Earth Ellipsoid, ellipsoidal coordinates, satellite coordinate systems

(Part I)

MSc Geodetic Engineering

Module Coordinate Systems
1st Semester, 2020/21

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Today

- Earth as a flattened body, Earth ellipsoid
- Ellipsoidal coordinates and transformations
- Earth's rotation and inertial coordinate system
- Kepler angles and satellite coordinate systems

References

- Kusche J, Lecture Notes -> ecampus
- Jekeli C., Geometric Reference Systems in Geodesy, Ohio State University (-> https://kb.osu.edu/handle/1811/77986)

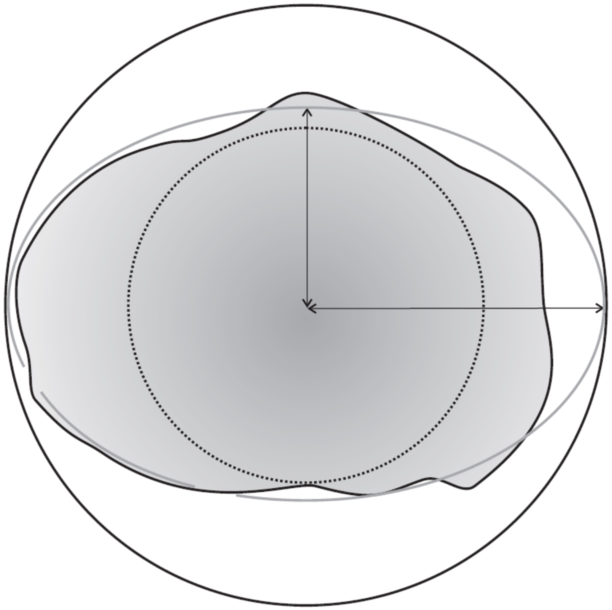
Earth as a flattened body

Some observations

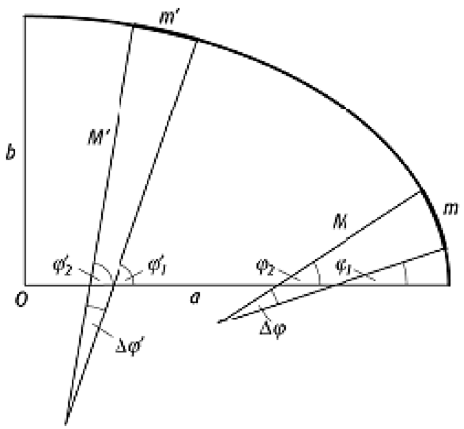
- Earth is not a perfect sphere.
- In first approximation, it resembles a sphere flattened at the poles
- Best-fitting polar radius 20 km less than equatorial radius (~ 6380km)
- Therefore, in geodesy we use an ellipsoid of revolution as a reference surface → latitude, longitude and height

How do we know this?

- History of geodesy: arc measurements, 17th and 18th century
- World-wide geodetic networks, geodetic datums
- Satellite observations of the (dynamical) flattening of the Earth: precession of satellite orbital plane observed already with Sputnik-1



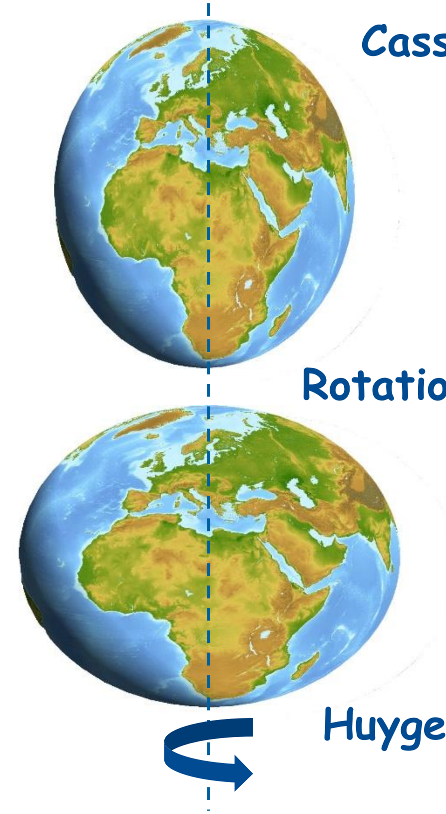
History of geodesy: arc measurements, 17th and 18th century



Semimajor axis

$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{\frac{3}{2}}}$$

Eccentricity parameter



Modern geodesy: precession of satellite orbital plane

Location	Latitude (θ)	Arc length (toises)
(1) Quito	0°0′	56,751
(2) Cape of Good Hope	33°18′	57,037
(3) Rome	42°59′	56,979
(4) Paris	49°23′	57,074
(5) Lapland	66°19′	57,422

Geodetic measurements enable to determine 3D positions with mm – accuracy

GPS installation for monitoring potential volcanic deformation (Japan, Photo: J. Kusche)

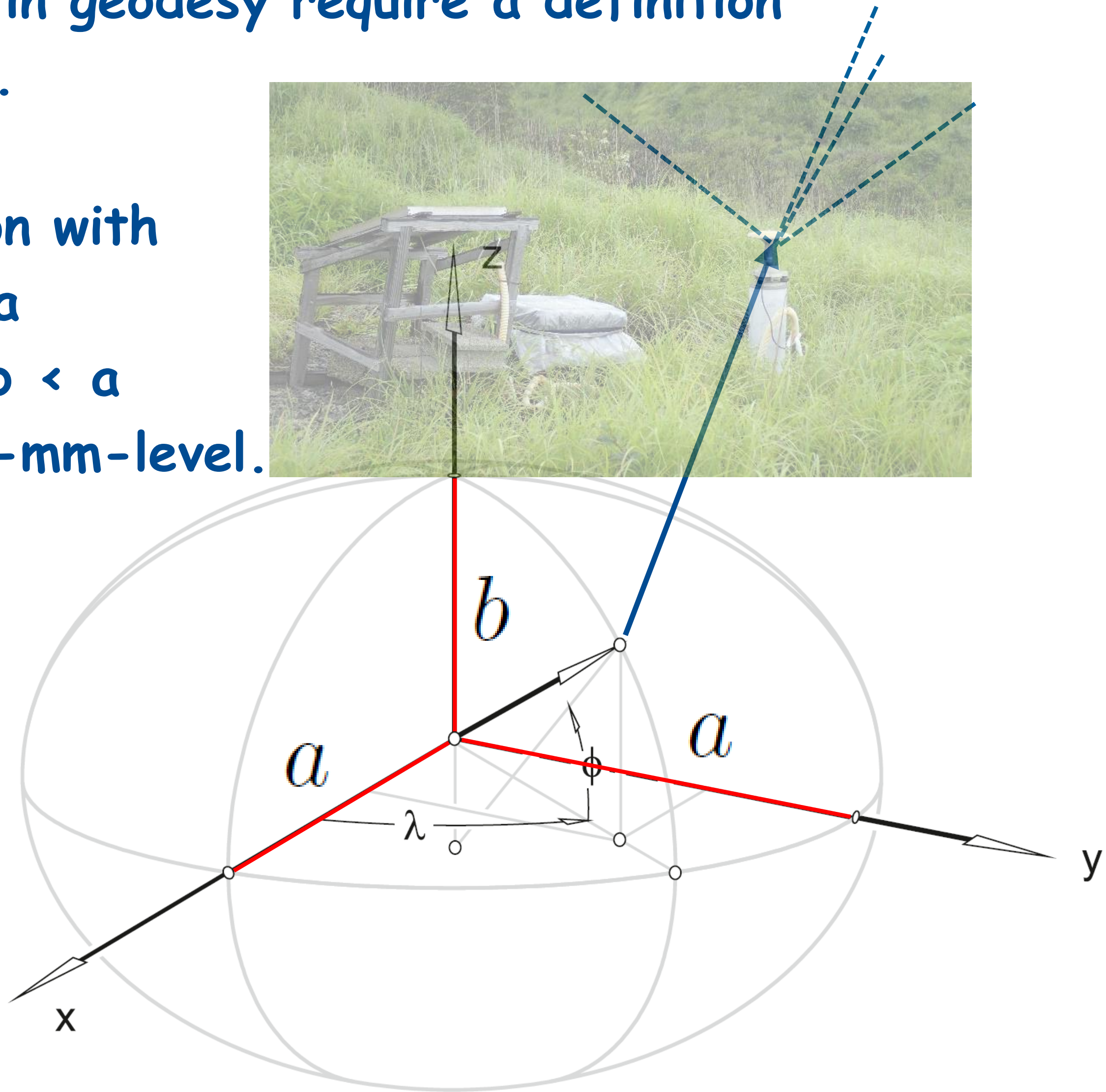
Reference surfaces in geodesy require a definition <u>precise at mm-level</u>.

