NEW ORIENTATION PROCEDURES

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1 ABSTRACT AND MOTIVATION

Orientation procedures are preceived as the central part of photogrammetry. During the last decade the problem of determining the interior and the exterior orientation of one or more cameras has found high attraction in Computer Vision.

The problem was formulated newly within a projective framework for several reasons: (1) often, the calibration of the cameras in use was not known, nor could be determined; (2) often, no approximate values for the orientation and calibration parameters were available; (3) often, self-calibration turned out to be instable, especially in case of image sequences or of variable focal length; (4) special boundary conditions, such as planar objects or the coplanarity of the projection centres allowed orientation and calibration with less corresponding points; (5) generating new views from given ones turned out ot be possible without calibration; (6) using more than two cameras with the same interior orientation was proven to allow selfcalibration, after projective reconstruction; (7) the epipolar constraint for image pairs turned out to be not sufficient for image triplets in practically relevant cases; last but not least: (8) orientation procedures were not documented for non-photogrammetrists in photogrammetric literature.

A set of new orientation and calibration procedures has evolved.

The imaging process is described in a projective framework (SEMPLE & KNEEBONE 1952), explicitely interpreting the 11 parameters of the direct linear transformation, being the basis for a direct determination of the 6 parameters of the exterior and 5 parameters of the interior orientation. These 5 parameters guarantee the projection to map straight lines into straight lines. Cameras with some of these 5 parameters unknown are called uncalibrated.

The relative orientation of two cameras with unknown calibration can be achieved by a direct solution from corresponding points, leading to the fundamental matrix F, having 7 degrees of freedom, establishing the coplanarity or epipolar constraint as matching constraint, and which can be used to determine the two principle distances. Restriction to calibrated cameras, F reduces to the essential matrix E with 5 degrees of freedom, already known in photogrammetry.

The relative orientation of three cameras with unknown calibration can also be achieved by a direct solution, in this case from corresponding points and lines, leading to the trifocal tensor T, having 18 degrees of freedom. It establishes matching constraints for points and straight lines, and can be used to determine a part of the calibration parameters of the three cameras. Restriction to calibrated cameras reduces to a metrical parametrization of the trifocal tensor, with 11 degrees of freedom, combining relative orientation of the first two cameras and spatial resection of the third.

The paper presents solutions to these problems useful for photogrammetric applications.

2 BASICS

We use homogeneous coordinates throughout the paper following the outline in (FAUGERAS & PAPADOPOULO 1998) indicating them with boldface upright letters. The vectors $\mathbf{x}$ and $\lambda\mathbf{x}$ with $\lambda \neq 0$ therefore represent the same object. We distinguish geometric objects, namely *points* $p(\mathbf{x})$ and *lines* $l(\mathbf{l})$ *in the plane*

$$\mathbf{x} = (u, v, w)^\mathsf{T} \cong (x, y, 1)^\mathsf{T} \qquad \mathbf{l} = (a, b, c)^\mathsf{T} \cong (\cos\phi, \sin\phi, -d_{l0})^\mathsf{T}$$

relating the representation of the line to the Hessian normal form with orientation ϕ and distance to the origin d_{l0}, and *points* $P(\mathbf{X})$, *planes* $\varepsilon(\mathbf{A})$

$$\mathbf{X} = (U, V, W, T)^\mathsf{T} \cong (X, Y, Z, 1) \qquad \mathbf{A} = (A, B, C, D)^\mathsf{T} \cong (n_x, n_y, n_z, -d_{\varepsilon 0})^\mathsf{T}$$

again relating the representation of the plane to its Hessian normal form with normal $\boldsymbol{n}$ and distance to the origin $d_{\varepsilon 0}$ and *lines* $L(\mathbf{L})$ in *Plücker coordinates* with their dual line $\overline{L}(\overline{\mathbf{L}})$ *in 3D space*

$$\mathbf{L} = (L_1, L_2, L_3, L_4, L_5, L_6)^\mathsf{T} \qquad \overline{\mathbf{L}} = (L_4, L_5, L_6, L_1, L_2, L_3)^\mathsf{T}$$

The line parameters have to fulfill the *Plücker condition*

$$L_1 L_4 + L_2 L_5 + L_3 L_6 = \frac{1}{2}\mathbf{L}^\mathsf{T}\overline{\mathbf{L}} = 0$$

It will be shown: the vector (L_1, L_2, L_3) is the direction of the line and the vector (L_4, L_5, L_6) is the normal of the plane through the line and the origin. The Plücker condition expresses the orthogonality condition of these two vectors.

Incidence of two objects can use inner products, namely for points $\mathbf{x}$and lines $\mathbf{l}$ in the plane, for points $\mathbf{X}$ and planes ε in 3D-space and for pairs $(\mathbf{L}, \mathbf{M})$ of 3D lines

$$< \mathbf{x}, \mathbf{l} >= \mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{x}\cdot\mathbf{l} = 0 \qquad < \mathbf{X}, \mathbf{A} >= \mathbf{X}^\mathsf{T}\mathbf{A} = \mathbf{X}\cdot\mathbf{A} = 0 \qquad < \mathbf{L}, \mathbf{M} >= \mathbf{L}^\mathsf{T}\overline{\mathbf{M}} = \mathbf{L}\cdot\overline{\mathbf{M}} = 0$$

The first 2 relation result from the definition of the 2D line and the plane in their Hessian form. The last relation will be proved below.

We can construct 2D lines l as *join* $\wedge$ of two points $\mathbf{x}$ and $\mathbf{y}$ and points $\mathbf{x}$ as *intersection* $\cap$ of two lines $\mathbf{l}_1$ and $\mathbf{l}_2$

$$\mathbf{l} = \mathbf{x} \wedge \mathbf{y} = \mathbf{x} \times \mathbf{y} \qquad \mathbf{x} = \mathbf{l}_1 \cap \mathbf{l}_2 = \mathbf{l}_1 \times \mathbf{l}_2$$

