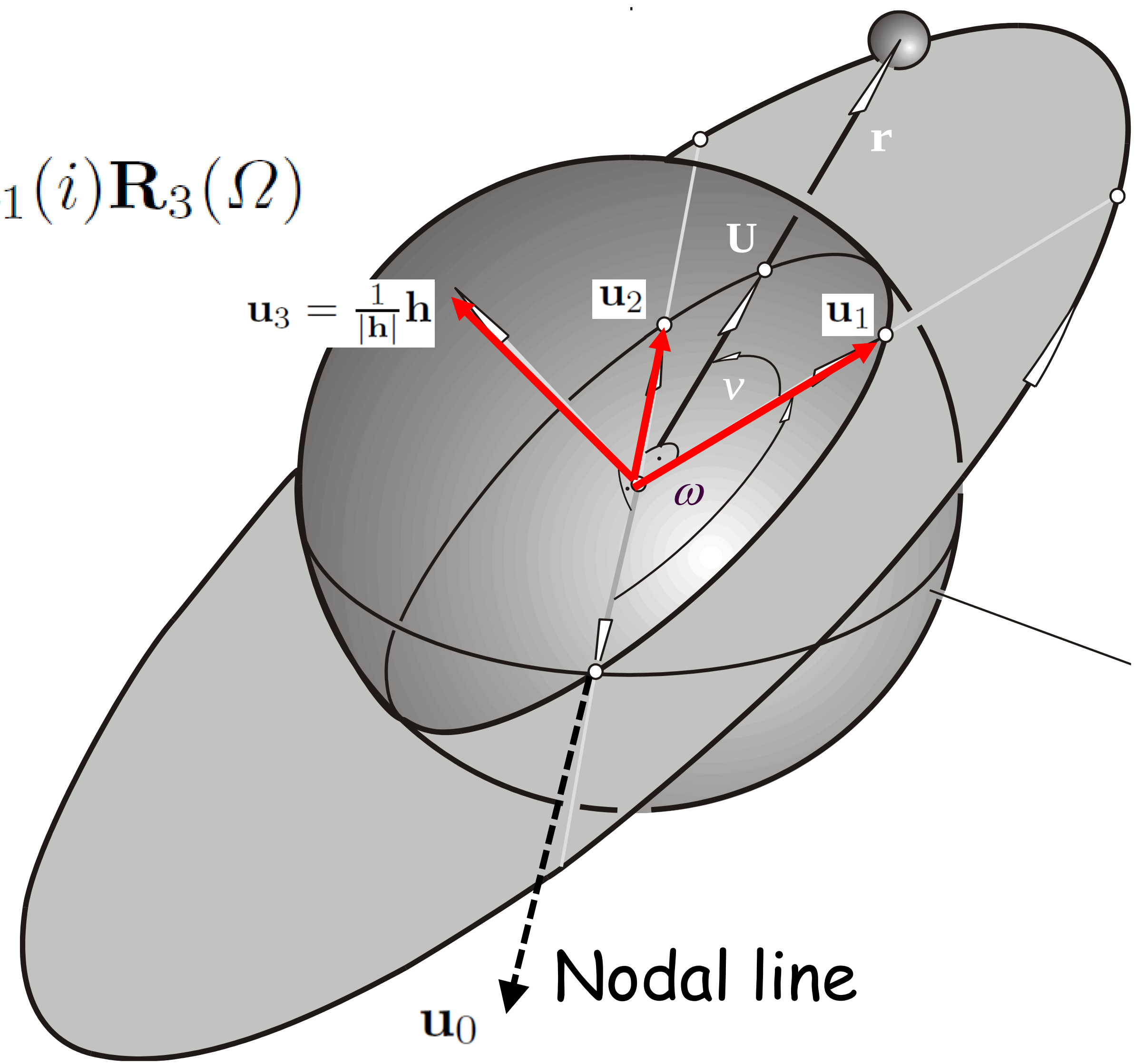
$$\mathbf{R} = \underline{\mathbf{R}_3(\omega)}\mathbf{R}_1(i)\mathbf{R}_3(\Omega)$$

