Efficient and Accurate Registration of Point Clouds with Plane to Plane Correspondences – Supplement

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1. Minimal representation of the normal

The minimal representation of the normal is realized by the 3-vector of the mean, a random 2-vector $\underline{\boldsymbol{N}}_r$ along the tangent plane spanned by $\mathsf{J}_r(\underline{\boldsymbol{N}}) = \text{null}(\underline{\boldsymbol{N}}^\mathsf{T})$ with $\mathsf{J}_r^\mathsf{T}\mathsf{J}_r = \mathsf{I}_2$, at $\mathbb{E}(\underline{\boldsymbol{N}})$, and subsequent normalization of the first 3-subvector:

$$\underline{\boldsymbol{N}} = \mathsf{N}^e(\boldsymbol{\mu}_N + \mathsf{J}_r(\boldsymbol{\mu}_N)\underline{\boldsymbol{N}}_r) \tag{1}$$

see [1, Eq. (10.24)].

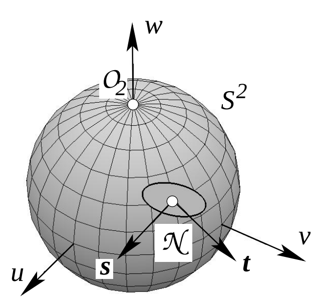

Figure 1: Reduced coordinates for representing an uncertain point $\mathcal{N}(\underline{\boldsymbol{N}})$ on the unit sphere S^2, which represents the projective plane $\mathbb{P}^2$. A point with mean $\boldsymbol{\mu}_N$, which is uncertain on the unit sphere, is projected into the tangent plane at the mean. Its uncertainty in the tangent space, which is the null space of $\boldsymbol{\mu}_N^\mathsf{T}$ and spanned by two basis vectors, say $\boldsymbol{s}$ and $\boldsymbol{t}$, has only two degrees of freedom and leads to a regular 2×2 covariance matrix (the ellipse shown in the figure) of the 2-vector $\boldsymbol{N}_r$ of the reduced coordinates in the tangent plane, adapted from [1, Fig. 10.10]

The random 2-vector $\underline{\boldsymbol{N}}_r$ with covariance matrix $\Sigma_{N_rN_r}$ is called the reduced normal vector, from which we can derive the covariance matrix of the homogeneous vector $\underline{\boldsymbol{N}}$, which is

$$\Sigma_{NN} = \mathsf{J}_r(\boldsymbol{\mu}_N)\,\Sigma_{N_rN_r}\,\mathsf{J}_r^\mathsf{T}(\boldsymbol{\mu}_N)\,. \tag{2}$$

Given a point

$$\underline{\boldsymbol{N}}^t = \boldsymbol{\mu}_N + \mathsf{J}_r(\boldsymbol{\mu}_A)\underline{\boldsymbol{N}}_r \tag{3}$$

in the tangent plane we have

$$\underline{\boldsymbol{N}}_r = \mathsf{J}_r^\mathsf{T}(\boldsymbol{\mu}_N)\,\underline{\boldsymbol{N}}^t \approx \mathsf{J}_r^\mathsf{T}(\boldsymbol{\mu}_N)\,\underline{\boldsymbol{N}}\,; \tag{4}$$

hence

$$\Sigma_{N_rN_r} = \mathsf{J}_r^\mathsf{T}(\boldsymbol{\mu}_N)\,\Sigma_{NN}\,\mathsf{J}_r(\boldsymbol{\mu}_N)\,. \tag{5}$$

2. The ML-solution for a model with constraints between the observations and the parameters

Given are observations, collected in the N-vector $\boldsymbol{y}$, being a sample of $\mathcal{N}(\tilde{\boldsymbol{y}}, \sigma_0^2\Sigma_{yy})$, where σ_0^2 is an unknown variance factor for the given covariance matrix Σ_{yy}. The unknown parameters are collected in the U-vector $\boldsymbol{\beta}$. They are related by the constraints $\boldsymbol{g}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{y}}) = \boldsymbol{0}$ between the true values, which also should hold for the estimated observations and parameters

$$\boldsymbol{g}(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{y}}) = [\boldsymbol{g}(\widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{y}}_i)] = \boldsymbol{0}\,. \tag{6}$$

We assume no two constraints contain the same group $\boldsymbol{y}_i$ of observations. The maximum likelihood solution minimizes the weighted sum

$$\Omega = \widehat{\boldsymbol{v}}^\mathsf{T}\Sigma_{yy}^{-1}\widehat{\boldsymbol{v}} \tag{7}$$

of the residuals,

$$\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{y}} - \boldsymbol{y}\,, \tag{8}$$

under the constraints (6).

Then the estimation can be summarized as follows (see [1, Sect. 4.8]): We start from approximate values $\widehat{\boldsymbol{\beta}}^a$ and $\widehat{\boldsymbol{y}}_i^a$ for the observations and parameters. With the residuals of the constraints

$$\boldsymbol{c}_{g_i} = -\boldsymbol{g}_i(\widehat{\boldsymbol{\beta}}^a, \widehat{\boldsymbol{y}}_i^a) + \mathsf{Z}_i^\mathsf{T}(\widehat{\boldsymbol{y}}_i^a - \boldsymbol{y}_i) \tag{9}$$

the normal equation system is

$$\mathsf{N} = \sum_i \mathsf{B}_i\mathsf{X}_i \quad \text{and} \quad \boldsymbol{n} = \sum_i \mathsf{B}_i\boldsymbol{c}_{g_i} \tag{10}$$

where

$$\mathsf{B}_i = \mathsf{X}_i^\mathsf{T}(\mathsf{Z}_i^\mathsf{T}\Sigma_{ii}\mathsf{Z}_i)^{-1} \tag{11}$$

and the covariance matrices – in our case – are

$$\Sigma_{ii} := \text{Diag}([\Sigma_{A_{ri}A_{ri}}, \Sigma_{A'_{ri}A'_{ri}}]) \,. \tag{12}$$

