Statistically Testing Uncertain Geometric Relations

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Abstract. This paper integrates statistical reasoning and Grassmann-Cayley algebra for making 2D and 3D geometric reasoning practical. The multi-linearity of the forms allows rigorous error propagation and statistical testing of geometric relations. This is achieved by representing all objects in homogeneous coordinates and expressing all relations using standard matrix calculus.[1]
Keywords. Spatial reasoning, uncertainty of geometric entities, statistical testing, Grassmann-Cayley algebra

1 Motivation

Many Computer Vision tasks involve grouping of geometric elements within one image or in 3D space. This requires testing geometric relations such as identity, incidence, parallelity or orthogonality. Due to uncertainty of the elements, checking these relation requires thresholds which in general are difficult to set.

The goal of this paper is to integrate statistical and geometric reasoning by integrating statistical testing theory and Grassmann-Cayley algebra. Grassmann-Cayley algebra has been introduced by [1] and [2] and showed to be useful for analyzing the geometry of image triplets [3]. Representing geometric entities in projective space, thus using homogeneous coordinates, leads to less singular cases, includes entities at infinity and in most cases leads to multi-linear relations, which itself allows to perform error propagation rigorously. On the other hand there is a profound knowledge about optimal hypothesis testing [9, 7] which is appropriate for checking the validity of geometric relations. The use of statistical testing theory reduces the choice of thresholds to the choice of a single value, the significance level.

What is lacking, is the integration of both concepts. This paper integrates statistical reasoning and Grassmann-Cayley algebra for making 2D and 3D geometric reasoning practical. The multi-linearity of the forms allows rigorous error propagation and statistical testing of geometric relations. This is achieved by representing all objects in homogeneous coordinates and expressing all relations using standard matrix calculus. The goal is to derive a simple rule for testing the basic relations between 2D points and lines, and 3D points, lines and planes, namely identity, incidence, parallelity and orthogonality.

The solution proposed have been developed parallel to the one given in [6]. They are equivalent to those but much more transparent.

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2 Geometric Elements and their Relations

2.1 Representation of Geometric Elements

We deal with points, (infinite) lines and planes in 2D and 3D space and represent them in projective space so that entities at infinity can be used, too. Because we are in projective space, we will use homogeneous vectors, denoted as "$\mathbf{v}$" in upright bold letters, then $\lambda\mathbf{v}$ represents the same element as $\mathbf{v}$. Euclidean vectors will be used with italic bold letters "$\boldsymbol{v}$". Furthermore we will use the convention that homogeneous resp. euclidean matrices are denoted as sans serif letters "S","S".

Elements in 2D have lowercase letters "$\mathbf{v}$", "$\boldsymbol{v}$"; elements in 3D have uppercase letters "$\mathbf{V}$", "$\boldsymbol{V}$". For points we use the letters "$\mathbf{x}$", "$\mathbf{y}$", "$\mathbf{X}$", "$\mathbf{Y}$", for lines "$\mathbf{l}$", "$\mathbf{m}$", "$\mathbf{L}$", "$\mathbf{M}$" and for planes "$\mathbf{A}$", "$\mathbf{B}$". The notation is shown in table 1.

	2D	3D
point	$\mathbf{x}^\mathsf{T} = (u, v; w) = (\boldsymbol{x}_0^\mathsf{T}, x)$	$\mathbf{X}^\mathsf{T} = (U, V, W; T) = (\boldsymbol{X}_0^\mathsf{T}, X)$
line	$\mathbf{l}^\mathsf{T} = (a, b; c) = (\boldsymbol{l}^\mathsf{T}, l_0)$	$\mathbf{L}^\mathsf{T} = (L_1, L_2, L_3; L_4, L_5, L_6) = (\boldsymbol{L}^\mathsf{T}, \boldsymbol{L}_0^\mathsf{T})$
plane	-	$\mathbf{A}^\mathsf{T} = (A, B, C; D) = (\boldsymbol{A}^\mathsf{T}, A_0)$

Table 1. *Homogeneous representation of points, lines and planes in 2D and 3D. A line in 3D is represented with Plücker coordinates, see text for details.*

Note that a 3D line $\mathbf{L}$ is represented in *Plücker coordinates* yielding a homogeneous vector $\mathbf{L} = (\boldsymbol{L}^\mathsf{T}, \boldsymbol{L}_0^\mathsf{T})$ which has to fulfill the *Plücker condition*

$$L_1 L_4 + L_2 L_5 + L_3 L_6 = \boldsymbol{L}^\mathsf{T} \boldsymbol{L}_0 = 0 \tag{1}$$

From the construction of a line given two points, cf. sec. 2.2, the interpretation of the line coordinates will be clear: the vector $\boldsymbol{L} = (L_1, L_2, L_3)^\mathsf{T}$ is the direction of the line and the vector $\boldsymbol{L}_0 = (L_4, L_5, L_6)^\mathsf{T}$ is the normal of the plane through the line and the origin. The Plücker condition expresses the orthogonality condition of these two vectors.

The dual of a geometric entity is overlined, so that e.g. the dual of a line $\mathbf{L}$ is denoted as $\overline{\mathbf{L}}$, which is defined as

$$\overline{\mathbf{L}}^\mathsf{T} = (L_4, L_5, L_6; L_1, L_2, L_3) = (\mathsf{Q}\mathbf{L})^\mathsf{T} = (\boldsymbol{L}_0^\mathsf{T}, \boldsymbol{L}^\mathsf{T}) \quad \text{with} \quad \mathsf{Q} \doteq \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2}$$

The index zero in the definition of the geometric entities in table 1 is chosen such that the distance $d_{\bullet,0}$ of the objects to the origin are:

$$d_{x,0} = \frac{|\boldsymbol{x}_0|}{|x|} \qquad d_{l,0} = \frac{|l_0|}{|\boldsymbol{l}|} \qquad d_{X,0} = \frac{|\boldsymbol{X}_0|}{|X|} \qquad d_{A,0} = \frac{|A_0|}{|\boldsymbol{A}|} \qquad d_{L,0} = \frac{|\boldsymbol{L}_0|}{|\boldsymbol{L}|}$$

2.2 Construction of Geometric Entities

We use two basic operations to construct a new geometric entity $\mathbf{u}$ from two given entities $\mathbf{v}_1$ and $\mathbf{v}_2$: (i) the *join* $\mathbf{u} = \mathbf{v}_1 \wedge \mathbf{v}_2$ yields the minimal linear space containing both entities $\mathbf{v}_1$ and $\mathbf{v}_2$, e. g. a line can be the join of two given points.