Angle of orbital ellipse perigee w.r.t. nodal line ω



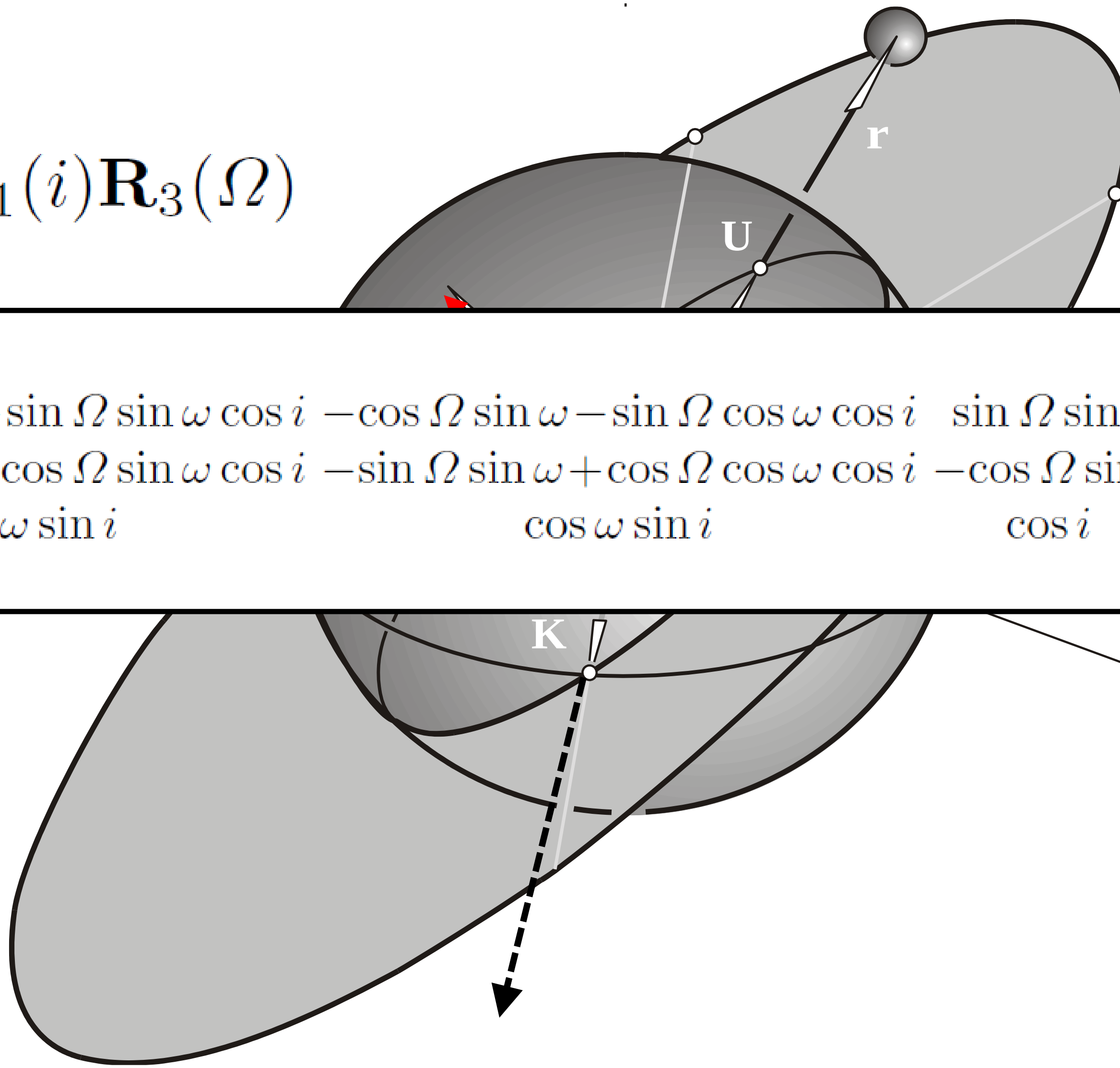
$$\mathbf{u}_i = \mathbf{R}\mathbf{e}_i$$

$$\mathbf{R} = \underline{\mathbf{R}_3(\omega)}\mathbf{R}_1(i)\mathbf{R}_3(\Omega)$$



$$\mathbf{u}_i = \mathbf{R}\mathbf{e}_i$$

$$\mathbf{R} = \mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{R}_3(\Omega)$$