We also can construct 3D-lines L as join $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ of two points or as intersection $\mathbf{L} = \mathbf{A} \cap \mathbf{B}$ of two planes, defined via the dual line $\overline{\mathbf{L}} = \overline{\mathbf{A} \cap \mathbf{B}} = \mathbf{A} \wedge \mathbf{B}$

$$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = \mathsf{A}(\mathbf{X})\mathbf{Y} = -\mathsf{A}(\mathbf{Y})\mathbf{X} \qquad \overline{\mathbf{L}} = \overline{\mathbf{A} \cap \mathbf{B}} = \mathbf{A} \wedge \mathbf{B} = \mathsf{A}(\mathbf{A})\mathbf{B} = -\mathsf{A}(\mathbf{B})\mathbf{P}$$

with the matrix at the same time being a Jacobian

$$\underbrace{\mathsf{A}(\mathbf{X})}_{6\times 4} \doteq \frac{\partial(\mathbf{X} \wedge \mathbf{Y})}{\partial \mathbf{Y}} = \begin{pmatrix} T & 0 & 0 & -U \\ 0 & T & 0 & -V \\ 0 & 0 & T & -W \\ 0 & -W & V & 0 \\ W & 0 & -U & 0 \\ -V & U & 0 & 0 \end{pmatrix}$$

using the convention homogeneous matrices to be upright sans serif letters. Observe $\mathsf{A}(\mathbf{X})\mathbf{X} = \mathbf{0}, \forall \mathbf{X}$ and rk$\mathsf{A}(\mathbf{X}) = 3$. The line coordinates obviously are bilinear in the homogeneous coordinates for the points and for the planes. Setting the fourth coordinate of the two homogeneous vectors to 1, we find $(L_1, L_2, L_3)^\mathsf{T} = \boldsymbol{Y} - \boldsymbol{X}$ and $(L_4, L_5, L_6)^\mathsf{T} = \boldsymbol{X} \times \boldsymbol{Y}$ with the Euklidean coordinates $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the two points indicated with slanted bold face letters. The relations for the planes exploit the duality of points and planes in 3D, specifically the duality of the join $\wedge$ and the intersection $\cap$.

We also obtain the plane coordinates as the join of a point and a line and the intersection of a line and a plane

$$\mathbf{A} = \mathbf{X} \wedge \mathbf{L} = \mathsf{A}^\mathsf{T}(\mathbf{X})\overline{\mathbf{L}} = -\mathsf{B}(\mathbf{L})\mathbf{X} \qquad \mathbf{X} = \mathbf{A} \cap \mathbf{L} = \mathsf{A}^\mathsf{T}(\mathbf{A})\mathbf{L} = -\mathsf{B}(\overline{\mathbf{L}})\mathbf{A}$$

with the Jacobian

$$\mathsf{B}(\mathbf{L}) \doteq \frac{\partial(\mathbf{L} \wedge \mathbf{X})}{\partial \mathbf{X}} = -\frac{\partial(\mathbf{X} \wedge \mathbf{L})}{\partial \mathbf{X}} = \begin{pmatrix} 0 & L_3 & -L_2 & -L_4 \\ -L_3 & 0 & L_1 & -L_5 \\ L_2 & -L_1 & 0 & -L_6 \\ L_4 & L_5 & L_6 & 0 \end{pmatrix} = \mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T}$$

where the last expression is valid for the line to be given by the intersection of two planes $\mathbf{L} = \mathbf{A} \cap \mathbf{B}$. Observe $\mathsf{B}(\overline{\mathbf{L}})\mathsf{B}(\mathbf{L}) = \mathbf{0}$ and rk$\mathsf{B}(\mathbf{L}) = 2$. The expressions for $\mathbf{A} = \mathbf{X} \wedge \mathbf{L}$ and $\mathbf{X} = \mathbf{A} \cap \mathbf{L}$ are consistent, as e. g. $\mathbf{X} \in \mathbf{A}$ due to $< \mathbf{X}, \mathbf{A} >= \mathbf{X}^\mathsf{T}\mathsf{A}^\mathsf{T}(\mathbf{X})\mathbf{L} = \mathbf{0}$ and $\mathbf{L} \in \mathbf{A}$ due to $(\mathbf{X} \wedge \mathbf{L}) \cap \mathbf{L} = (-\mathsf{B})(\overline{\mathbf{L}})(-\mathsf{B}(\mathbf{L})\mathbf{X}) = \mathbf{0}$.

We now can prove the condition $< \mathbf{L}, \mathbf{M} >= 0$ for two lines to intersect. Let $\mathbf{L}$ be given as the join $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} = -\mathsf{A}(\mathbf{Y})\mathbf{X}$. Then the intersection condition is equivalent to the condition the point $\mathbf{X}$ to lie in the plane $\mathbf{A} = \mathbf{M} \wedge \mathbf{Y} = \mathsf{A}^\mathsf{T}(\mathbf{Y})\overline{\mathbf{M}}$ which leads to $\mathbf{X}^\mathsf{T}\mathbf{A} = (\mathbf{X}^\mathsf{T}\mathsf{A}^\mathsf{T}(\mathbf{Y}))\,\overline{\mathbf{M}} = -\mathbf{L}^\mathsf{T}\overline{\mathbf{M}} = 0$.

We finally need conditions for two lines to intersect, in case they are given by two points or two planes.

$$(|\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4| = 0 \qquad |\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4| = 0 \qquad (\mathbf{X} \wedge \mathbf{Y}) \cap (\mathbf{A} \cap \mathbf{B}) = \mathbf{X}^\mathsf{T}(\mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T})\mathbf{Y} = 0 \qquad (1)$$

The first and second condition results from the coplanarity of the points or from the intersection condition for four planes. The last condition uses $\mathbf{X}^\mathsf{T}\mathbf{R} = \mathbf{X}^\mathsf{T}\mathsf{B}(\mathbf{L})\mathbf{Y} = 0$ where the plane $\mathbf{R} = \mathbf{L} \wedge \mathbf{Y}$ is the join of $\mathbf{L} = \mathbf{A} \cap \mathbf{B}$ and $\mathbf{Y}$.