We used the Jacobians

$$X_i^{\mathsf{T}} := \partial \boldsymbol{g}_i / \partial \boldsymbol{\beta}|_{\beta=\widehat{\beta}, y=\widehat{y}} \tag{13}$$

and

$$Z_i^{\mathsf{T}} := \partial \boldsymbol{g}_i / \partial \boldsymbol{y}|_{\beta=\widehat{\beta}, y=\widehat{y}} \tag{14}$$

to be evaluated at the estimated observations and the estimated parameters. The updates of the parameters usually are $\widehat{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}}^a + \widehat{\Delta\boldsymbol{\beta}}$; in our case we use

$$\widehat{\mathsf{M}}^{(\nu+1)} = \begin{bmatrix} R(\widehat{\Delta \boldsymbol{r}}) & \widehat{\Delta \boldsymbol{t}} \\ \boldsymbol{0}^{\mathsf{T}} & 1 \end{bmatrix} \widehat{\mathsf{M}}^{(\nu)} \,. \tag{15}$$

The observations are corrected to achieve the fitted observations. The corrections are

$$\widehat{\Delta \boldsymbol{y}}_i = \Sigma_{ii} Z (Z_i^{\mathsf{T}} \Sigma_{ii} Z_i)^{-1} (\boldsymbol{c}_{g_i} - X^{\mathsf{T}} \widehat{\Delta\boldsymbol{\beta}}) + (\boldsymbol{y}_i - \widehat{\boldsymbol{y}}_i^a) \,,$$

which finally leads to

$$\begin{bmatrix} \widehat{\mathbf{A}}_i^{(\nu+1)} \\ \widehat{\mathbf{B}}_i^{(\nu+1)} \end{bmatrix} = \begin{bmatrix} u(\widehat{\mathbf{A}}_i^{(\nu)}, \widehat{\Delta \boldsymbol{A}}_{ri}) \\ u(\widehat{\mathbf{B}}_i^{(\nu)}, \widehat{\Delta \boldsymbol{B}}_{ri}) \end{bmatrix} \tag{16}$$

where the update function usually is $\widehat{\boldsymbol{y}}^{(\nu+1)} = \widehat{\boldsymbol{y}}^{(\nu)} + \widehat{\Delta \boldsymbol{y}}$. Here we have for the plane parameters

$$\widehat{\mathbf{A}}_i^{(\nu+1)} = \begin{bmatrix} \mathsf{N}(\boldsymbol{N}^{(\nu)} + \boldsymbol{J}_r(\boldsymbol{N}^{(\nu),\mathsf{T}})\Delta \boldsymbol{N}_r \\ S^{(\nu)} + \Delta S \end{bmatrix} , \tag{17}$$

and similarly for $\mathbf{A}'$. The covariance matrix of the estimated parameters is

$$\Sigma_{\widehat{\xi}\widehat{\xi}} = N^{-1} = \left(\sum_i X_i^{\mathsf{T}} (Z_i^{\mathsf{T}} \Sigma_{ii} Z_i)^{-1} X_i \right)^{-1} . \tag{18}$$

It captures the geometric structure of the observational setup via N^{-1} – in our case the spatial distribution and accuracy of the observed planes. It does not depend on the actual observations, thus can be used for predicting the achievable precision of the motion, given the geometric setup of the observed planes and their uncertainty.

The estimated variance factor is

$$\widehat{\sigma}_0^2 = \frac{\Omega}{3I - 6} \,. \tag{19}$$

It can be used for testing, since $\underline{\widehat{\sigma}_0}^2 | H_0 \sim F(G-U, \infty)$. If the test is not rejected, we also can provide the empirical covariance matrix

$$\widehat{\Sigma}_{\widehat{\xi}\widehat{\xi}} = \widehat{\sigma}_0^2 \Sigma_{\widehat{\xi}\widehat{\xi}} \,. \tag{20}$$

It captures both, the geometric structure of the observational setup via $\Sigma_{\widehat{\xi}\widehat{\xi}}$ but also the consistency of the plane pairs with a spatial motion via the estimated variance factor $\widehat{\sigma}_0^2$.

3. Testing an empirical covariance matrix

Given an $U \times U$ covariance matrix $\widehat{\Sigma}_{\beta\beta}$ estimated from K samples $\boldsymbol{\beta}_k$. Under the hypothesis $\widehat{\Sigma_{\beta\beta}} = \Sigma_{\beta\beta}$, the test statistic X^2(CovM)

$$(K-1) \left[\ln \left(\det \Sigma_{\beta\beta} / \det \underline{\widehat{\Sigma}}_{\beta\beta} \right) - U + \text{tr} \left(\underline{\widehat{\Sigma}}_{\beta\beta} \Sigma_{\beta\beta}^{-1} \right) \right] \tag{21}$$

given the null-hypothesis, is approximately χ^2-distributed with $U(U+1)/$ degrees of freedom, needed for specifying the $U \times U$ covariance matrix $\Sigma_{\beta\beta}$, see [2], Sects. 2.8.7, 4.1.212.

References

[1] W. Förstner and B. P. Wrobel. *Photogrammetric Computer Vision – Statistics, Geometry, Orientation and Reconstruction*. Springer, 2016. 1

[2] K.-R. Koch. *Parameter Estimation and Hypothesis Testing in Linear Models*. Springer, 2nd edition, 1999. 2