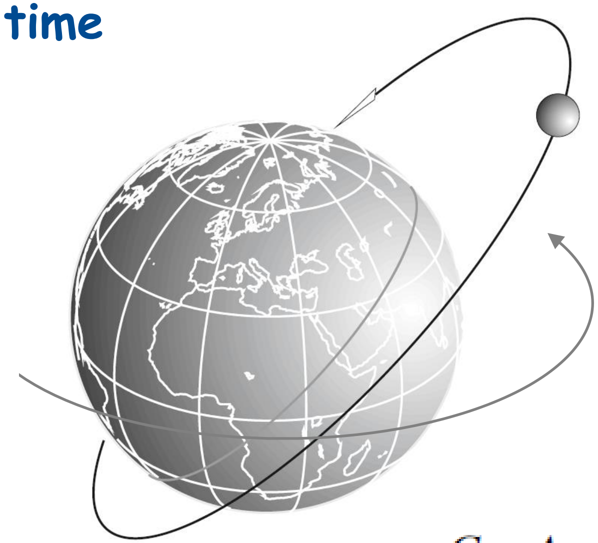
$$\mathbf{R} = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{pmatrix}$$

Kepler's laws state that

- Orientation of satellite orbital plane remains fixed in space

But that's true only if the Earth would have a perfectly spherical gravity field. → Due to Earths dynamical flatting J_2, satellite orbital precession in space → Ω changes with time

$$\frac{d}{dt}\Omega = -\frac{3R^2 n J_2}{2(1-e^2)^2 a^2}\cos i$$



And with respect to the rotating Earth

$$J_2 = \frac{C-A}{M_e a^2}$$

$$\Omega_e = \frac{d\Omega}{dt} - \omega = -\frac{3R^2 n J_2}{2(1-e^2)^2 a^2}\cos i - \omega$$

Ground tracks computation

(Approximate transformation of satellite Keplerian elements to geographical coordinates of sub-satellite point)

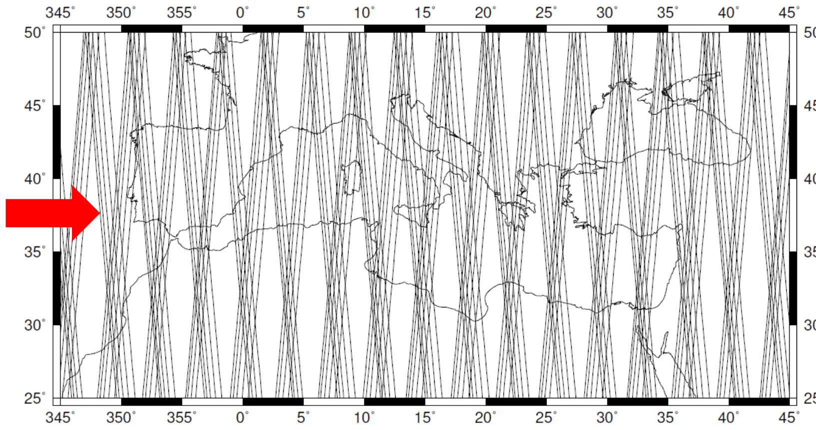
Satellite orbits commonly described in inertial coordinate frame by

- Kepler elements, or
- TLEs ("Two-line elements"), a variant of Kepler elements

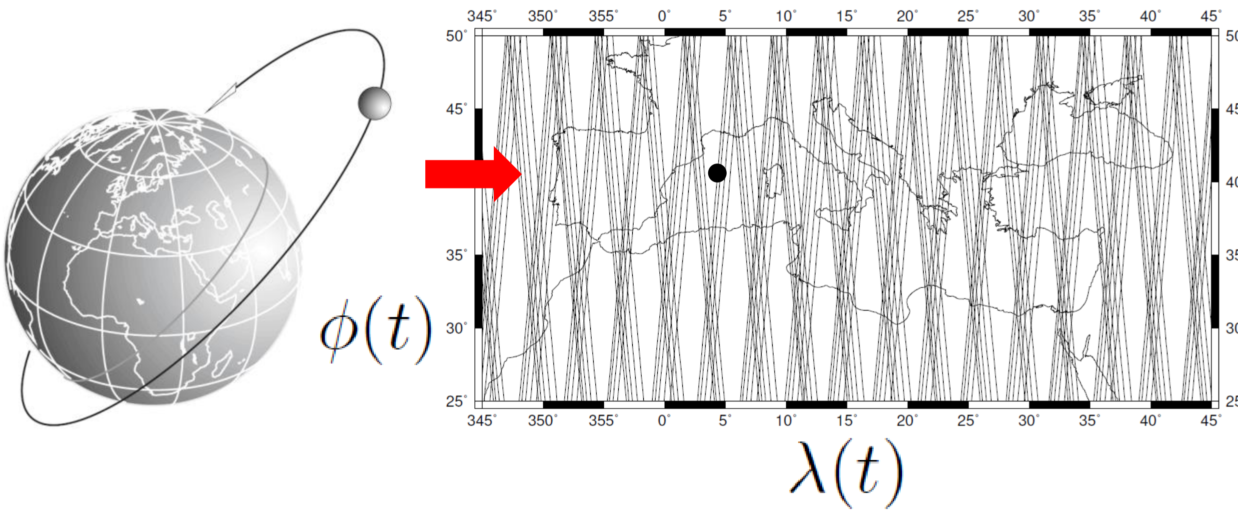
We need a simple way to create ground-track plots, e.g. for visibility planning.