(ii) the *intersection* $\mathbf{u} = \mathbf{v}_1 \cap \mathbf{v}_2$ yields the maximal common linear subspace of $\mathbf{v}_1$ ad $\mathbf{v}_2$, e. g. a point can be the intersection of two given lines. Note that join and intersection are dual operators, i. e. $\overline{\mathbf{u} = \mathbf{v}_1 \wedge \mathbf{v}_2} \Leftrightarrow \overline{\mathbf{u}} = \overline{\mathbf{v}}_1 \cap \overline{\mathbf{v}}_2$.

2D Entities. We can construct a 2D line $\mathbf{l}$ as the join of two points $\mathbf{x}$ and $\mathbf{y}$ and a point $\mathbf{x}$ as the intersection of two lines $\mathbf{l}$ and $\mathbf{m}$

$$\mathbf{l} = \mathbf{x} \wedge \mathbf{y} \doteq \mathbf{x} \times \mathbf{y} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x} = -\mathbf{y} \wedge \mathbf{x} \tag{3}$$

$$\mathbf{x} = \mathbf{l} \cap \mathbf{m} = \mathbf{l} \times \mathbf{m} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l} = -\mathbf{m} \wedge \mathbf{l} \tag{4}$$

with the skew matrix $\mathsf{S}(\mathbf{x})$ of a 3-vector $\mathbf{x}$ and its nullspace $\mathcal{N}(\mathsf{S}(\mathbf{x}))$

$$\mathsf{S}(\mathbf{x}) = \frac{\partial(\mathbf{x} \wedge \mathbf{y})}{\partial \mathbf{y}} = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix} \qquad \mathcal{N}(\mathsf{S}(\mathbf{x})) = \mathbf{x} \tag{5}$$

Eq. (4) follows from (3), because of duality: $\overline{\mathbf{l} = \mathbf{x} \wedge \mathbf{y}} \Leftrightarrow \overline{\mathbf{l}} = \overline{\mathbf{x}} \cap \overline{\mathbf{y}}$. The matrix $\mathsf{S}(\mathbf{x})$ induces the cross product and at the same time is equal to the Jacobian of the cross product useful for error propagation.

3D Entities. We can construct a 3D line $\mathbf{L}$ as a join of two points or as intersection of two planes with the antisymmetric forms

$$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} \doteq \qquad \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X} = -\mathbf{Y} \wedge \mathbf{X} \tag{6}$$

$$\mathbf{L} = \mathbf{A} \cap \mathbf{B} = \overline{\overline{\mathbf{A}} \wedge \overline{\mathbf{B}}} = \mathsf{Q}(\mathbf{A} \wedge \mathbf{B}) = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A} = -\mathbf{B} \cap \mathbf{A} \tag{7}$$

with Q from (2), the Jacobians and the nullspace of its transpose

$$\underbrace{\Pi(\mathbf{X})}_{6\times 4} \doteq \frac{\partial(\mathbf{X} \wedge \mathbf{Y})}{\partial \mathbf{Y}} = \begin{pmatrix} XI & -\boldsymbol{X}_0 \\ S(\boldsymbol{X}_0) & \mathbf{0} \end{pmatrix} \qquad \mathcal{N}(\Pi(\mathbf{X})^{\mathsf{T}}) = \mathbf{X} \tag{8}$$

$$\overline{\Pi}(\mathbf{A}) \doteq \frac{\partial(\mathbf{A} \cap \mathbf{B})}{\partial \mathbf{B}} = \mathsf{Q}\Pi(\mathbf{A}) \qquad \mathcal{N}(\overline{\Pi}(\mathbf{A})^{\mathsf{T}}) = \mathbf{A} \tag{9}$$

Observe $\Pi(\mathbf{X})\mathbf{X} = \mathbf{0}, \forall \mathbf{X}$ and $\mathrm{rk}\Pi(\mathbf{X}) = 3$. The line coordinates obviously are bilinear in the homogeneous coordinates for points and for planes. Setting the fourth coordinate of the two homogeneous vectors to 1, we find $\boldsymbol{L} = \boldsymbol{Y} - \boldsymbol{X}$ and $\boldsymbol{L}_0 = \boldsymbol{X} \times \boldsymbol{Y}$ with the Euclidean coordinates $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the two points. Eq. (7) follows from (6) because of duality reasons.

We also obtain the intersection of a line and a plane and the join of a point and a line with anti-symmetric forms

$$\mathbf{X} = \mathbf{A} \cap \mathbf{L} \doteq \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L} = -\overline{\Gamma}(\mathbf{L})\mathbf{A} = \overline{\Gamma}^{\mathsf{T}}(\mathbf{L})\mathbf{A} = -\mathbf{L} \cap \mathbf{A} \tag{10}$$

$$\mathbf{A} = \mathbf{X} \wedge \mathbf{L} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L} = -\Gamma(\mathbf{L})\mathbf{X} = \Gamma^{\mathsf{T}}(\mathbf{L})\mathbf{X} = -\mathbf{L} \wedge \mathbf{X} \tag{11}$$

with the skew symmetric Jacobians and their null space

$$\underbrace{\Gamma(\mathbf{L})}_{4\times 4} \doteq \frac{\partial(\mathbf{L} \wedge \mathbf{X})}{\partial \mathbf{X}} = \begin{pmatrix} -S(\boldsymbol{L}) & -\boldsymbol{L}_0 \\ \boldsymbol{L}_0^{\mathsf{T}} & 0 \end{pmatrix} \quad \mathcal{N}(\Gamma(\mathbf{L})) = \left(\begin{array}{c|c} \boldsymbol{L} & \boldsymbol{L} \times \boldsymbol{L}_0 \\ 0 & |\boldsymbol{L}|^2 \end{array}\right) \tag{12}$$

$$\overline{\Gamma}(\mathbf{L}) \doteq \frac{\partial(\mathbf{L} \cap \mathbf{A})}{\partial \mathbf{A}} = \Gamma(\mathsf{Q}\mathbf{L}) \qquad \mathcal{N}(\overline{\Gamma}(\mathbf{L})) = \left(\begin{array}{c|c} \boldsymbol{L}_0 & \boldsymbol{L}_0 \times \boldsymbol{L} \\ 0 & |\boldsymbol{L}_0|^2 \end{array}\right) \tag{13}$$