→

ellipsoid of revolution with

- **semi-major axis a**
- **semi-minor axis b < a**

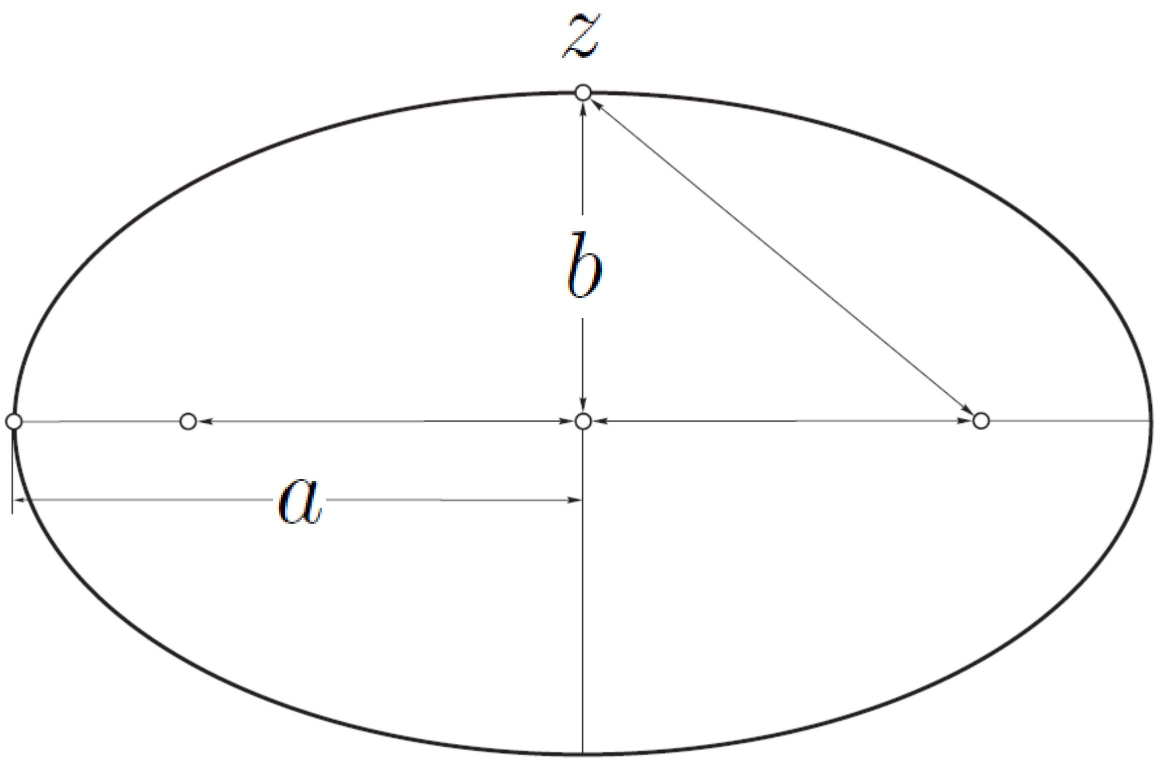
a, b defined at sub-mm-level.



Ellipsoid of revolution

Parametric description

$$p = (x^2 + y^2)^{1/2}$$

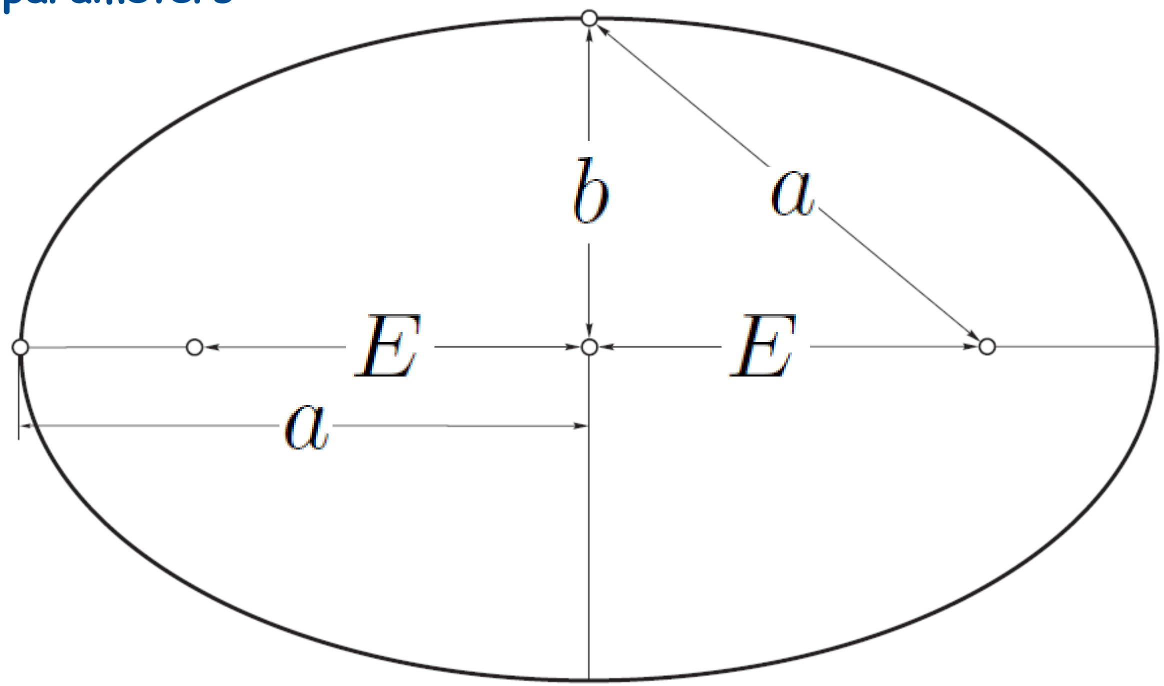


For an ellipsoid of revolution it is sufficient to consider the meridional ellipse, with the p-coordinate

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = \left(\frac{p}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1$$

Ellipsoid of revolution

Use of other ellipsoid parameters



Linear eccentricity

$$E = ea = e'b$$

2nd numerical eccentricity

1st numerical eccentricity

Geometrical flattening $f = \frac{a-b}{a}$

Ellipsoid parameters in practice († derived from other parameters)

Ellipsoid		
Bessel (1841)	$a = 6377397.155^\dagger$	$f = 1/299.1528128^\dagger$
Hayford (1909)	$a = 6378388$	$f = 1/297$
Krassowsky (1940)	$a = 6378245$	$f = 1/298.3$
GRS80	$a = 6378137$	$f = 1/298.257222101^\dagger$
WGS84	$a = 6378137$	$f = 1/298.257223563^\dagger$
Topex/Poseidon	$a = 6378136.3$	$f = 1/298.257$

There are several relations between the parameters, e.g.