3 PROJECTION

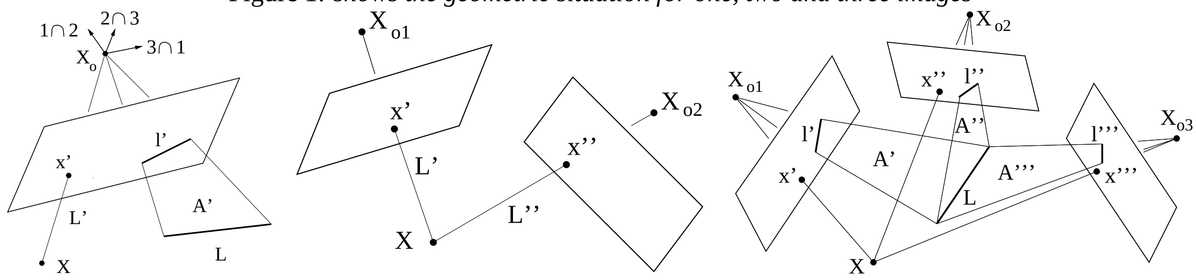
3.1 POINTS

The projection of a 3D point $P(\mathbf{X})$ onto the image plane yields the image point $p'(\mathbf{x}')$ via a direct linear transformation (cf. Fig. 1).

$$\mathbf{x}' = \mathsf{P}\mathbf{X} \qquad \text{or} \qquad (u', v', w')^\mathsf{T} = (\mathbf{1}, \mathbf{2}, \mathbf{3})^\mathsf{T}\mathbf{X} = (\mathbf{1}\cdot\mathbf{X}, \mathbf{2}\cdot\mathbf{X}, \mathbf{3}\cdot\mathbf{X})^\mathsf{T} \qquad \text{with} \qquad \mathsf{P} = \mathsf{K}\boldsymbol{R}(\boldsymbol{I}| - \boldsymbol{X}_o)$$

where $(\cdot|\cdot)$ denotes concatenation. The 3×4 *projection matrix* P can be explicitely related to the 6 parameters of the exterior orientation and 5 parameters of the interior orientation namely the Euclidean coordinates $\boldsymbol{X}_o$ of the projection centre $O(\boldsymbol{X}_o)$, the rotation matrix $\boldsymbol{R}$, the principle distance c, the coordinates (x'_H, y'_H) of the principle point, the shear s and the scale difference of the x'- and the y'-coordinates. The parameters of the interior orientation are collected in the 3×3 *calibration matrix*

Figure 1: *shows the geometric situation for one, two and three images*



$$\boldsymbol{K} \doteq \begin{pmatrix} c & cs & x'_H \\ 0 & c(1+m) & y'_H \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s & x'_H \\ 0 & 1+m & y'_H \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

It is an upper diagonal matrix and can be arbitrarily scaled, if no interpretation of its elements is required. Observe the part $\boldsymbol{R}(\boldsymbol{I}| - \boldsymbol{X}_o)$ transforms the object coordinates into the camera system, the second factor $Diag(c, c, 1)$ of the calibration matrix performs the projection and the first factor the calibration. The projection matrix in general has rank 3 and its null space is the projection centre as $\mathsf{P}\mathbf{X}_o = \mathbf{0}$. Therefore the three row vectors $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ of the projection matrix P can be interpreted as the parameters of planes. The vector $\mathbf{1}$ is a plane through the line $u' = 0$ as $u' = \mathbf{1}{\cdot}\mathbf{X} =< \mathbf{1}, \mathbf{X} >= 0, \forall \mathbf{X}$ and passes through the projection centre. Similarily $\mathbf{2}$ is a plane through $v' = 0$, and $\mathbf{3}$ is the focal plane parallel to the image plane, as then $w' = \mathbf{3}{\cdot}\mathbf{X} = 0$. The three planes intersect in the projection centre: $\mathbf{X}_o = \mathbf{1} \cap \mathbf{2} \cap \mathbf{3}$.

3.2 LINES

A similar projection relation holds for 3D lines. The image line $\mathbf{l}' = \mathbf{x}' \wedge \mathbf{y}'$ of a 3D line $\mathbf{L} = \mathbf{X} \wedge \mathbf{Y}$ can be expressed as a function of the images $\mathbf{x}' = \mathsf{P}\mathbf{X} = (\mathbf{1}{\cdot}\mathbf{X}, \mathbf{2}{\cdot}\mathbf{X}, \mathbf{3}{\cdot}\mathbf{X})^\mathsf{T}$ and $\mathbf{y}' = \mathsf{P}\mathbf{Y} = (\mathbf{1}{\cdot}\mathbf{Y}, \mathbf{2}{\cdot}\mathbf{Y}, \mathbf{3}{\cdot}\mathbf{Y})^\mathsf{T}$ of two object points $\mathbf{X}$ and $\mathbf{Y}$, namely $\mathbf{l}' = \mathbf{x}' \times \mathbf{y}'$ or

$$\mathbf{l}' = (a', b', c')^\mathsf{T} = (\mathbf{1}{\cdot}\mathbf{X}, \mathbf{2}{\cdot}\mathbf{X}, \mathbf{3}{\cdot}\mathbf{X})^\mathsf{T} \times (\mathbf{1}{\cdot}\mathbf{Y}, \mathbf{2}{\cdot}\mathbf{Y}, \mathbf{3}{\cdot}\mathbf{Y})^\mathsf{T}$$

This expression can be simplified. E. g. the first element a' is $a' = (\mathbf{2}{\cdot}\mathbf{X})(\mathbf{3}{\cdot}\mathbf{Y}) - (\mathbf{2}{\cdot}\mathbf{Y})(\mathbf{3}{\cdot}\mathbf{X}) = \mathbf{X}^\mathsf{T}(\mathbf{2}\mathbf{3}^\mathsf{T} - \mathbf{3}\mathbf{2}^\mathsf{T})\mathbf{Y} = (\mathbf{X} \wedge \mathbf{Y}) \cap (\mathbf{2} \cap \mathbf{3}) = (\mathbf{2} \cap \mathbf{3}){\cdot}\overline{\mathbf{L}}$. Similarily we obtain expressions for b' and c'. We therefore obtain the direct linear transformation of 3D lines

$$\mathbf{l}' = \tilde{\mathsf{P}}\overline{\mathbf{L}} \qquad \text{or} \qquad \mathbf{l}' = (\tilde{\mathbf{1}}, \tilde{\mathbf{2}}, \tilde{\mathbf{3}})^\mathsf{T}\overline{\mathbf{L}} = (\tilde{\mathbf{1}}{\cdot}\overline{\mathbf{L}}, \tilde{\mathbf{2}}{\cdot}\overline{\mathbf{L}}, \tilde{\mathbf{3}}{\cdot}\overline{\mathbf{L}})^\mathsf{T} \qquad \text{with} \qquad \tilde{\mathsf{P}} = (\mathbf{2} \cap \mathbf{3}, \mathbf{3} \cap \mathbf{1}, \mathbf{1} \cap \mathbf{2})^\mathsf{T}$$

with a 3×6 projection matrix $\tilde{\mathsf{P}}$. Its three rows are 6-vectors representing 3D lines, namely the intersections of the principle planes, thus the three coordinate axes of the camera system.