Note: What is described in the following is NOT accurate at the cm-level, i.e. it is not intended for positioning purposes.

Orbits and groundtracks



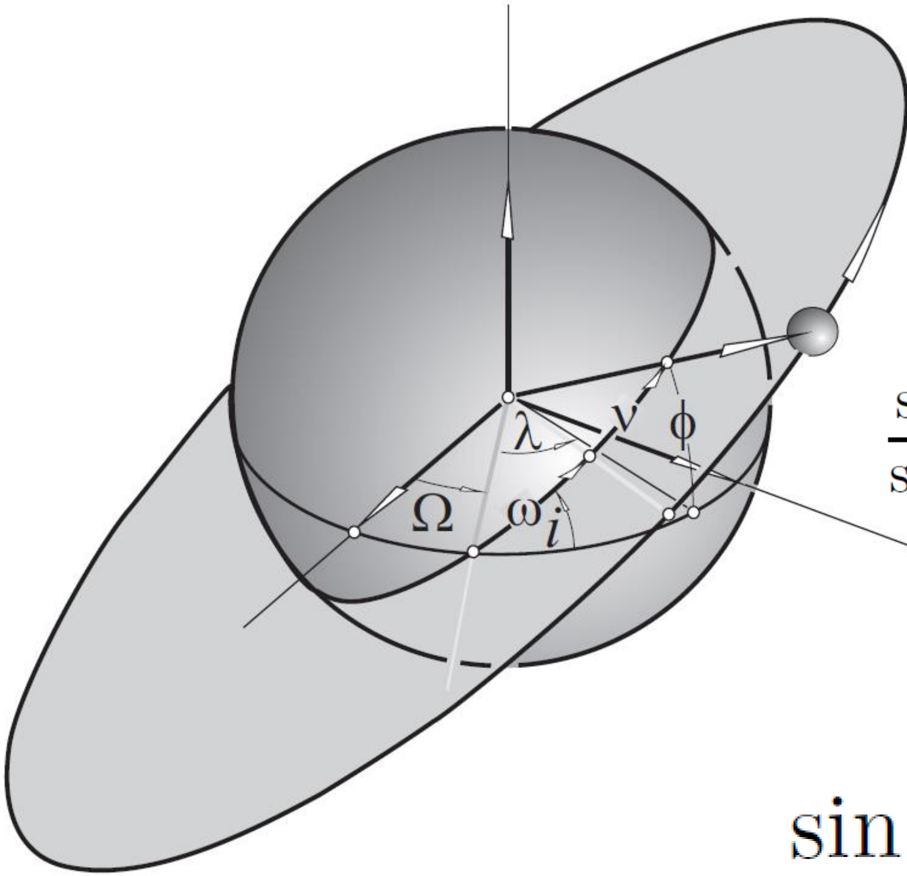
Coordinate Systems

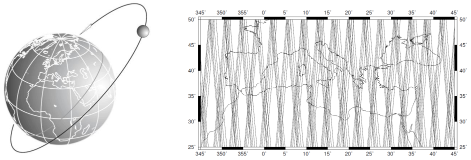


For each satellite, we assume

- circular orbit
- orbital plane described by constant Ω, i, ω
- position along orbit: mean anomaly $\nu(t)$