Again eq. (11) follows from (10) because of duality.

If the line is given by the intersection of two planes $\mathbf{L} = \mathbf{A} \cap \mathbf{B}$ we have the anti-symmetric matrix

$$\Gamma(\mathbf{A} \cap \mathbf{B}) = \mathbf{A}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{A}^{\mathsf{T}} = -\Gamma(\mathbf{B} \cap \mathbf{A}) \qquad (14)$$

which can be easily verified using (7) and (9). Observe the null space of the matrices to be of rank 2, therefore $\mathrm{rk}\Gamma(\mathbf{L}) = \mathrm{rk}\overline{\Gamma}(\mathbf{L}) = 2$, and $\overline{\Gamma}(\mathbf{L})\Gamma(\mathbf{L}) = \mathbf{0}$, due to (1).

Finally we have two constructors using three entities: determining a plane $\mathbf{A}$ from three points $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ resp. a point $\mathbf{X}$ from three planes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ leads to the trilinear forms:

$$\mathbf{A} = (\mathbf{Y} \wedge \mathbf{Z}) \wedge \mathbf{X} = \Gamma(\Pi(\mathbf{Y})\mathbf{Z})\ \mathbf{X} = \Gamma(\Pi(\mathbf{Z})\mathbf{X})\ \mathbf{Y} = \Gamma(\Pi(\mathbf{X})\mathbf{Y})\ \mathbf{Z} \qquad (15)$$

$$\mathbf{X} = (\mathbf{B} \cap \mathbf{C}) \cap \mathbf{A} = \Gamma(\Pi(\mathbf{B})\mathbf{C})\ \mathbf{A} = \Gamma(\Pi(\mathbf{C})\mathbf{A})\ \mathbf{B} = \Gamma(\Pi(\mathbf{A})\mathbf{B})\ \mathbf{C} \qquad (16)$$

Table 2 summarizes the expressions for constructing new geometric entities.

entities	construction	expression	eq.
points $\mathbf{x}, \mathbf{y}$	$\mathbf{l} = \mathbf{x} \wedge \mathbf{y}$	$\mathbf{l} = S(\mathbf{x})\mathbf{y} = -S(\mathbf{y})\mathbf{x}$	(3)
lines $\mathbf{l}, \mathbf{m}$	$\mathbf{x} = \mathbf{l} \cap \mathbf{m}$	$\mathbf{x} = S(\mathbf{l})\mathbf{m} = -S(\mathbf{m})\mathbf{l}$	(3)
points $\mathbf{X}, \mathbf{Y}$	$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$	$\mathbf{L} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$	(6)
planes $\mathbf{A}, \mathbf{B}$	$\mathbf{L} = \mathbf{A} \cap \mathbf{B}$	$\mathbf{L} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A}$	(7),(2)
point $\mathbf{X}$, line $\mathbf{L}$	$\mathbf{A} = \mathbf{X} \wedge \mathbf{L}$	$\mathbf{A} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L} = -\Gamma(\mathbf{L})\mathbf{X}$	(11)
plane $\mathbf{A}$, line $\mathbf{L}$	$\mathbf{X} = \mathbf{A} \cap \mathbf{L}$	$\mathbf{X} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L} = -\overline{\Gamma}(\mathbf{L})\mathbf{A}$	(10)
points $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$	$\mathbf{A} = \mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z}$	$\Gamma(\mathbf{X} \wedge \mathbf{Y})\mathbf{Z} = \Gamma(\mathbf{Y} \wedge \mathbf{Z})\mathbf{X} = \Gamma(\mathbf{Z} \wedge \mathbf{X})\mathbf{Y}$	(15)
planes $\mathbf{A}, \mathbf{B}, \mathbf{C}$	$\mathbf{X} = \mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$	$\Gamma(\mathbf{A} \cap \mathbf{B})\mathbf{C} = \Gamma(\mathbf{B} \cap \mathbf{C})\mathbf{A} = \Gamma(\mathbf{C} \cap \mathbf{A})\mathbf{B}$	(16)

Table 2. *Construction of new geometric entities. The matrices S, Π, $\overline{\Pi}$, Γ and $\overline{\Gamma}$ are given in eqs.* (5), (8), (9), (12) *and* (13) *resp. All forms are linear in the coordinates of the given entities allowing rigorous error propagation.*

2.3 Geometric Relations between Entities

We explore four types of geometric relationships between entities: identity, incidence, parallelity and orthogonality. Identity can be checked by the difference of the vectors representing the entities, parallelity and orthogonality can be checked easily using the direction vectors of the lines and planes, see table 3. Here we want to focus on possible *incidence* relations.