3.3 INVERSION

Inversion of the projection leads to projection rays $\mathbf{L}'$ for image points $\mathbf{x}'$ and projection planes $\mathbf{A}'$ for image lines $\mathbf{l}'$

$$\mathbf{L}' = \tilde{\mathsf{P}}^\mathsf{T}\mathbf{x}' = u'\mathbf{2} \cap \mathbf{3} + v'\mathbf{3} \cap \mathbf{1} + w'\mathbf{1} \cap \mathbf{2} \qquad \mathbf{A}' = \mathsf{P}^\mathsf{T}\mathbf{l}' = a'\mathbf{1} + b'\mathbf{2} + c'\mathbf{3}$$

The expression for $\mathbf{L}'$ results from the incidence relation ${\mathbf{x}'}^\mathsf{T}\mathbf{l}' = 0$ for all lines $\mathbf{l}'$ passing through $\mathbf{x}'$, leading to $({\mathbf{x}'}^\mathsf{T}\tilde{\mathsf{P}})\ \overline{\mathbf{L}} = < \mathbf{L}', \mathbf{L} >= 0$. The expression for $\mathbf{A}'$ results from the incidence relation ${\mathbf{l}'}^\mathsf{T}\mathbf{x}' = 0$ for all points $\mathbf{x}'$ on the line $\mathbf{l}'$, leading to $({\mathbf{l}'}^\mathsf{T}\mathsf{P})\ \mathbf{X} =< \mathbf{A}', \mathbf{X} >= 0$. The projection ray and the projection plane can be expressed as a function of the 3D point and the 3D line resp. showing the concatenated matrix $\tilde{\mathsf{P}}^\mathsf{T}\mathsf{P}$ only to depend on the projection centre

$$\mathbf{L}' = \tilde{\mathsf{P}}^\mathsf{T}\mathsf{P}\mathbf{X} = \mathbf{X}_o \wedge \mathbf{X} = \mathsf{A}(\mathbf{X}_o)\mathbf{X}_o \qquad \mathbf{A}' = \mathsf{P}^\mathsf{T}\tilde{\mathsf{P}}\overline{\mathbf{L}} = \mathbf{X}_o \wedge \mathbf{L} = \mathsf{A}^\mathsf{T}(\mathbf{X}_o)\overline{\mathbf{L}} \qquad \text{with} \qquad \tilde{\mathsf{P}}^\mathsf{T}\mathsf{P} = \mathsf{A}(\mathbf{X}_o)$$

4 ONE IMAGE

4.1 OBSERVATION EQUATIONS AND CONSTRAINTS

We now easily can write down the *observation equations* for points in one image, i. e. the collinearity equations

$$x' = \frac{u'}{w'} = \frac{\mathbf{1}{\cdot}\mathbf{X}}{\mathbf{3}{\cdot}\mathbf{X}} \qquad y' = \frac{v'}{w'} = \frac{\mathbf{2}{\cdot}\mathbf{X}}{\mathbf{3}{\cdot}\mathbf{X}}$$

from which two contraints can be derived

$$\mathbf{A}(x') \cdot \mathbf{X} = 0 \qquad \mathbf{B}(y') \cdot \mathbf{X} = 0 \qquad \text{with} \qquad \mathbf{A}(x') = x'\mathbf{3} - \mathbf{1} \cong u'\mathbf{3} - w'\mathbf{1} \qquad \mathbf{B}(y') = \mathbf{2} - y'\mathbf{3} \cong w'\mathbf{2} - v'\mathbf{3}$$

The planes $\mathbf{A}$ and $\mathbf{B}$ pass through the image point and the projection centre and span the image ray as $\mathbf{A} \cap \mathbf{B} = (u'\mathbf{3} - w'\mathbf{1}) \cap (w'\mathbf{2} - v'\mathbf{3}) = -w'(u'\mathbf{2} \cap \mathbf{3} + v'\mathbf{3} \cap \mathbf{1} + w'\mathbf{1} \cap \mathbf{2})$ using $\mathbf{C} \cap \mathbf{D} = -\mathbf{D} \cap \mathbf{C}, \forall \mathbf{C}, \mathbf{D}$, the factor $-w' \neq 0$ has no effect. A similar derivation of observation equations and constraints can be performed for observed lines.

4.2 ORIENTATION OF ONE IMAGE

The projection matrix P can easily be determined, if $n \geq 6$ corresponding points $\mathbf{x}_i$ and $\mathbf{X}_i$ are given. Its row vectors can be collected in the 12-vector $\mathbf{u}^\mathsf{T} = (\mathbf{1}^\mathsf{T}, \mathbf{2}^\mathsf{T}, \mathbf{3}^\mathsf{T})$ leading to

$$\begin{pmatrix} x'_i\,(\mathbf{3} \cdot \mathbf{X}_i) - (\mathbf{1} \cdot \mathbf{X}_i) \\ y'_i\,(\mathbf{3} \cdot \mathbf{X}_i) - (\mathbf{2} \cdot \mathbf{X}_i) \end{pmatrix} = \begin{pmatrix} -\mathbf{X}_i^\mathsf{T} & \mathbf{0}^\mathsf{T} & x'_i \mathbf{X}_i^\mathsf{T} \\ \mathbf{0}^\mathsf{T} & -\mathbf{X}_i^\mathsf{T} & y'_i \mathbf{X}_i^\mathsf{T} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix} = \mathbf{0} \qquad \text{or} \qquad \mathsf{C}_i \mathbf{u} = \mathbf{0} \qquad \forall i = 1, ..., n$$

An estimate for $\mathbf{u}$ is the adequately normalized eigenvector $\widehat{\mathbf{u}}$ corresponding to the smallest eigenvalue of the matrix $\mathsf{C} = (\mathsf{C}_i)$, leading to an estimated projection matrix

$$\widehat{\mathsf{P}} = (\widehat{\mathbf{1}}\ \widehat{\mathbf{2}}\ \widehat{\mathbf{3}})^\mathsf{T} = (\mathsf{D}|\mathbf{d}) = (\widehat{\mathsf{K}}\widehat{\boldsymbol{R}}| - \widehat{\mathsf{K}}\widehat{\boldsymbol{R}}\widehat{\boldsymbol{X}}_o)$$