$$e^2 = \frac{a^2-b^2}{a^2} \quad e'^2 = \frac{a^2-b^2}{b^2} \quad E^2 = a^2 - b^2$$

$$E = ea = e'b$$

$$b^2 = a^2(1-e^2)$$

$$b^2 = \frac{a^2}{1+e'^2}$$

Important

Ellipsoid		
Bessel (1841)	$a = 6377397.155^{\dagger}$	$f = 1/299.1528128^{\dagger}$
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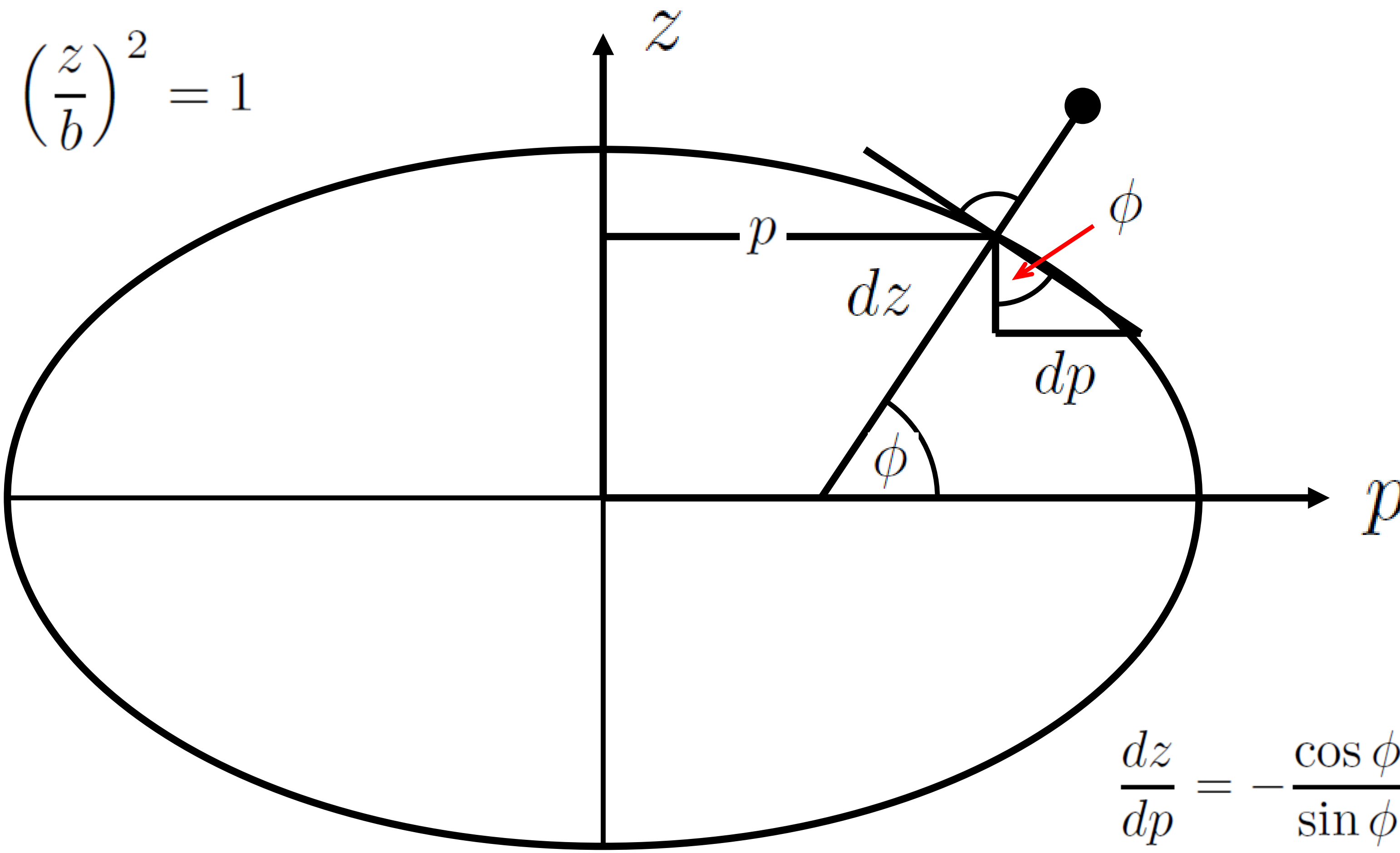
- For a given ellipsoid only two parameters are independent
- All others are "derived" and can (and must) be computed from the two given ones with maximum precision
- "Maximum precision" means all parameters must be given or computed with sufficient precision (digits) that allow transformation of coordinates with 0.1 mm accuracy.
- This must be considered in particular for numerical eccentricy or flattening parameters

Earth ellipsoid

(introducing the ellipsoidal latitude)

$$\left(\frac{p}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1$$



$$\frac{dz}{dp} = -\frac{\cos\phi}{\sin\phi}$$

Ellipsoidal latitude φ and the slope of the tangent

$$\frac{dz}{dp} = -\frac{\cos\phi}{\sin\phi} \qquad \left(\frac{p}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1$$

$$\frac{dz}{dp} = b\frac{-2\frac{p}{a^2}}{2\sqrt{1-\frac{p^2}{a^2}}} = \frac{-bp}{a^4 - a^2p^2}$$

(all derivations can be found in lecture notes J. Kusche)

$$p = \frac{a^2\cos\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}\cos\phi$$

$$z = b\sqrt{1 - \frac{a^2\cos^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi}} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}\frac{1}{1+e'^2}\sin\phi$$

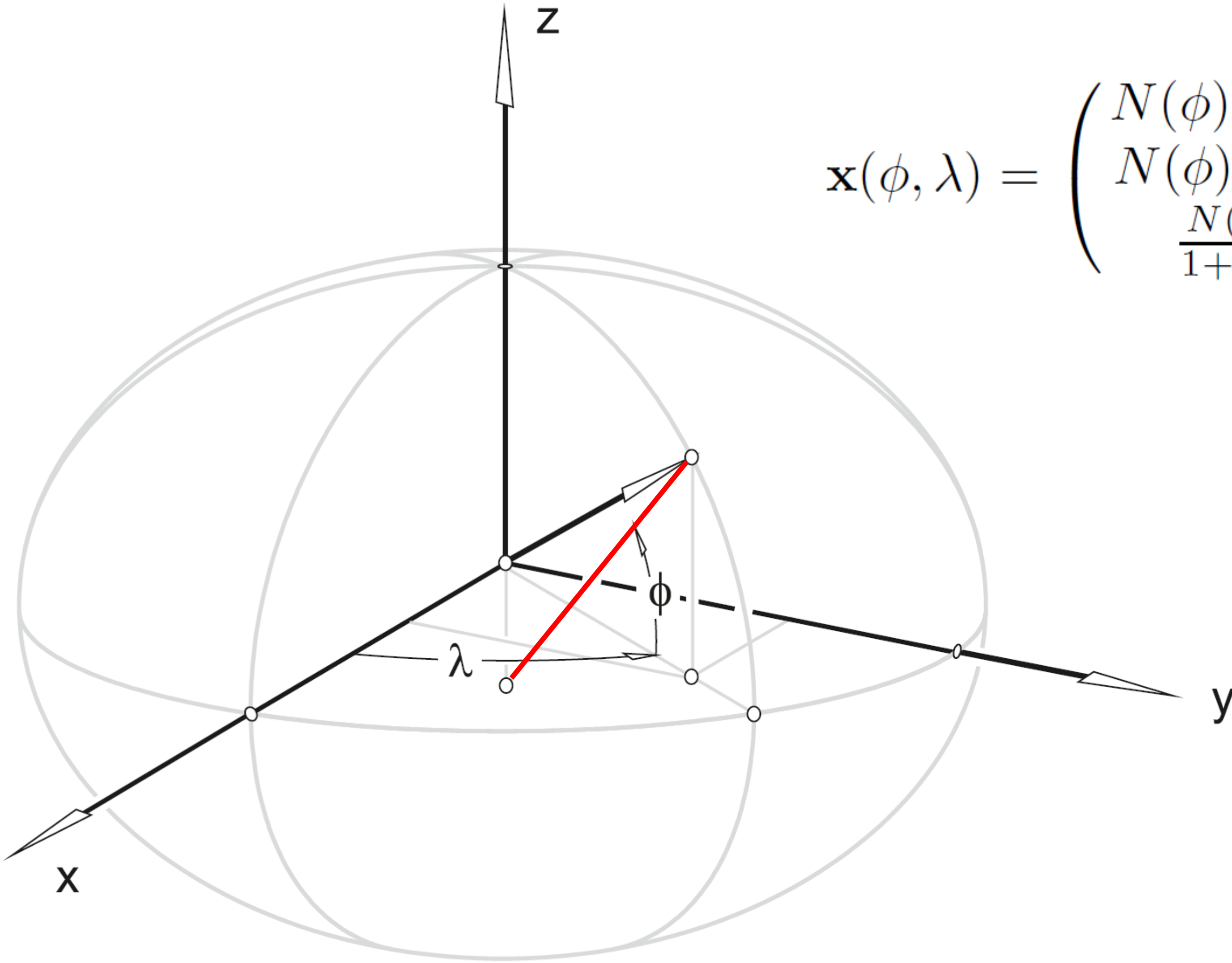
$\varphi \rightarrow p, z$

$$N(\phi) = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

$$r^2 = p^2 + z^2 = \frac{a^4\cos^2\phi + b^4\sin^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi}$$