Incidence of two objects can use the inner products, namely for points and lines in the plane, for points and planes in 3D-space and for pairs of lines

$$< \mathbf{x}, \mathbf{l} >= \mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \qquad < \mathbf{X}, \mathbf{A} >= \mathbf{X}^{\mathsf{T}}\mathbf{A} = 0 \qquad < \mathbf{L}, \mathbf{M} > \doteq \mathbf{L}^{\mathsf{T}}\overline{\mathbf{M}} = 0 \quad (17)$$

The first two relations directly follow from the Hessian form of the 2D line and the plane. We can prove the last relation $< \mathbf{L}, \mathbf{M} >= 0$ easily: let $\mathbf{L}$ the join $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$. Then the intersection condition is equivalent to the condition that the point $\mathbf{X}$ to lie in the plane $\mathbf{A} = \mathbf{Y} \wedge \mathbf{M} = \Pi^{\mathsf{T}}(\mathbf{Y})\overline{\mathbf{M}}$ which leads to $\mathbf{X}^{\mathsf{T}}\mathbf{A} = (\mathbf{X}^{\mathsf{T}}\Pi^{\mathsf{T}}(\mathbf{Y}))\ \overline{\mathbf{M}} = -\mathbf{L}^{\mathsf{T}}\overline{\mathbf{M}} = 0$.

Finally we want to test if two lines intersect, in case they are given by two points $\mathbf{X}, \mathbf{Y}$ or two planes $\mathbf{A}, \mathbf{B}$, being 4-linear forms

$$|\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4| = 0 \qquad |\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4| = 0 \qquad \mathbf{X}^\mathsf{T}(\mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T})\mathbf{Y} = 0$$

The first and second condition results from the coplanarity of the points or from the intersection condition for four planes. The last condition uses $\mathbf{X}^\mathsf{T}\mathbf{R} = \mathbf{X}^\mathsf{T}(-\Gamma(\mathbf{L})\mathbf{Y}) = 0$ where the plane $\mathbf{R} = \mathbf{Y} \wedge \mathbf{L}$ is the join of $\mathbf{Y}$ and $\mathbf{L} = \mathbf{A} \cap \mathbf{B}$.

3 Statistical Tests

3.1 Error Propagation

We represent the *uncertainty of a vector* $\underline{\boldsymbol{x}}$ using the second moments of its probability distribution, namely its covariance matrix $\boldsymbol{\Sigma}_{xx} = E[(\boldsymbol{x} - \mu_x)(\boldsymbol{x} - \mu_x)^\mathsf{T}]$ and write its first and second moments as $\underline{\boldsymbol{x}} \sim M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$.

The covariance matrix contains on its diagonals the variances $\sigma^2_{x_i} = \boldsymbol{\Sigma}_{x_i x_i}$, with standard deviations σ_{x_i} and covariances $\sigma_{x_i x_j}$ describing the mutual statistical dependencies, observe $\sigma^2_{x_i} \doteq \sigma_{x_i x_i}$.

We use the standard technique for *error propagation*: Given a stochastical vector with first and second moments $\underline{\boldsymbol{y}} \sim M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ and a vector valued function $\underline{\boldsymbol{y}} = \boldsymbol{f}(\underline{\boldsymbol{x}})$ with Jacobian $J = (\partial \boldsymbol{f}(\boldsymbol{x})/\partial \boldsymbol{x})$ then the second moments of $\underline{\boldsymbol{y}}$ are $\underline{\boldsymbol{y}} \sim M(\boldsymbol{f}(\boldsymbol{\mu}_x), J\boldsymbol{\Sigma}_{xx}J^\mathsf{T})$ (cf. [7] , eq. 233.2).

Our tests all work on bilinear functions $\underline{\boldsymbol{z}} = \underline{\boldsymbol{z}}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})$ of two stochastical vectors $\underline{\boldsymbol{x}} \sim M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ and $\underline{\boldsymbol{y}} \sim M(\boldsymbol{\mu}_y, \boldsymbol{\Sigma}_{yy})$. They can be written in the form

$$\underline{\boldsymbol{z}} = U(\underline{\boldsymbol{y}})\underline{\boldsymbol{x}} = V(\underline{\boldsymbol{x}})\underline{\boldsymbol{y}} \tag{18}$$

and give the Jacobians $U = (\partial \boldsymbol{z}(\boldsymbol{x}, \boldsymbol{y}))/\partial \boldsymbol{x}$ and $V = (\partial \boldsymbol{z}(\boldsymbol{x}, \boldsymbol{y}))/\partial \boldsymbol{y}$ (cf. table 3, column 5). For *uncorrelated* vectors $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$ we therefore have $\boldsymbol{\Sigma}_{zz} = U(\boldsymbol{\mu}_y)\boldsymbol{\Sigma}_{xx}U^\mathsf{T}(\boldsymbol{\mu}_y) + V(\boldsymbol{\mu}_y)\boldsymbol{\Sigma}_{yy}V^\mathsf{T}(\boldsymbol{\mu}_y)$. If the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are observed, then these are best estimators for their means $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$, therefore

$$\boldsymbol{\Sigma}_{zz} = U(\boldsymbol{y})\boldsymbol{\Sigma}_{xx}U^\mathsf{T}(\boldsymbol{y}) + V(\boldsymbol{x})\boldsymbol{\Sigma}_{yy}V^\mathsf{T}(\boldsymbol{x}) \tag{19}$$

For trilinear forms $\underline{\boldsymbol{u}} = U(\underline{\boldsymbol{y}}, \underline{\boldsymbol{z}})\underline{\boldsymbol{x}} = V(\underline{\boldsymbol{x}}, \underline{\boldsymbol{z}})\underline{\boldsymbol{y}} = W(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}})\underline{\boldsymbol{z}}$ we obtain by analogy

$$\boldsymbol{\Sigma}_{uu} = U\boldsymbol{\Sigma}_{xx}U^\mathsf{T} + V\boldsymbol{\Sigma}_{yy}V + W\boldsymbol{\Sigma}_{zz}W^\mathsf{T}$$

which can be used to derive the uncertainty of planes from three points or of the point from three planes.

3.2 Uncertainty of Homogeneous Coordinates

Homogeneous vectors $\mathbf{x}$ represent the same object if multiplied with an arbitrary factor $\lambda \neq 0$. We do not want to normalize during geometric reasoning. But we need to fix the length of the vector in order the elements not to be uncertain due to scaling. For 3D-lines we in addition have to take the Plücker condition into account. As we need covariances and their inverse we have to solve two tasks: (i) impose restrictions on a given covariance matrix and (ii) impose restrictions on the inverse of a given covariance matrix.