This requires at least $n \geq 6$ image points to be given. It can be partitioned into a left 3×3-matrix D and a right 3×1-vektor $\mathbf{d}$ from which the parameters of the exterior and the interior orientation can be directly computed in four steps:

1. $\widehat{\boldsymbol{X}}_o = -\mathsf{D}^{-1}\mathbf{d}$ 2. $\widehat{\mathsf{K}}\widehat{\mathsf{K}}^\mathsf{T} = \mathsf{D}\mathsf{D}^\mathsf{T}$ using a Choleski partitioning 3. $\widehat{\boldsymbol{R}} = \widehat{\mathsf{K}}^{-1}\mathsf{D}$ 4. $\widehat{\boldsymbol{K}} = \widehat{\mathsf{K}}/\widehat{\mathsf{K}}_{33}$

The normalization of the calibration matrix is necessary only if one wants to interprete its entries. Due to the generality of the model this procedure is much simpler than the one given in (Bopp & Kraus 1978). It only works in case the points are not coplanar and do not sit on an algebraic curve of third order (Faugeras 1993).

A similar estimation procedure can be developed for observed lines, leading to $\widehat{\widehat{\mathsf{P}}}$ from which the rows of $\widehat{\mathsf{P}}$ can be determined by joining the corresponding principle rays, e. g. $\widehat{\mathbf{1}} = \widehat{\widehat{\mathbf{2}}} \wedge \widehat{\widehat{\mathbf{3}}} = \widehat{\mathbf{3} \cap \mathbf{1}} \wedge \widehat{\mathbf{1} \cap \mathbf{2}}$.

5 TWO IMAGES

We model the geometry of two images using the projection matrices

$$\mathsf{P}_1 = (\mathbf{1}, \mathbf{2}, \mathbf{3})^\mathsf{T} = \mathsf{K}_1 \boldsymbol{R}_1 (\boldsymbol{I}|\mathbf{0}) \qquad \mathsf{P}_2 = (\mathbf{4}, \mathbf{5}, \mathbf{6})^\mathsf{T} = \mathsf{K}_2 \boldsymbol{R}_2 (\mathsf{I}| - \boldsymbol{T})$$

thus putting the origin of the object coordinate system into the first projection centre. We will also use the *reduced image coordinates*

$${}^k\mathbf{x}' = \boldsymbol{R}_1^{-1}\mathsf{K}_1^{-1}\mathbf{x}' \qquad {}^k\mathbf{x}'' = \boldsymbol{R}_2^{-1}\mathsf{K}_2^{-1}\mathbf{x}''$$

which represent the intersections of the image rays with two normalized cameras looking downwards, having horizontal image planes, and principle distance $c_1 = c_2 = 1$.

We do not need to model the projection of 3D lines as they do not contribute to the orientation of an image pair.

5.1 EPIPLOAR LINES AND COPLANARITY

For each object point the two projecting lines $\mathbf{L}'$ and $\mathbf{L}''$ need to intersect which can be expressed in two ways as a function of the rows in the projection matrices (cf. Fig. 1)

$$\mathbf{L}' \cap \mathbf{L}'' = (u'\mathbf{2} \cap \mathbf{3} + v'\mathbf{3} \cap \mathbf{1} + w'\mathbf{1} \cap \mathbf{2}) \cap (u''\mathbf{5} \cap \mathbf{6} + v''\mathbf{6} \cap \mathbf{4} + w''\mathbf{4} \cap \mathbf{5}) = (\mathbf{A}_1(x') \cap \mathbf{B}_1(y')) \cap (\mathbf{A}_2(x'') \cap \mathbf{B}_2(y'')) = 0$$

This *coplanarity condition* is linear in all image coordinates and can be expressed as

$$\mathbf{x}'^\mathsf{T} \mathsf{F} \mathbf{x}'' = 0 \qquad \text{with} \qquad \mathsf{F} = \begin{pmatrix} |\mathbf{2}, \mathbf{3}; \mathbf{5}, \mathbf{6}| & |\mathbf{2}, \mathbf{3}; \mathbf{6}, \mathbf{4}| & |\mathbf{2}, \mathbf{3}; \mathbf{4}, \mathbf{5}| \\ |\mathbf{3}, \mathbf{1}; \mathbf{5}, \mathbf{6}| & |\mathbf{3}, \mathbf{1}; \mathbf{6}, \mathbf{4}| & |\mathbf{3}, \mathbf{1}; \mathbf{4}, \mathbf{5}| \\ |\mathbf{1}, \mathbf{2}; \mathbf{5}, \mathbf{6}| & |\mathbf{1}, \mathbf{2}; \mathbf{6}, \mathbf{4}| & |\mathbf{1}, \mathbf{2}; \mathbf{4}, \mathbf{5}| \end{pmatrix}$$

where the 3×3-matrix F is the *fundamental matrix*. We also may use the coplanarity of the three directions $\mathbf{x}'$, $\mathbf{x}''$ and $\boldsymbol{T}$, given in the same coordinate system

$$^k\mathbf{x}' \cdot (\,^k\mathbf{T} \times \,^k\mathbf{x}'') = \,^k\mathbf{x}'^{\mathsf{T}} \mathsf{S}_T \,^k\mathbf{x}'' \quad \text{or} \quad \mathbf{x}'^{\mathsf{T}} \mathsf{K}_1^{-1\mathsf{T}} \boldsymbol{R}_1 \mathsf{S}_T \boldsymbol{R}_2^{-1} \mathsf{K}_2^{-1} \mathbf{x}'' = 0 \qquad \text{with} \qquad \boldsymbol{S}_T = \begin{pmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{pmatrix}$$

where the skewsymmetric matrix $\boldsymbol{S}_T$ of the vector $\boldsymbol{T}$ can be used to express the cross product $\boldsymbol{T} \times \boldsymbol{X} = \boldsymbol{S}_T \boldsymbol{X} = -\boldsymbol{S}_X \boldsymbol{T}$. We therefore have an alternative expression for the fundamental matrix in dependency of the parameters of the interior and exterior orientation of the two cameras. In case of known calibration thus $\mathsf{K}_i = \boldsymbol{I}$ and $\boldsymbol{R}_1 = \boldsymbol{I}$ and $\boldsymbol{R}_2 = \boldsymbol{R}$ it reduces to the *essential matrix* E yielding the coplanarity condition $^k\mathbf{x}'^{\mathsf{T}} \mathsf{E} \,^k\mathbf{x}'' = 0$. Thus we have

$$\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0 \qquad \text{with} \qquad \mathsf{F} \doteq \mathsf{K}_1^{-1\mathsf{T}} \boldsymbol{R}_1 \mathsf{S}_T \boldsymbol{R}_2^{-1} \mathsf{K}_2^{-1} \qquad \text{and} \qquad ^k\mathbf{x}'^{\mathsf{T}} \mathsf{E} \,^k\mathbf{x}'' = 0 \qquad \text{with} \qquad \mathsf{E} \doteq \mathsf{S}_T \boldsymbol{R}^{-1} \tag{3}$$