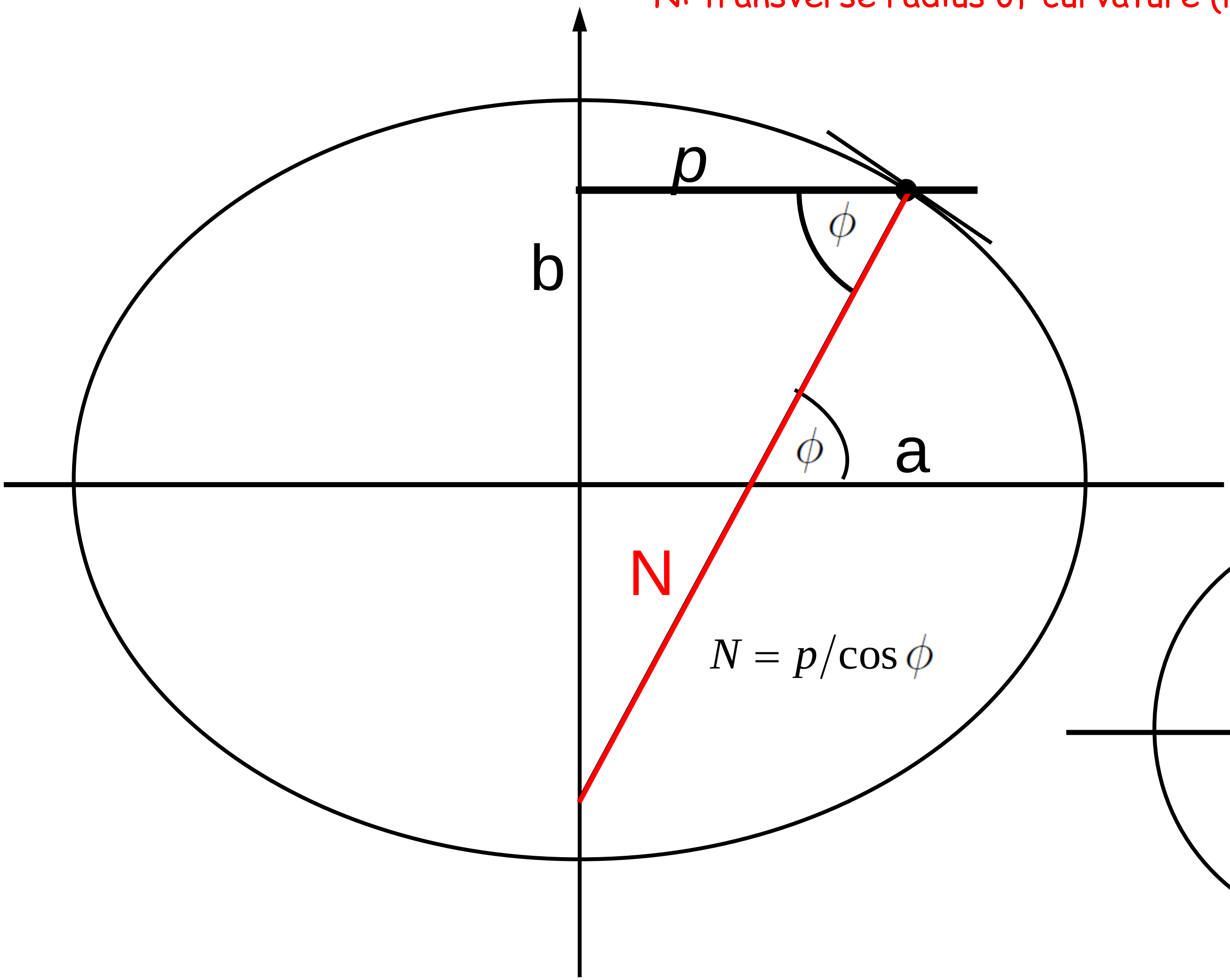
("radius equation") $\varphi \rightarrow r$