In both cases it is of advantage to know the nullspace H of the matrices in advance. We just assume, that either each given covariance matrix has full rank or its nullspace is contained in the required nullspace. This assumption will be no restriction.

Imposing restrictions onto a covariance matrix: Given a stochastic vector $\underline{\boldsymbol{x}}^{(0)} \sim M(\boldsymbol{\mu}_x^{(0)}, \boldsymbol{\Sigma}_{xx}^{(0)})$ and constraints $\boldsymbol{g}(\underline{\boldsymbol{x}}^{(0)}) = 0$, determine the vector $\boldsymbol{x}$ and its covariance matrix $\boldsymbol{\Sigma}_{xx}$ fulfilling these constraints

$$\boldsymbol{\Sigma}_{xx} = P_g \boldsymbol{\Sigma}_{xx}^{(0)} P_g, \text{ with } \quad P_g = I - H(H^\mathsf{T} H)^{-1} H^\mathsf{T} \quad \text{and} \quad H^\mathsf{T} = \frac{\partial \boldsymbol{g}(\boldsymbol{x}^{(0)})}{\partial \boldsymbol{x}^{(0)}} \quad (20)$$

Obviously $\boldsymbol{\Sigma}_{xx}$ has null space $\mathcal{N}(\boldsymbol{\Sigma}_{xx}) = H$. The results follow from [7], eq. 355.2, observing that in [7] $\widehat{\gamma}$ are the residuals of the estimates.

We use this relation to force a covariance matrix to have the correct nullspace and thus the correct rank, namely for points, planes and lines

$$\mathcal{N}(\boldsymbol{\Sigma}_{xx}) = \mathbf{x},\ \mathcal{N}(\boldsymbol{\Sigma}_{ll}) = \mathbf{l},\ \mathcal{N}(\boldsymbol{\Sigma}_{XX}) = \mathbf{X},\ \mathcal{N}(\boldsymbol{\Sigma}_{AA}) = \mathbf{A},\ \mathcal{N}(\boldsymbol{\Sigma}_{LL}) = (\mathbf{L}, \overline{\mathbf{L}}) \quad (21)$$

The parts $\mathbf{x}$, $\mathbf{l}$, $\mathbf{X}$, $\mathbf{A}$ and $\mathbf{L}$ in the null spaces result from the constraint $\frac{1}{2}\underline{\mathbf{x}}^\mathsf{T}\underline{\mathbf{x}} = c$ with $\partial(\frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{x})/\partial\mathbf{x} = \mathbf{x}$ etc. The additional part $\overline{\mathbf{L}}$ in the nullspace of $\boldsymbol{\Sigma}_{LL}$ results from the Plücker condition (1). In our application we only force the length of the vectors to be non-stochastic, thus the vectors are not changed in value, they just obtain the correct stochastic properties, i. e. covariance matrix. We also assume the Plücker condition to hold for a given line vector.

Imposing restrictions onto the inverse of a covariance matrix: Given a stochastic vector $\underline{\boldsymbol{x}} \sim M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ fulfilling constraints $\underline{\boldsymbol{g}}(\boldsymbol{x}) = 0$ determine the pseudoinverse $\boldsymbol{\Sigma}_{xx}^+$ of its covariance matrix (cf.[7], eq. 155.21):

$$\begin{pmatrix} \boldsymbol{\Sigma}_{xx}^+ & H(H^\mathsf{T} H) \\ (H^\mathsf{T} H) H^\mathsf{T} & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{xx} & H \\ H^\mathsf{T} & \mathbf{0} \end{pmatrix}^{-1} \qquad \text{with} \qquad H^\mathsf{T} = \frac{\partial \boldsymbol{g}(\boldsymbol{x})}{\partial \boldsymbol{x}} \qquad (22)$$

We use this procedure for inverting a singular matrix with known nullspace and for inverting a matrix while imposing the given nullspace and rank condition onto it. I. e. (22) can also be used if $\boldsymbol{\Sigma}_{xx}$ has full rank and we would like to impose the rank condition with the correct null space onto its pseudo inverse.

3.3 Testing with Singular Covariance Matrices of Known Nullspace

In the follwing we assume all variables to be normally distributed, replacing $M(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$ by $N(\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_{xx})$. This is reasonable as long as the random errors are small and follows from the maximum entropy principle if only the first and second moments are known.

We use the following theorem from statistical testing theory (cf. [7], sect. 272):

Test of $\boldsymbol{x} = \boldsymbol{\mu}$: Given a n-vector $\boldsymbol{x}$ with normal distribution $\underline{\boldsymbol{x}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\text{rk}\boldsymbol{\Sigma} = r \le n$, and known nullspace $\mathcal{N}(\boldsymbol{\Sigma}) = H$, being a $n \times (n - r)$-matrix, the the optimal test statistic for the hypothesis $H_o : \boldsymbol{x} = \boldsymbol{\mu}$ is given by

$$\underline{T} = (\underline{\boldsymbol{x}} - \boldsymbol{\mu})^\mathsf{T} \boldsymbol{\Sigma}^+ (\underline{\boldsymbol{x}} - \boldsymbol{\mu}) \sim \chi_r^2 \quad (23)$$

where χ^2_r denotes the χ^2_r-distribution with r degrees of freedom and the pseudo inverse is determined from (22). In the case of full rank the pseudo inverse is to be replaced by the normal inverse. In case of $n = 1$ the test statistic may be replaced by

$$\underline{t} = \frac{\underline{x} - \mu_x}{\sigma_x} \sim N(0,1)$$

4 Testing Geometric Relations

4.1 The Tests

Tests based on an Inner Product: Tests based on an inner product are used to check incidence or orthogonality of vectors (cf. table 3): no. 2 $\mathbf{x} \in \mathbf{l}$, no. 5 $\mathbf{l} \perp \mathbf{m}$, no. 8 $\mathbf{X} \in \mathbf{A}$, no. 11 $\mathbf{L} \cap \mathbf{M} \neq \emptyset$, no. 12 $\mathbf{L} \perp \mathbf{M}$, no. 15 $\mathbf{L} \parallel \mathbf{A}$, no. 18 $\mathbf{A} \perp \mathbf{B}$.