As S_T has rank 2, also F has rank two or $|\mathsf{F}| = 0$. As F is only defined up to scale it has only 7 degrees of freedom. It captures the complete geometry of the image pair, as without knowledge of the interior orientation the 3D structure, i. e. the photogrammetric model, can only be captured up to a projective transformation with 15 degrees of freedom (**?**), leaving $2 \cdot 11 - 15 = 7$ free parameters.

The essential matrix E has only 5 degrees of freedom, 2 for the direction of the basis and 3 for the rotation matrix, requiring two additional constraints (HUANG & FAUGERAS 1989). This relation has already been published in (THOMPSON 1968).

Given a point in one of the image its projection ray maps to the *epipolar line* in the other image. They are given by

$$\mathbf{l}'(\mathbf{x}'') = \mathsf{F}\mathbf{x}'' \qquad \mathbf{l}''(\mathbf{x}') = \mathsf{F}^{\mathsf{T}}\mathbf{x}'$$

They intersect on the *epipoles*. Therefore the coordinates of the epipoles $\mathbf{e}_2'$ and $\mathbf{e}_1''$ in the first in the second image are left and right eigenvectors of the fundamental matrix

$$\mathsf{F}\mathbf{e}_1'' = \mathbf{0} \qquad \mathsf{F}^{\mathsf{T}}\mathbf{e}_2' = \mathbf{0}$$

The epipoles also are the images of the projection centres $\mathbf{X}_{01} = \mathbf{1} \cap \mathbf{2} \cap \mathbf{3}$ and $\mathbf{X}_{02} = \mathbf{4} \cap \mathbf{5} \cap \mathbf{6}$, thus

$$\mathbf{e}_2' = \mathsf{P}_1 \mathbf{X}_{02} = (|\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{3}|, |\mathbf{5},\mathbf{1},\mathbf{2},\mathbf{3}|, |\mathbf{6},\mathbf{1},\mathbf{2},\mathbf{3}|)^{\mathsf{T}} \qquad \text{and} \qquad \mathbf{e}_1'' = \mathsf{P}_2 \mathbf{X}_{01} = (|\mathbf{1},\mathbf{4},\mathbf{5},\mathbf{6}|, |\mathbf{2},\mathbf{4},\mathbf{5},\mathbf{6}|, |\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}|)^{\mathsf{T}}$$

5.2 RELATIVE ORIENTATION

The coplanarity condition is linear in the unknown parameters of the fundamental matrix. Therefore we can estimate the parameter vector $\mathbf{f} = \text{vec}\mathsf{F}$ from

$$(\mathbf{x}_i' \otimes \mathbf{x}_i'')^{\mathsf{T}} \text{vec}\mathsf{F} = 0 \qquad i = 1, ..., n \qquad \rightarrow \qquad \mathsf{A}\mathbf{f} = [(\mathbf{x}_i' \otimes \mathbf{x}_i'')^{\mathsf{T}}]\mathbf{f} = \mathbf{0}$$

as the eigenvector $\widehat{\mathbf{f}}$ of A corresponding to its smallest eigenvalue, leading to $\widehat{\mathsf{F}}$. This would require ≥ 8 corresponding points $(\mathbf{x}_i', \mathbf{x}_i'')$ and would not take the rank condition $|\widehat{\mathsf{F}}| = 0$ into account. Therefore we use the matrix $\widehat{\widehat{\mathsf{F}}}$ which is closest to $\widehat{\mathsf{F}}$ using a singular value decomposition. The three steps of this procedure are

$$1. \quad \widehat{\mathsf{F}} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V} \quad \text{with} \quad \boldsymbol{\Lambda} = Diag(\lambda_1, \lambda_2, \lambda_{min}) \qquad 2. \quad \widehat{\boldsymbol{\Lambda}} = Diag(\lambda_1, \lambda_2, 0) \qquad 3. \quad \widehat{\widehat{\mathsf{F}}} = \boldsymbol{U}\widehat{\boldsymbol{\Lambda}}\boldsymbol{V}$$

with the orthogonal matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ and the diagonal matrix $\boldsymbol{\Lambda}$ of the singular values. It works for arbitrary $n \geq 8$. A procedure with 7 points can use a different procedure, as A then has two eigenvalues close to zero. It uses the two eigenvectors $\widehat{\mathbf{f}}_1 = \text{vec}\widehat{\mathsf{F}}_1$ and $\widehat{\mathbf{f}}_2 = \text{vec}\widehat{\mathsf{F}}_2$ corresponding to the two smallest eigenvalues of A leading to $\widehat{\mathsf{F}}_1$ and $\widehat{\mathsf{F}}_2$. The condition

$$|\widehat{\mathsf{F}}| = |s\widehat{\mathsf{F}}_1 + (1-s)\widehat{\mathsf{F}}_2| = 0$$

leads to polynomial of third degree in s with 1 or 3 zeros, thus to one or three solutions for F with 7 points.