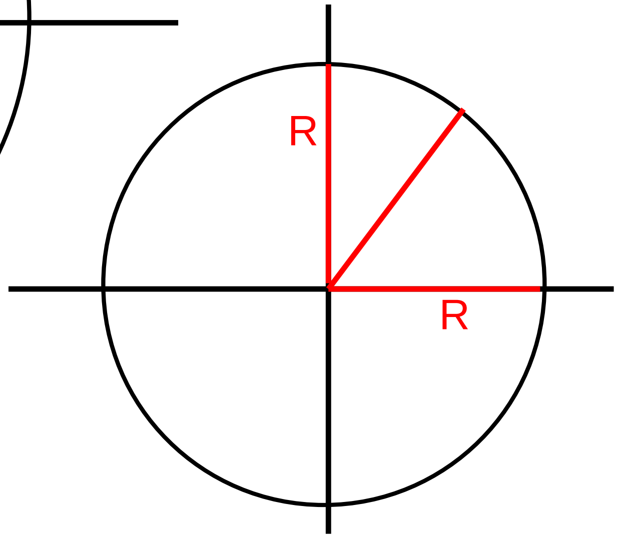
Ellipsoidal longitude and latitude

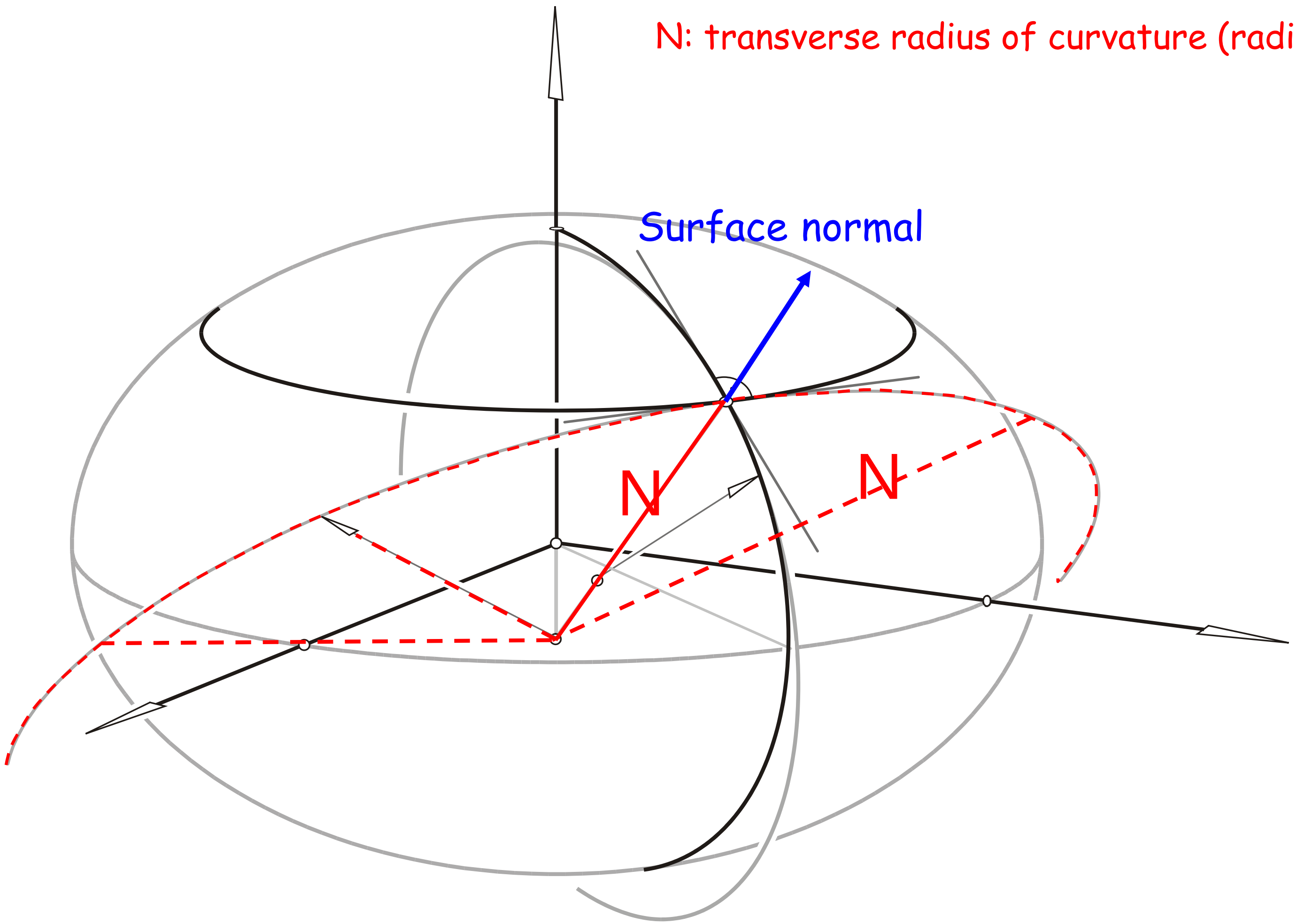


$$\mathbf{x}(\phi,\lambda) = \begin{pmatrix} N(\phi)\cos\lambda\cos\phi \\ N(\phi)\sin\lambda\cos\phi \\ \frac{N(\phi)}{1+e'^2}\sin\phi \end{pmatrix}$$

N: transverse radius of curvature (radius of prime vertical)

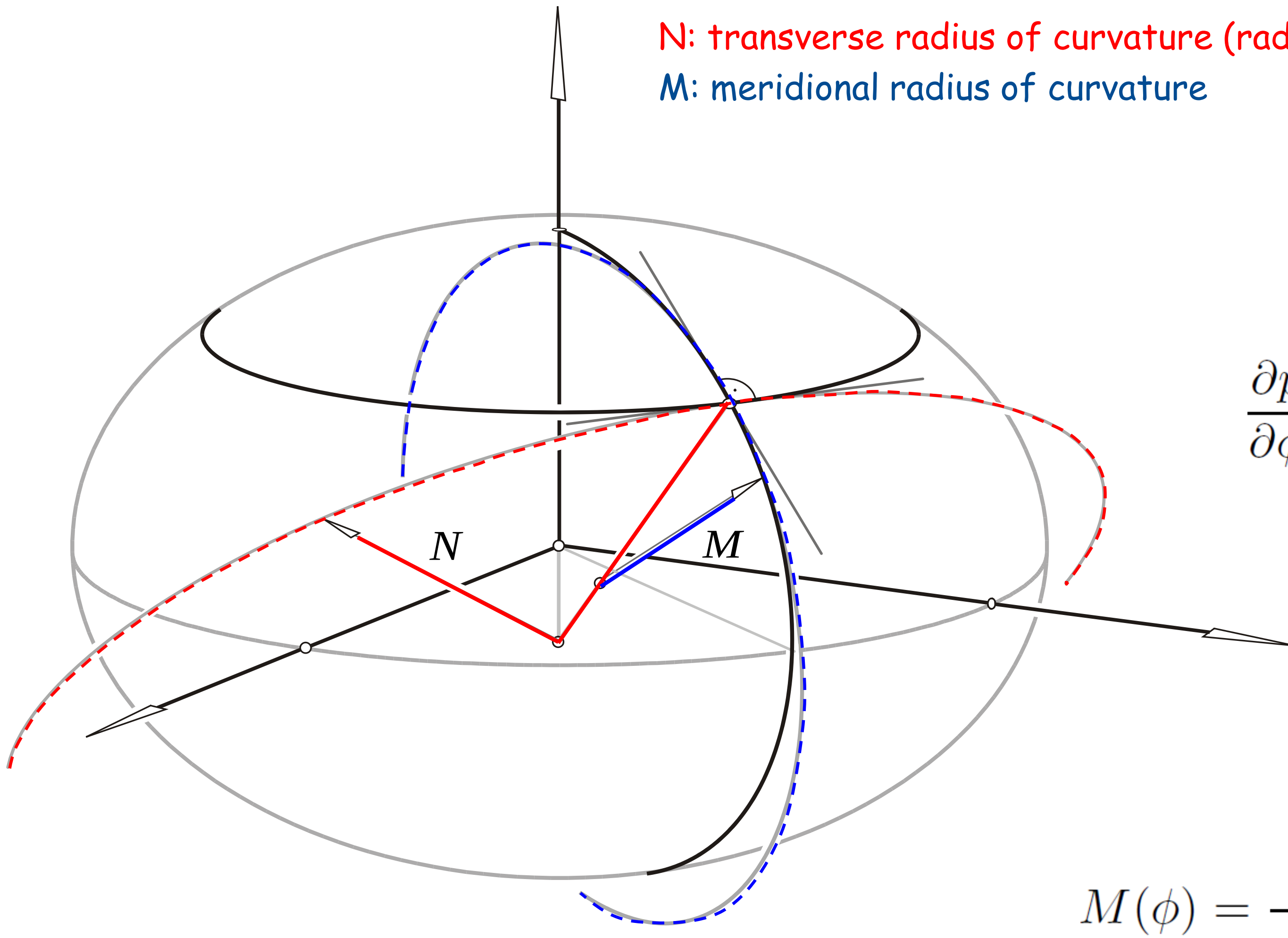





N: transverse radius of curvature (radius of prime vertical)
Surface normal
N
N

N: transverse radius of curvature (radius of prime vertical)