Test of the Identity of two Homogeneous Vectors: Identity of two homogeneous n-vectors is equivalent to checking $\mathbf{U} = \lambda\mathbf{V}$ or $\mathbf{U} \wedge \mathbf{V} = \mathbf{0}$ It thus can be based on proportionality or on the outer product of the vectors which should be zero. The outer product has dimension $\binom{n}{2}$ containing all different 2×2 subdeterminants of $(\mathbf{U}, \mathbf{V})$.

(i) In the simplest *case* $n = 2$ of checking the parallelity of two 2D-lines $\mathbf{l} \parallel \mathbf{m}$, no. 4 in table 3 we use the determinant of the two not necessarily normalized 2D-directions $\boldsymbol{l}$ and $\boldsymbol{m}$

$$d = |\boldsymbol{l}, \boldsymbol{m}| = a_l b_m - a_m b_l = (-b_l, a_l)\boldsymbol{m} = -(-b_m, a_m)\boldsymbol{l} = (\boldsymbol{l}^\perp)^\mathsf{T}\boldsymbol{m} = (\boldsymbol{m}^\perp)^\mathsf{T}\boldsymbol{l}$$

inducing the vectors $\boldsymbol{l}^\perp$ and $\boldsymbol{m}^\perp$ being perpendicular to $\boldsymbol{l}$ and $\boldsymbol{m}$.

(ii) The *case* $n = 3$ occurs when checking the identity of 2D lines and points and when checking the parallelity of 3D vectors. Here we also check the outer product, equivalent to the cross product of the entities, cf. tests no. 1 $\mathbf{x} \equiv \mathbf{y}$, no. 3 $\mathbf{l} \equiv \mathbf{m}$, no. 10 $\mathbf{L} \parallel \mathbf{M}$, no. 14 $\mathbf{L} \perp \mathbf{A}$ and no. 17 $\mathbf{A} \parallel \mathbf{B}$ in table 3 using the skew matrix $S(\boldsymbol{x})$ of a 3-vector from eq. (5).

(iii) The *case* $n > 3$ occurs when checking the identity of 3D points no. 6 $\mathbf{X} \equiv \mathbf{Y}$, 3D planes no. 16 $\mathbf{A} \equiv \mathbf{B}$ and 3D lines no. 9 $\mathbf{L} \equiv \mathbf{M}$. Then we would need to check 6- or 15-vectors of all 2×2 subdeterminants contained in the outer product. But the tests actually have only 3 and 4 degrees of freedom resp., indicating all these determinants to be statistically dependent. We therefore develop a test statistic with lower dimension based on the proportionality but taking all elements of $\mathbf{U}$ and $\mathbf{V}$ into account, in order to obtain a sufficient test statistic (cf. [9]). This way, we gain numerical efficiency at the expense of some symmetry in the test.

We choose an index $i \in (1, ..., n)$ such that $|U_i V_i| >> 0$ and solve for λ yielding $\lambda = U_i / V_i$. Then we determine the bilinear form

$$\mathbf{D} = V_i \mathbf{U} - U_i \mathbf{V} \qquad \text{with} \qquad E(\mathbf{D}) = \mathbf{0}$$

The Jacobians $\partial\mathbf{D}/\partial\mathbf{V} = \mathsf{C}_i(\mathbf{U})$ and $\partial\mathbf{D}/\partial\mathbf{U} = -\mathsf{C}_i(\mathbf{V})$ can be used to write

$$\mathbf{D} = \mathsf{C}_i(\mathbf{U})\mathbf{V} = -\mathsf{C}_i(\mathbf{V})\mathbf{U} \quad \text{with } \mathsf{C}_i(\mathbf{U}) = \mathbf{U}\boldsymbol{e}_i^\mathsf{T} - U_i I \quad \text{and} \quad \boldsymbol{e}_i = (0, ..., \underset{i}{1}, ..., 0)$$

The covariance matrix of $\mathbf{D}$ is $\boldsymbol{\Sigma}_{DD} = \mathsf{C}_i(\mathbf{U})\boldsymbol{\Sigma}_{VV}\mathsf{C}_i^\mathsf{T}(\mathbf{U}) + \mathsf{C}_i(\mathbf{V})\boldsymbol{\Sigma}_{UU}\mathsf{C}_i^\mathsf{T}(\mathbf{V})$ Observe, in general it has null space $\boldsymbol{e}_i$ (cf. table 3, rows 6 and 16) as $\boldsymbol{e}_i^\mathsf{T}\mathsf{C}_i(\mathbf{X}) = \mathbf{0}^\mathsf{T}, \ \forall \mathbf{X}$ For 3D-lines we in addition have $\overline{\mathbf{L}}^\mathsf{T}\mathsf{C}_i(\mathbf{L}) = \overline{\mathbf{M}}^\mathsf{T}\mathsf{C}_i(\mathbf{L}) = \lambda_L \mathbf{L}^\mathsf{T} = \lambda_M \mathbf{M}^\mathsf{T}, \ \forall \mathbf{L}$ if $< \mathbf{L}, \mathbf{M} >= 0$, which then are in the null space of $\boldsymbol{\Sigma}_{LL}$ and $\boldsymbol{\Sigma}_{MM}$. Thus we have null space $(\boldsymbol{e}_i, \overline{\mathbf{L}})$ or $(\boldsymbol{e}_i, \overline{\mathbf{M}})$ (cf. table 3, line 9).