From spherical trigonometric relations…



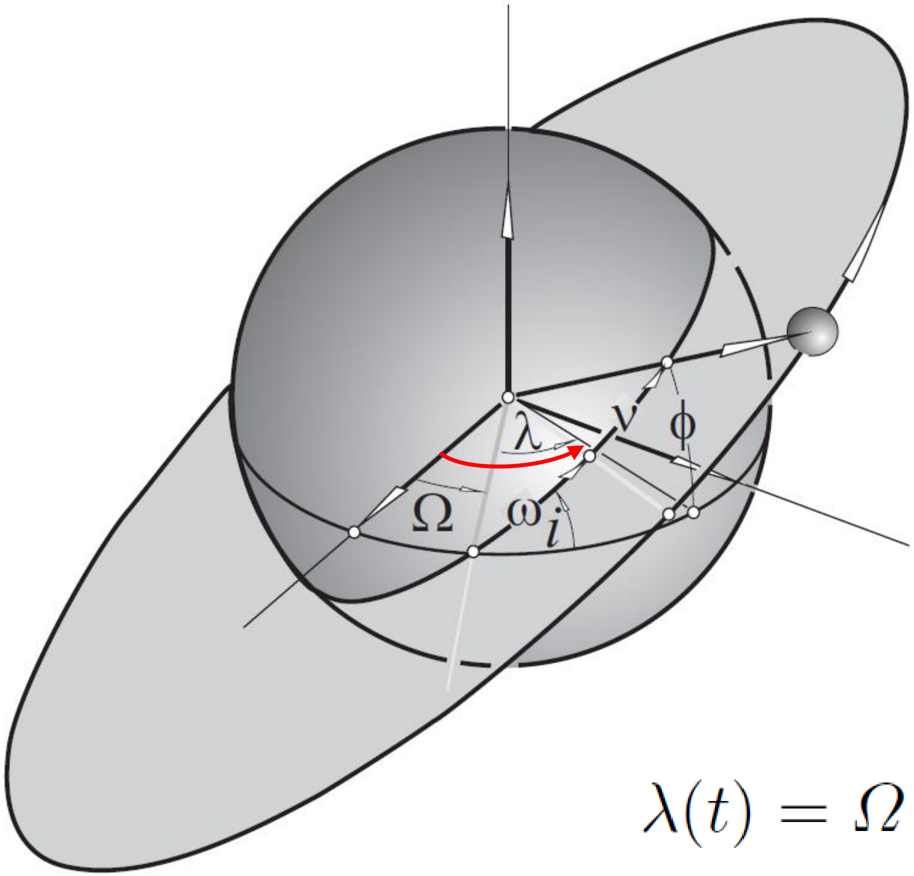


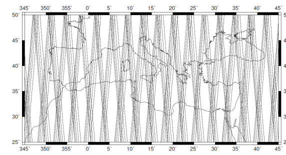
$$\frac{\sin i}{\sin \phi} = \frac{\sin \frac{\pi}{2}}{\sin(\omega + \nu)} = \frac{1}{\sin(\omega + \nu)}$$

$$\sin \phi = \sin i \sin(\omega + \nu)$$

… we find the spherical latitude as a function of the mean anomaly, for given orbit inclination and argument of the perigee