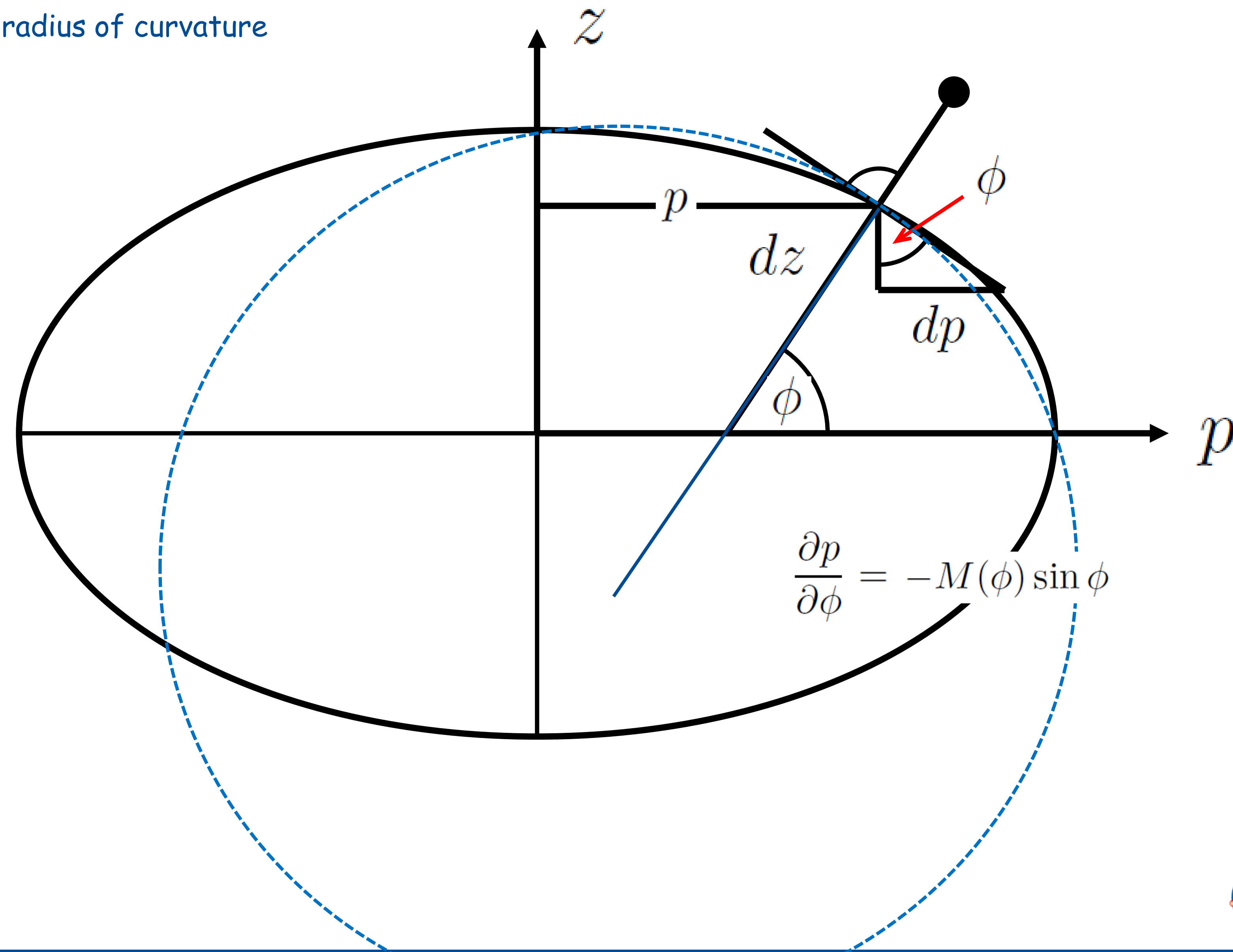
M: meridional radius of curvature

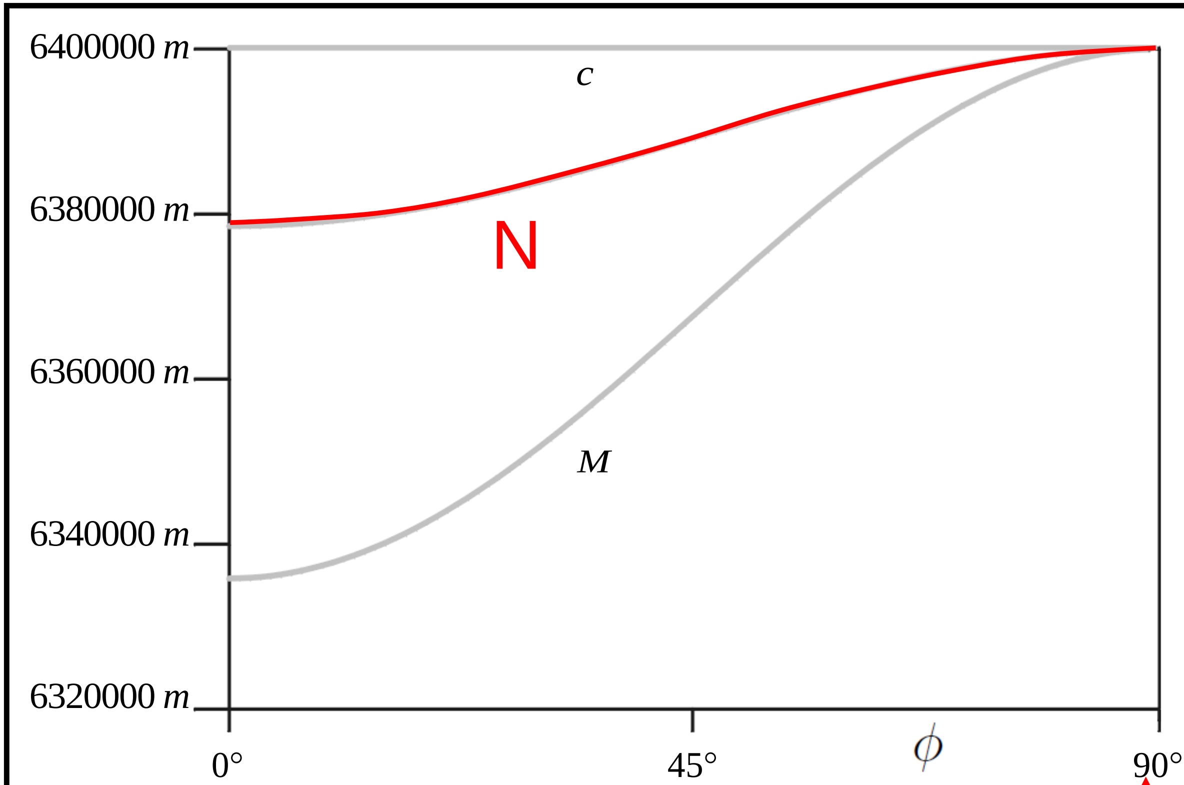


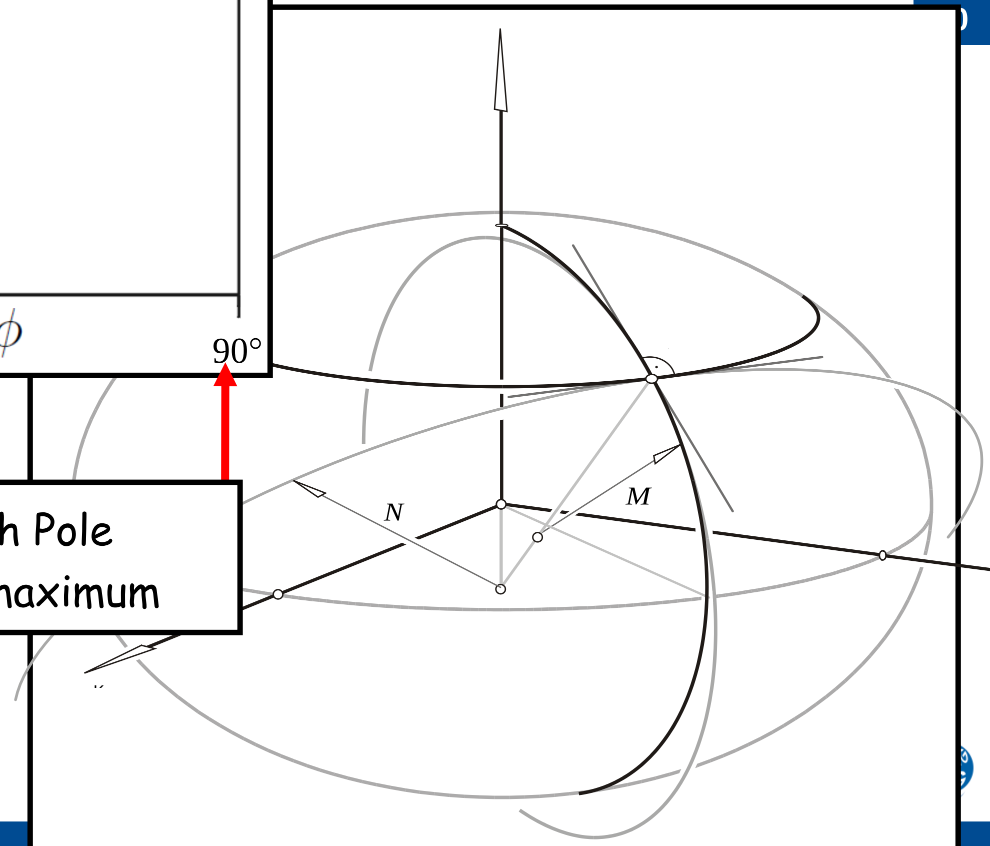
$$\frac{\partial p}{\partial \phi} = -M(\phi)\sin\phi$$