1	2	3	4	5	6
No.	entities	relation	dof	test	nullspace of $\boldsymbol{\Sigma}_{dd}$
1	points $\mathbf{x}, \mathbf{y}$	$\mathbf{x} \equiv \mathbf{y}$	2	$\mathbf{d} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$	$\mathbf{x}$ or $\mathbf{y}$
2	point $\mathbf{x}$, line $\mathbf{l}$	$\mathbf{x} \in \mathbf{l}$	1	$d = \mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{l}^\mathsf{T}\mathbf{x}$	
3	lines $\mathbf{l}, \mathbf{m}$	$\mathbf{l} \equiv \mathbf{m}$	2	$\mathbf{d} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$	$\mathbf{l}$ or $\mathbf{m}$
4		$\mathbf{l} \parallel \mathbf{m}$	1	$d = (\boldsymbol{l}^\perp)^\mathsf{T}\boldsymbol{m} = -(\boldsymbol{m}^\perp)^\mathsf{T}\boldsymbol{l}$	
5		$\mathbf{l} \perp \mathbf{m}$	1	$d = \boldsymbol{l}^\mathsf{T}\boldsymbol{m} = \boldsymbol{m}^\mathsf{T}\boldsymbol{l}$	
6	points $\mathbf{X}, \mathbf{Y}$	$\mathbf{X} \equiv \mathbf{Y}$	3	$\mathbf{D} = \mathsf{C}_i(\mathbf{X})\mathbf{Y} = -\mathsf{C}_i(\mathbf{Y})\mathbf{X}$	$\boldsymbol{e}_i$
7	point $\mathbf{X}$, line $\mathbf{L}$	$\mathbf{X} \in \mathbf{L}$	2	$\mathbf{D} = \overline{\Pi}^\mathsf{T}(\mathbf{X})\mathbf{L} = -\overline{\Gamma}(\mathbf{L})\mathbf{X}$	$((\boldsymbol{L}^\mathsf{T}, 0)^\mathsf{T}, \mathbf{X})$
8	point $\mathbf{X}$, line $\mathbf{A}$	$\mathbf{X} \in \mathbf{A}$	1	$d = \mathbf{X}^\mathsf{T}\mathbf{A} = \mathbf{A}^\mathsf{T}\mathbf{X}$	
9	lines $\mathbf{L}, \mathbf{M}$	$\mathbf{L} \equiv \mathbf{M}$	4	$\mathbf{D} = \mathsf{C}_i(\mathbf{L})\mathbf{M} = -\mathsf{C}_i(\mathbf{M})\mathbf{L}$	$(\boldsymbol{e}_i, \overline{\mathbf{L}})$ or $(\boldsymbol{e}_i, \overline{\mathbf{M}})$
10		$\mathbf{L} \parallel \mathbf{M}$	2	$\boldsymbol{D} = S(\boldsymbol{L})\boldsymbol{M} = -S(\boldsymbol{M})\boldsymbol{L}$	$\boldsymbol{L}$ or $\boldsymbol{M}$
11		$\mathbf{L} \cap \mathbf{M} \neq \emptyset$	1	$d = \overline{\mathbf{L}}^\mathsf{T}\mathbf{M} = \overline{\mathbf{M}}^\mathsf{T}\mathbf{L}$	
12		$\mathbf{L} \perp \mathbf{M}$	1	$d = \boldsymbol{L}^\mathsf{T}\boldsymbol{M} = \boldsymbol{M}^\mathsf{T}\boldsymbol{L}$	
13	line $\mathbf{L}$, plane $\mathbf{A}$	$\mathbf{L} \in \mathbf{A}$	2	$\mathbf{D} = \Pi^\mathsf{T}(\mathbf{A})\mathbf{L} = -\overline{\Gamma}(\mathbf{L})\mathbf{A}$	$((\boldsymbol{L}_0^\mathsf{T}, 0)^\mathsf{T}, \mathbf{A})$
14		$\mathbf{L} \perp \mathbf{A}$	2	$\boldsymbol{D} = S(\boldsymbol{L})\boldsymbol{A} = -S(\boldsymbol{A})\boldsymbol{L}$	$\boldsymbol{A}$ or $\boldsymbol{L}$
15		$\mathbf{L} \parallel \mathbf{A}$	1	$d = \boldsymbol{L}^\mathsf{T}\boldsymbol{A} = \boldsymbol{A}^\mathsf{T}\boldsymbol{L}$	
16	planes $\mathbf{A}, \mathbf{B}$	$\mathbf{A} \equiv \mathbf{B}$	3	$\mathbf{D} = \mathsf{C}_i(\mathbf{A})\mathbf{B} = -\mathsf{C}_i(\mathbf{B})\mathbf{A}$	$\boldsymbol{e}_i$
17		$\mathbf{A} \parallel \mathbf{B}$	2	$\boldsymbol{D} = S(\boldsymbol{A})\boldsymbol{B} = -S(\boldsymbol{B})\boldsymbol{A}$	$\boldsymbol{A}$ or $\boldsymbol{B}$
18		$\mathbf{A} \perp \mathbf{B}$	1	$d = \boldsymbol{A}^\mathsf{T}\boldsymbol{B} = \boldsymbol{B}^\mathsf{T}\boldsymbol{A}$	

Table 3. *shows 18 relationships between points, lines and planes useful for 2D and 3D grouping, together with the degrees of freedom (dof) and the essential part of the test statistic. The index i in the condition $\mathbf{X} \equiv \mathbf{Y}$, $\mathbf{L} \equiv \mathbf{M}$ and $\mathbf{A} \equiv \mathbf{B}$ is to be chosen such that $|X_i Y_i| >> 0$ etc. Observe, all tests are bilinear in the coordinates of the involved entities, thus allow rigorous error propagation. Column five implicitly contains the Jacobians of the differences d etc. as all are of the form $\boldsymbol{z} = U(\boldsymbol{y})\boldsymbol{x} = V(\boldsymbol{x})\boldsymbol{y}$, cf* (18).