In a similar way we find the longitude



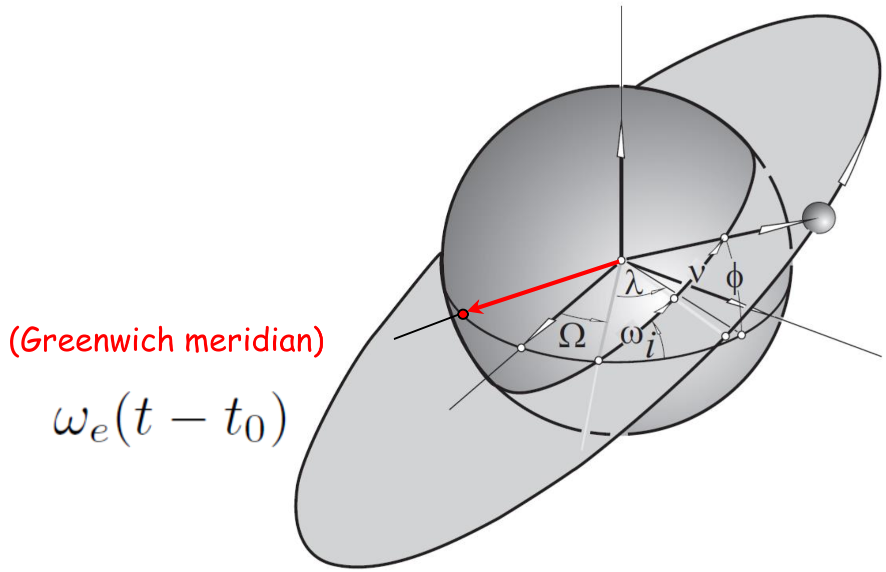


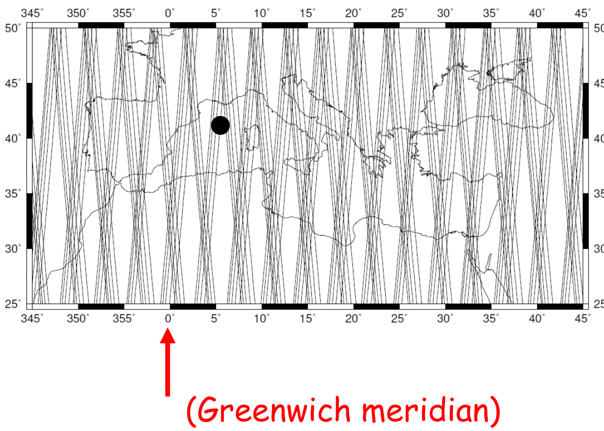
$$\tan\lambda = \frac{\cos(\omega+\nu)\tan i}{\sin\phi}$$

$$\lambda(t) = \Omega + \arctan\left(\frac{\cos(\omega+\nu)\tan i}{\sin\phi}\right)$$

… but we have to add the RAAN Ω to refer the longitude the x-axis of the (inertial) coordinate system

Geographical latitude refers to an Earth-fixed x-axis (Greenwich meridian)





$$\lambda(t) = \omega_e(t-t_0) + \Omega + \arctan\left(\frac{\cos(\omega+\nu)\tan i}{\sin\phi}\right)$$

This means we have to add the Earth's rotation angle, to arrive at Earth-fixed longitude.

(note Earth rotation is simplified here, assuming $\omega_e = 2\pi$ / sidereal day)

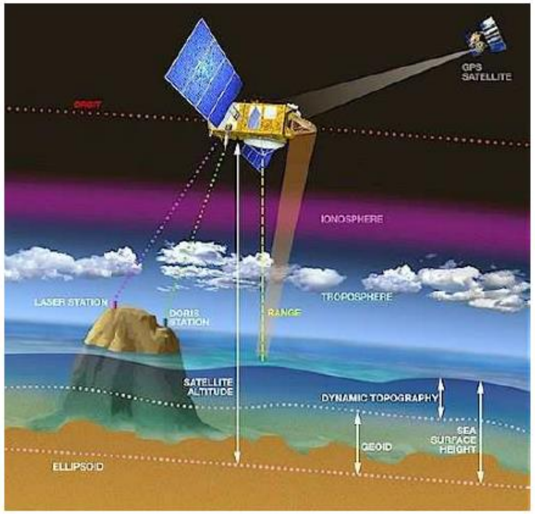
Wrap-up: At this point, you should be able to

- Explain the concepts of ellipsoidal reference surface and ellipsoidal coordinates
- Implement transformations Cartesian w.r.t. ellipsoidal
- Explain the concept of satellite orbital reference frame
- Implement simple mappings from orbital coordinates to Earth-fixed coordinates

Outlook

A more detailed treatment will need to take into account issues of reference frame realization (where exactly is the ITRF w.r.t. Earth crust?), Earth rotation and its variations (motion of Earth spin axis in space and w.r.t. solid Earth, and satellite orbit perturbations (orbits are not exactly ellipsoidal)

Homework Assignement



- is provided via ecampus or via email
- must be handed in within 2 weeks

Your solution must include

- a written, type-set report (i.e. not scanned handwritten) showing command of technical language
- step-by-step explanation of the way of solving (what equations are used)
- all intermediate results
- all results must be provided with (the correct) units
- all results must be provided with the relevant number of digits
- all codes that you used
- if required, figures or drawings