$$M(\phi) = \frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\phi}^3}$$

M: meridional radius of curvature





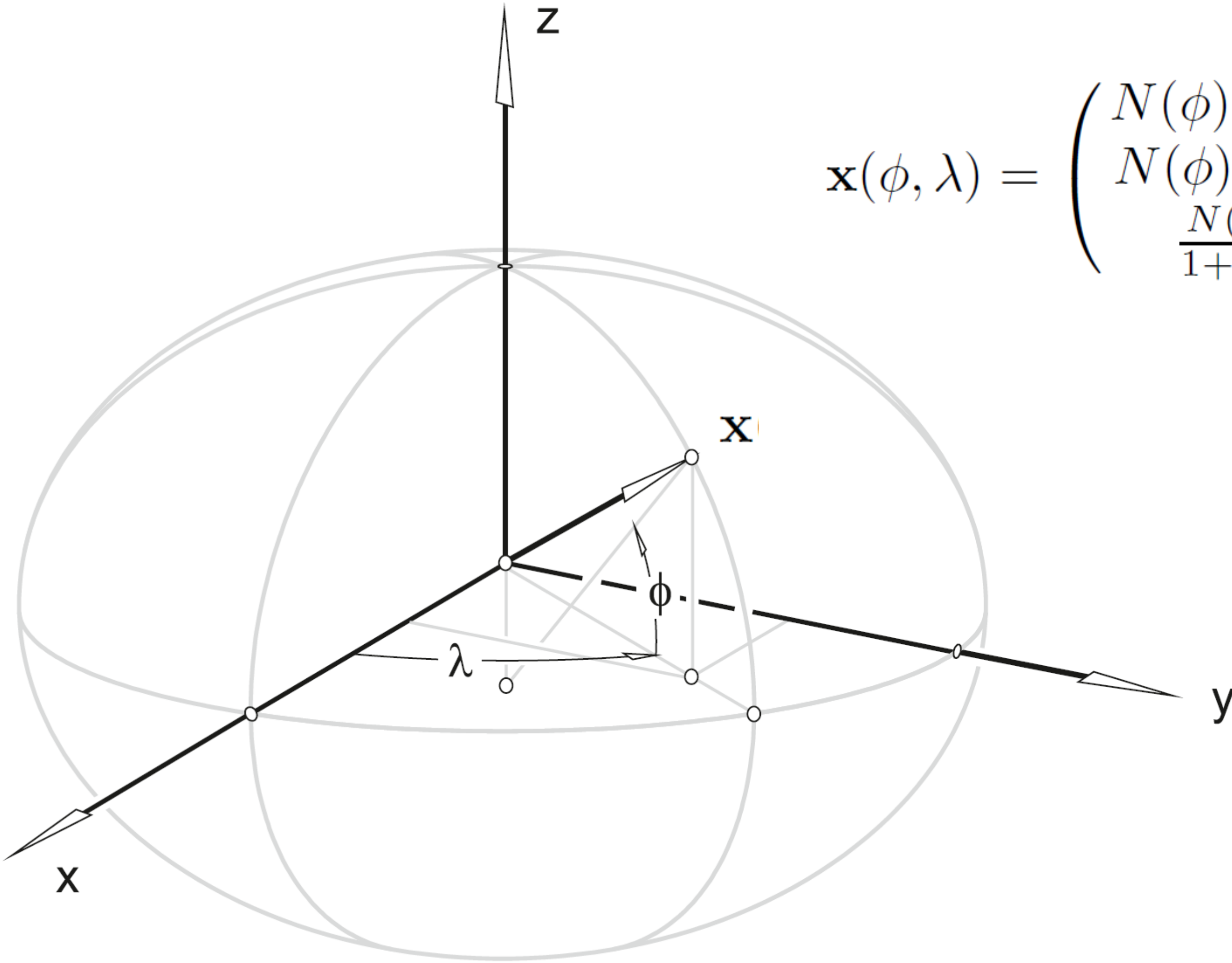


Equator
N = semi-major axis a

North Pole
N = maximum

Ellipsoidal coordinates

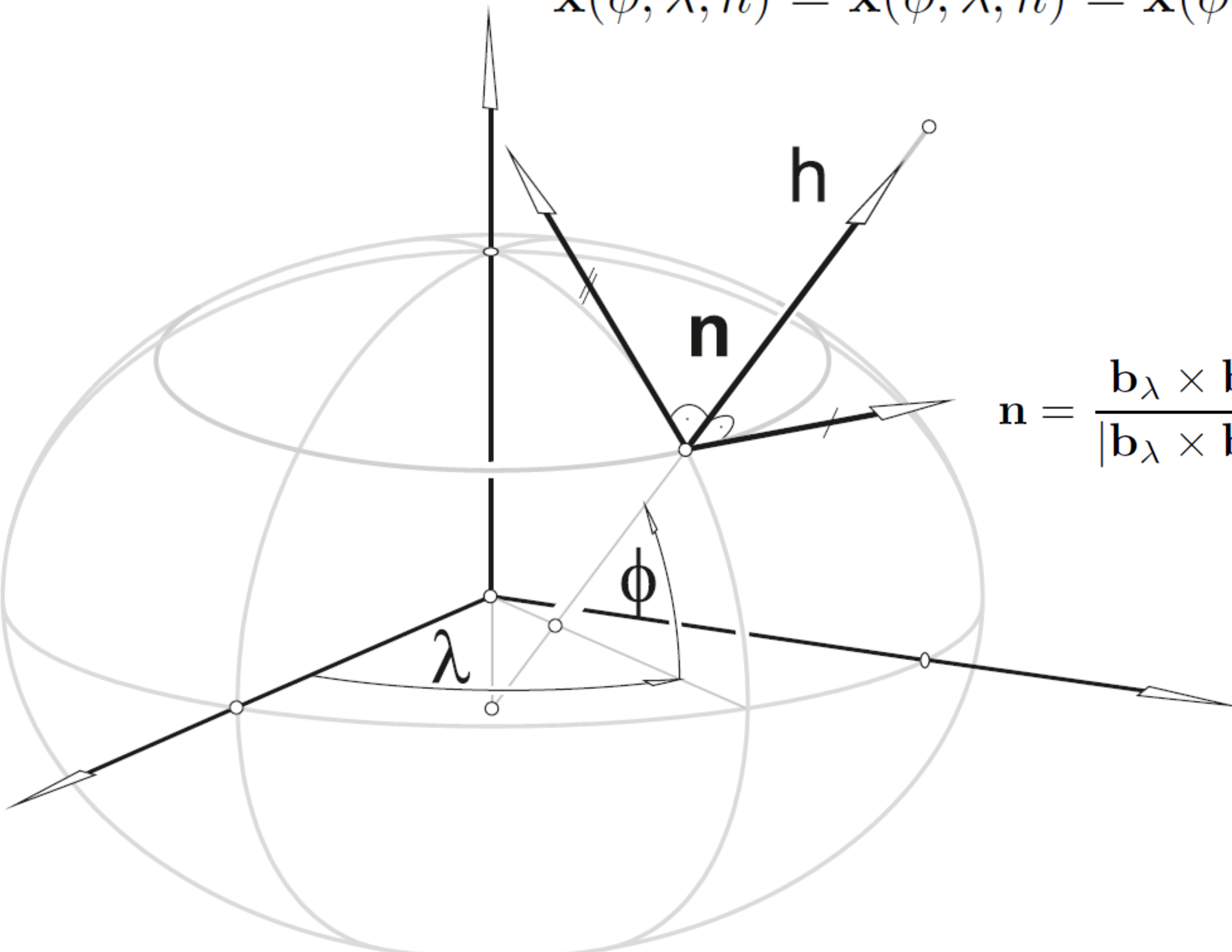
Ellipsoidal (also called geographical) longitude and latitude



$$\mathbf{x}(\phi, \lambda) = \begin{pmatrix} N(\phi)\cos\lambda\cos\phi \\ N(\phi)\sin\lambda\cos\phi \\ \frac{N(\phi)}{1+e'^2}\sin\phi \end{pmatrix}$$