We want to give an explicit expression for the relative orientation for the case of known interior orientation. Based on an estimate for the essential matrix E we can derive estimates for $\mathbf{T}$ and $\boldsymbol{R}$. This can be performed in three steps:

$$1. \quad \widehat{\mathbf{T}}^{\mathsf{T}}\widehat{\mathsf{E}} = \mathbf{0}^{\mathsf{T}} \qquad 2. \quad \widehat{\mathsf{E}}^{\mathsf{T}}\mathsf{S}_{\widehat{T}} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^{\mathsf{T}} \qquad 3. \quad \widehat{\boldsymbol{R}} = \boldsymbol{U}\boldsymbol{V}^{\mathsf{T}}$$

As $\mathbf{T}^{\mathsf{T}}\mathsf{S}_T = \mathbf{0}^{\mathsf{T}}$ the estimated base vector is the adequately normalized left eigenvector of $\widehat{\mathsf{E}}$ corresponding to its smallest eigenvalue, allowing to determine $\mathsf{S}_{\widehat{T}}$. Following (ARUN et al. 1987) we can determine the rotation matrix $\widehat{\boldsymbol{R}}$ minimizing the Frobenius norm $|\widehat{\mathsf{E}} - \mathsf{S}_{\widehat{T}}\widehat{\boldsymbol{R}}^{-1}|$ using the singular value decomposition of $\widehat{\mathsf{E}}^{\mathsf{T}}\mathsf{S}_{\widehat{T}}$.

In case more than two parameters of the interior orientation is not known relative orientation is not possible. A direct solution for the case where besides the direction of the basis and the rotation matrix also the two principle distance c' and c'' are unknown is given in (PAN 1999).

Observed 3D lines cannot be used to support relative orientation as they do not put a constraint on the relative orientation of two images: Any two lines l' and l'' in the two images of arbitrary orientation can be caused by a 3D line.

6 THREE IMAGES

The orientation of image triplet, already explored by Mikhail (cf. (MIKHAIL 1962, MIKHAIL 1963)), has quite some advantages over image pairs.

• The orientation can be based on homologeous points and lines, which can be extracted easily and with high precision. (SPETSAKIS & ALOIMONOS 1990).

• The constraints for homologeous points as well as for homologeous lines are linear in their homogeneous coordinates. For homologeous points this already has been shown by (MIKHAIL 1963). In addition they linearly depend on the elements of a $3 \times 3 \times 3$ tensor T – thus from three 3×3-matrices – the socalled trifocal tensor (HARTLEY 1995). This tensor plays the same role for the image triplet as the fundamental matrix for image pairs.

• The prediction of points and lines into a third image is easy with this trifocal tensor, whithout going via 3D space. Again this is in full analogy to predicting the position of a point in the second image of an image pair leading to the epipolar line. Here, however, the prediction generally leads to a unique result.

In general the prediction of points could be performed by intersection the epipolar lines with respect to the first two images (FAUGERAS & ROBERT 1994). But this generally is not possible in case the projection centres are collinear, which is the standard case in a photogrammetric strip, as then the epipolar planes coincide. In this important case the prediction with the trifocal tensor is possible again without going via the determination of the 3D point.

The prediction of lines still has some prerequisites: they should not go through the one of the projection centers O' or O'' or lie in an epipolare plane through $O'O''$.

6.1 PREDICTION AND CONSTRAINTS FOR POINTS AND LINES

We first discuss the case when a 3D line is observed in thre images. The condition that the image lines $\mathbf{l}'$, $\mathbf{l}''$ und $\mathbf{l}'''$ are homologeous can be easily written down using the projection matrices P_1, P_2 und P_3 of the three images.

Following fig. 1 we need to express the condition that the three projection planes $\mathbf{P}' = \mathsf{P}_1\mathbf{l}'$, $\mathbf{P}'' = \mathsf{P}_2\mathbf{l}'$ and $\mathbf{P}''' = \mathsf{P}_3\mathbf{l}'''$ intersect in a 3D line. They intersect in case they meet two mutually different planes in two single points. If we use one of the threee planes $\mathbf{1}$, $\mathbf{2}$ or $\mathbf{3}$ then, using (1.2), at least two of the following three conditions must be valid

$$|\mathbf{1}, \mathsf{P}_1^{\mathsf{T}}\mathbf{l}', \mathsf{P}_2^{\mathsf{T}}\mathbf{l}'', \mathsf{P}_3^{\mathsf{T}}\mathbf{l}'''| = 0 \qquad |\mathbf{2}, \mathsf{P}_1^{\mathsf{T}}\mathbf{l}', \mathsf{P}_2^{\mathsf{T}}\mathbf{l}'', \mathsf{P}_3^{\mathsf{T}}\mathbf{l}'''| = 0 \qquad |\mathbf{3}, \mathsf{P}_1^{\mathsf{T}}\mathbf{l}', \mathsf{P}_2^{\mathsf{T}}\mathbf{l}'', \mathsf{P}_3^{\mathsf{T}}\mathbf{l}'''| = 0 \tag{4}$$

We now want to give an explicit expression for the parameters of the line $\mathbf{l}'$ which only depends on the two image lines $\mathbf{l}''$ and $\mathbf{l}'''$ and the orientation parameters.

For simplicity we assume $\mathsf{P}_1 = (\boldsymbol{I}|\mathbf{0})$, $\mathsf{P}_2 = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ and $\mathsf{P}_3 = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4)$ With $\mathbf{l}' = (a', b', c')^{\mathsf{T}}$ we obtain for the first constraint $|\mathbf{e}_1, \mathsf{P}_1^{\mathsf{T}}\mathbf{l}', \mathsf{P}_2^{\mathsf{T}}\mathbf{l}'', \mathsf{P}_3^{\mathsf{T}}\mathbf{l}'''| = 0$ or

$$\begin{vmatrix} 1 & a' & \boldsymbol{r}_1^{\mathsf{T}}\mathbf{l}'' & \boldsymbol{s}_1^{\mathsf{T}}\mathbf{l}''' \\ 0 & b' & \boldsymbol{r}_2^{\mathsf{T}}\mathbf{l}'' & \boldsymbol{s}_2^{\mathsf{T}}\mathbf{l}''' \\ 0 & c' & \boldsymbol{r}_3^{\mathsf{T}}\mathbf{l}'' & \boldsymbol{s}_3^{\mathsf{T}}\mathbf{l}''' \\ 0 & 0 & \boldsymbol{r}_4^{\mathsf{T}}\mathbf{l}'' & \boldsymbol{s}_4^{\mathsf{T}}\mathbf{l}''' \end{vmatrix} = 0$$