Checking Line-Point and Line-Plane-Incidence: Checking 3D line-point and line-plane incidence (line 7 and 13 in table 3) require some elaboration. The idea is to check whether the join $\mathbf{D}_X = \mathbf{X} \wedge \mathbf{L}$ or the intersection $\mathbf{D}_A = \mathbf{A} \cap \mathbf{L}$ yields an undefined object, plane or point resp., i. e. the resulting entity is $\mathbf{0}$. Both conditions have 2 degrees of freedom. This can be seen if we choose the 3D line to be the X-axis. Then the point needs to have coordinates (1) $Y = 0$ and (2) $Z = 0$, whereas the plane needs (1) to be parallel to the line and (2) have distance 0 to the origin. The nullspaces of $\boldsymbol{\Sigma}_{D_X D_X}$ and $\boldsymbol{\Sigma}_{D_A D_A}$ therefore have dimension 2. We easily can verify that

$$\begin{pmatrix} (\boldsymbol{L}^\mathsf{T}, 0) \\ \mathbf{X}^\mathsf{T} \end{pmatrix} \Pi^\mathsf{T}(\mathbf{X}) = \begin{pmatrix} (\boldsymbol{L}^\mathsf{T}, 0) \\ \mathbf{X}^\mathsf{T} \end{pmatrix} \Gamma(\mathbf{L}) = \mathbf{0}$$

For the point $\mathbf{X}$ we have $\Pi(\mathbf{X})\mathbf{X} = \mathbf{0}$ and $\Gamma(\mathbf{L})\mathbf{X} = \mathbf{0}$. This proves $\mathbf{X} \in \mathcal{N}(\boldsymbol{\Sigma}_{D_X D_X})$. On the other hand we have $(\boldsymbol{L}^\mathsf{T}, 0)\Gamma(\mathbf{L}) = \mathbf{0}$ and – with some intermediate steps – $(\boldsymbol{L}^\mathsf{T}, 0)\Pi(\mathbf{X}) = \overline{\mathbf{L}}$ which is in the nullspace of $\boldsymbol{\Sigma}_{LL}$ due to (21). A similar reasoning holds for the plane parameters $\mathbf{A}$.

4.2 Performing the Statistical Tests

We are now able to give a general scheme for testing geometric relations. For any test do the following

1. determine the difference d, $\mathbf{d}$, $\boldsymbol{D}$ or $\mathbf{D}$ using one of the two equations in column 5 in table 3.
2. determine the covariance matrices of the two geometric entities by imposing the length and for 3D lines the Plücker constraint following eq. (20) using (21). This is not necessary if the covariance matrix already has the correct rank and nullspace.
3. determine the covariance matrix of the difference d, $\mathbf{d}$, $\boldsymbol{D}$ or $\mathbf{D}$ using eq. (19) and the Jacobians from table 3 in column 5. The Jacobians can be taken from these equations all having the structure of eq. (18).
4. determine the inverse covariance matrix, either by direct inversion of the variance or using eq. (22) and the null space given in column 6 of table 3.
5. determine the test statistic T from eq. (23) being χ^2_r-distributed with the degrees of freedom (dof) r given in column 4 of table 3.
6. choose a significance number α and compare T with the critical value $\chi^2_{r,\alpha}$. If $T > \chi^2_{r,\alpha}$ then the hypothesis that the spatial relation holds can be rejected.

It is advisable to normalize the coordinates of the entities such that the homogenous coordinates are of comparable magnitude, compare the discussion in [4].

5 Example and Conclusions

We are working on reconstructing polyhedral objects from multiple images, cf. [5]. After feature extraction we first determine 3D nodes by stereo analysis, (cf. fig. 1(a),(b)) where nodes are corner points with two or more half lines. Then we start with a 3D grouping process on these nodes to find polyhedral surfaces. We use both the neighborhood relations from the feature extraction and the stereo analysis and the geometric relations of the various involved 3D entities, namely points, lines and planes.

During the grouping process we sequentially perform various geometric tests, which are induced by the known neighborhood relations. We first test the coplanarity of pairs of planes, where each plane is induced by a corner point and two half lines. We then search for additional half lines belonging to such a plane. We finally test for collinearity of half lines belonging to two different 3D nodes in order to merge those half lines to edges of the polyhedral, cf. fig. 1(c).

As the used points and lines are uncertain one needs thresholds for testing. Experiences showed that an adhoc definition of these thresholds gives unsatisfactory results, namely inconsistencies, asymmetries of the decisions and unpredictable dependencies of the results on the choice of the thresholds. Last not least new datasets required new settings.

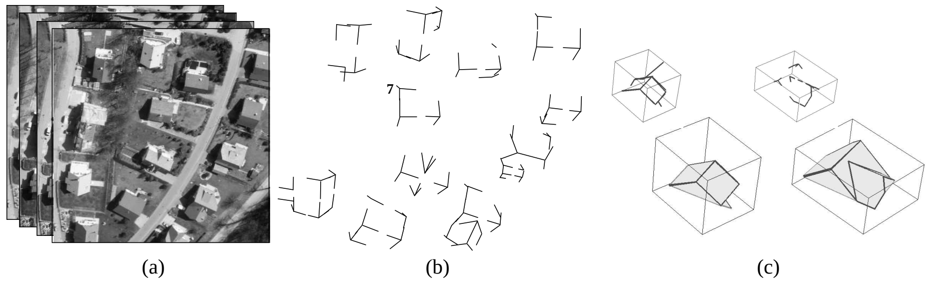

Fig. 1. 3D grouping of 3D nodes (b) extracted from multiple images (a), cf. [8], (c) shows resulting planar surfaces of two buildings.

Using the proposed statistical tests greatly simplifies the control of the grouping process as only the significance level has to be fixed.

Observe, statistical tests in general only answer the question whether the hypotheses should be rejected. Thus we only get objective arguments *against* grouping hypotheses, but no positive confirmation. Thus the function of the statistical tests can be seen as a filter rejecting wrong grouping hypotheses. An example are short nearly collinear line segments which are very far apart: though the statistical test might result in a small, i. e. statistically insignificant test statistic, one might not like to group them to one long straight line. The decision on such two line segments needs to be based on other criteria.

The proposed tools have been implemented in C++ and used within our grouping software. The current version only works on unconstraint geometric entities and in case the coordinate systems of the entities including their covariance matrices are consistent. We are currently extending the software to handle these cases.

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