With the three 3×3-matrices or a $3 \times 3 \times 3$-tensor

$$\mathsf{T}_i = \boldsymbol{r}_i\boldsymbol{s}_4^{\mathsf{T}} - \boldsymbol{r}_4\boldsymbol{s}_i^{\mathsf{T}} \qquad \text{or} \qquad T_{ijk} = r_{ij}s_{k4} - r_{4j}s_{ki} \tag{5}$$

we obtain after expanding the expression $b'\mathbf{l}''^{\mathsf{T}}\mathsf{T}_3\mathbf{l}''' - c'\mathbf{l}''^{\mathsf{T}}\mathsf{T}_2\mathbf{l}''' = 0$ thus $b' : c' = (\mathbf{l}''^{\mathsf{T}}\mathsf{T}_2\mathbf{l}''') : (\mathbf{l}''^{\mathsf{T}}\mathsf{T}_3\mathbf{l}''')$. Analogically we may handle the other constraints in (4) and obtain $a' : b' : c' = (\mathbf{l}''^{\mathsf{T}}\mathsf{T}_1\mathbf{l}''') : (\mathbf{l}''^{\mathsf{T}}\mathsf{T}_2\mathbf{l}''') : (\mathbf{l}''^{\mathsf{T}}\mathsf{T}_3\mathbf{l}''')$. This can be written as *prediction* for the image line $\mathbf{l}'$ in two ways using the tensor (5)

$$\mathbf{l}' = (a', b', c')^{\mathsf{T}} = (\mathbf{l}''^{\mathsf{T}}\mathsf{T}_1\mathbf{l}''', \mathbf{l}''^{\mathsf{T}}\mathsf{T}_2\mathbf{l}''', \mathbf{l}''^{\mathsf{T}}\mathsf{T}_3\mathbf{l}''')^{\mathsf{T}} \qquad \text{or} \qquad l_i' = \sum_{j=1}^{3}\sum_{k=1}^{3} l_j'' l_k''' T_{ijk} \quad i = 1, 2, 3 \tag{6}$$

Thus the prediction of homologeous image lines is extremely simple using bilinear forms and the using the trifocal tensor (5).

The relations for homologeous points $P'(\mathbf{x}')$, $P''(\mathbf{x}'')$ und $P'''(\mathbf{x}''')$ are somewhat more complicated.

We need three constraints for the three projection lines to intersect in one 3D point, as we have 6 observed coordinates and 3 independent unknown point coordinates. For points not lying on the trifocal plane through the three projection centres we could just use the three pairwise epipolar constraints. Otherwise at least one constraint including coordinates are necessary. An example set of contraints is the following (FÖRSTNER 2000a):

$$|\mathbf{A}_1(x'), \mathbf{B}_1(y'), \mathbf{A}_2(x''), \mathbf{B}_2(y'')| = 0,\ |\mathbf{A}_1(x'), \mathbf{B}_1(y'), \mathbf{A}_2(x''), \mathbf{A}_3(x''')| = 0,\ |\mathbf{A}_1(x'), \mathbf{B}_1(y'), \mathbf{A}_2(x''), \mathbf{B}_3(y''')| = 0$$

In all constraints the two first planes $\mathbf{A}_1$ and $\mathbf{B}_1$ span the first projection ray. The intersection with the x''-plane defines the 3D point uniquely. This point must lie on the y''-plane of the second camera, being identical with the epipolar constraint. Moreover, the point must lie on the ray defined by the two planes $\mathbf{A}_3$ and $\mathbf{B}_3$.

Prediction of points is a bit more complicated. The coordinates x_k''' of the point in the third image can be derived from the coordinates in the first two images by (HARTLEY 1995):

$$x_l''' = \sum_{k=1}^{3} x_k'(x_i''T_{kjl} - x_j''T_{kil}), \qquad i,j = 1,2,3, \text{ choosable} \tag{7}$$

The prediction is not unique in case of noisy data, as the indices i und j can be schosen freely under the constraint $i \neq j$, unless the homologeous point fulfill the epipolar constraint in the first two images. For standard cases, especially with collinear projection centres one can give rules for selecting constraints (FÖRSTNER 2000a).

6.2 ESTIMATION OF THE RELATIVE ORIENTATION OF IMAGE TRIPLETS

Linear Parametrization: As the constraints linearily depend on the tensor coefficients one can directly estimate the tensor coefficients, in full analogy to estimating the fundamental matrix of stereo pairs in case at least 26 constraints are available. This solution, however, turned out to be quite instable (HARTLEY 1994b), as the trifocal tensor overparametrizes the geometry of the image triplet.

Minimal Parametrization: In reality the trifocal tensor only has 18 degrees of freedom, as the projection matrices of the three cameras (33 parameters) can only be determined up to a 3D projective transformation (15 parameters). Therefore there exist 9=27-18 independent nonlinear constraints on the parameters of the trifocal tensor, which need to be taken into account (FAUGERAS & PAPADOPOULO 1998) or one minimally parametrizes the trifocal tensor as e. g. in (TORR & ZISSERMAN 1997).

In case the interior orientation of the cameras is known one may parametrize the trifocal tensor Euclidean, as then the 3*6=18 parameters of the exterior orientation of the three cameras can be only determined up to a similarity transformation with 7 degrees of freedom, these are 5 for the relative orientation of the first two cameras and 6 for the exterior orientation of the third cameras, a standard procedure in Photogrammetry and used for trilinear constraints already in (MIKHAIL 1962, MIKHAIL 1963). We used it for analysing triplets in image sequences (STEINES & ABRAHAM 1999, ABRAHAM 2000). Of course approximate values are required for the estimation, but one directly obtains orientation parameters.

Determining the orientation parameters: There are rich relations between the coefficients of the trifocal tensor, the fundamental matrices, and the projection matrices. They can be used to derive the parameters of the relative orientation of the three image pairs and, after fixing a coordinate system, for determining then projection matrices. (HARTLEY 1994a, HARTLEY 1994b, FAUGERAS & PAPADOPOULO 1998).

Our experience confirms that the local geometry can be determined much more reliably and more robust from image triplets than from image pairs, which are only weakly overdetermined (ABRAHAM 2000)

7 OUTLOOK

We discussed the geometric relations useful for orienting single images, pairs and triplets. Whereas the singular cases for the single image and the image pair are well understood, not all critical configurations for triplets are known. There also exist interesting procedures for orienting streams of images (TOMASI & KANADE 1997, KANADE & MORRIS 1998) or of image pairs, including general analyses on critical configurations (STURM 1997).

Parallel to these very important geometric findings, results on automating the orientation using matching techniques are available, both on points, and lines, though orientation with lines turn out to be not as stable as with points. In all cases robust estimators with high break-down point as well as classical M-estimators (FISCHLER & BOLLES 1981, ROUSSEEUW & LEROY 1987) are applied.

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