Exterior to the ellipsoid: Ellipsoidal longitude, latitude, and height

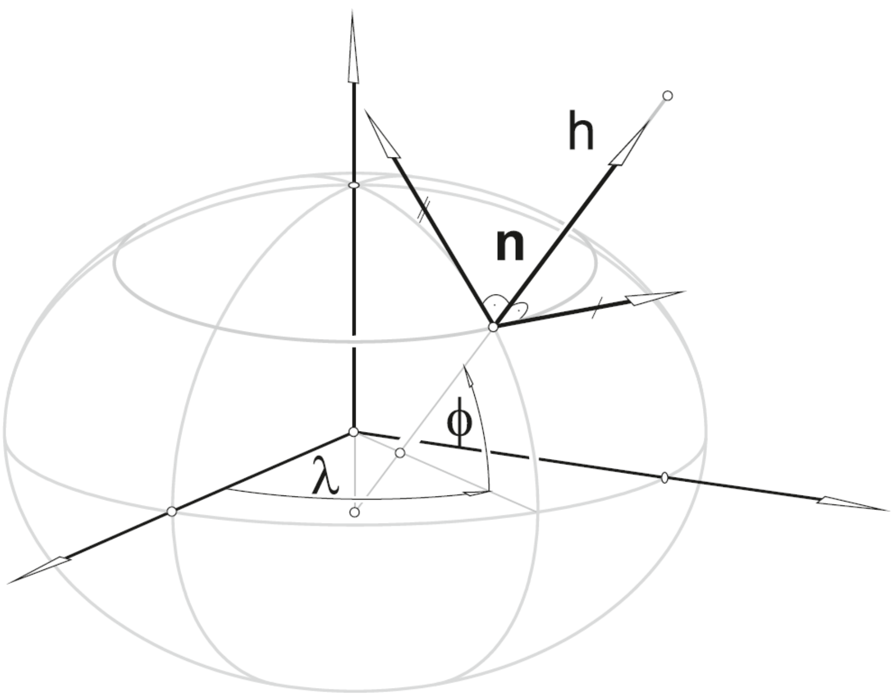
$$\mathbf{x}(\phi, \lambda, h) = \mathbf{x}(\phi, \lambda, h) = \mathbf{x}(\phi, \lambda) + h\mathbf{n}(\phi, \lambda)$$



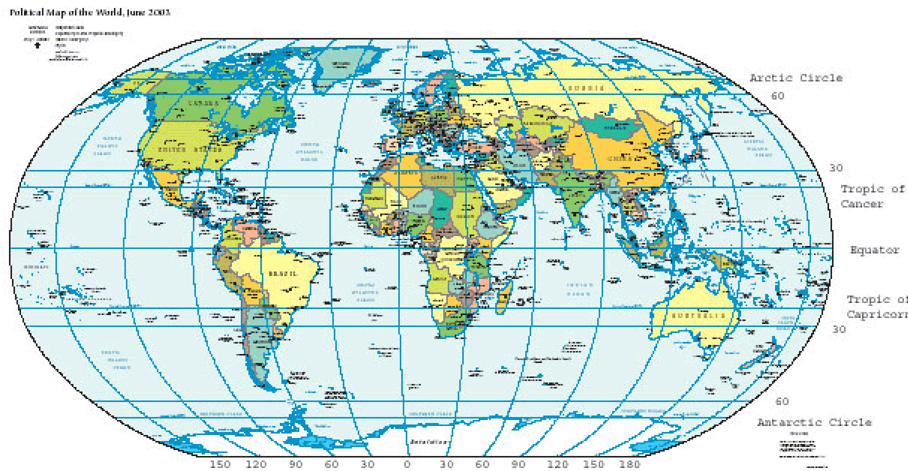
$$\mathbf{n} = \frac{\mathbf{b}_\lambda \times \mathbf{b}_\phi}{|\mathbf{b}_\lambda \times \mathbf{b}_\phi|} = \begin{pmatrix} \cos\lambda\cos\phi \\ \sin\lambda\cos\phi \\ \sin\phi \end{pmatrix}$$

Ellipsoidal longitude, latitude, and height

$$\mathbf{x}(\phi,\lambda,h) = \begin{pmatrix} (N(\phi)+h)\cos\lambda\cos\phi \\ (N(\phi)+h)\sin\lambda\cos\phi \\ \left(\frac{N(\phi)}{1+e'^2}+h\right)\sin\phi \end{pmatrix}$$

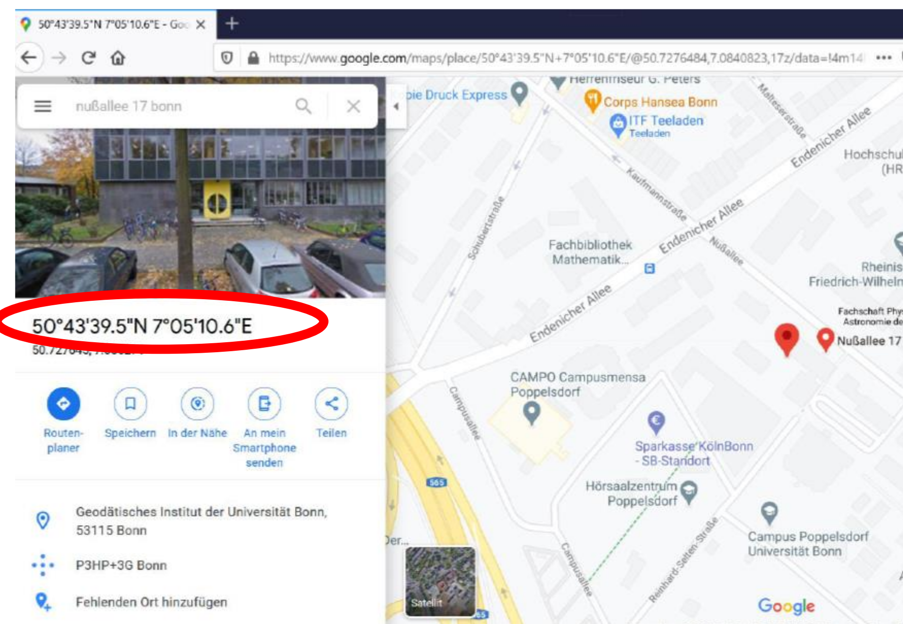


Ellipsoidal longitude and latitude ("GPS-coordinates") (e.g. ITRF)



Always need to know

- Ellipsoid parameter
- Location & orientation of ellipsoid w.r.t. solid Earth = reference frame (e.g. ITRF) → lecture MGE-05

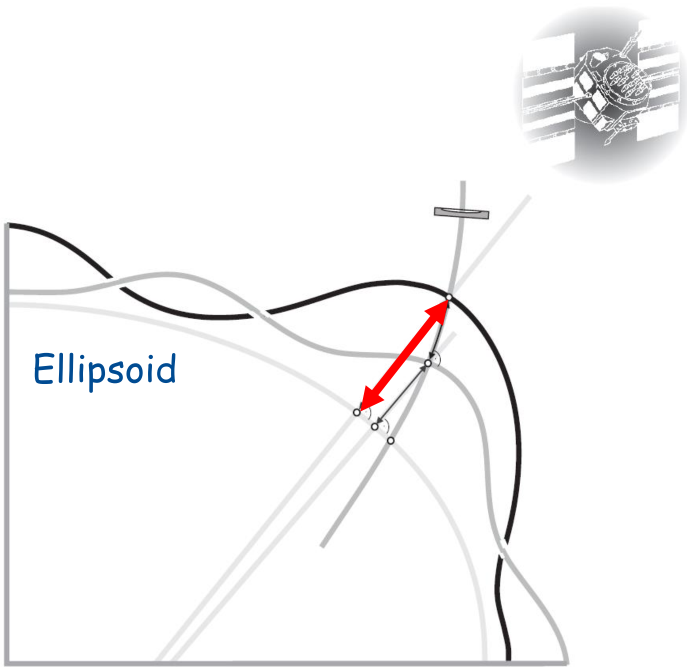


Google maps

Ellipsoidal heights (GNSS) and physical heights

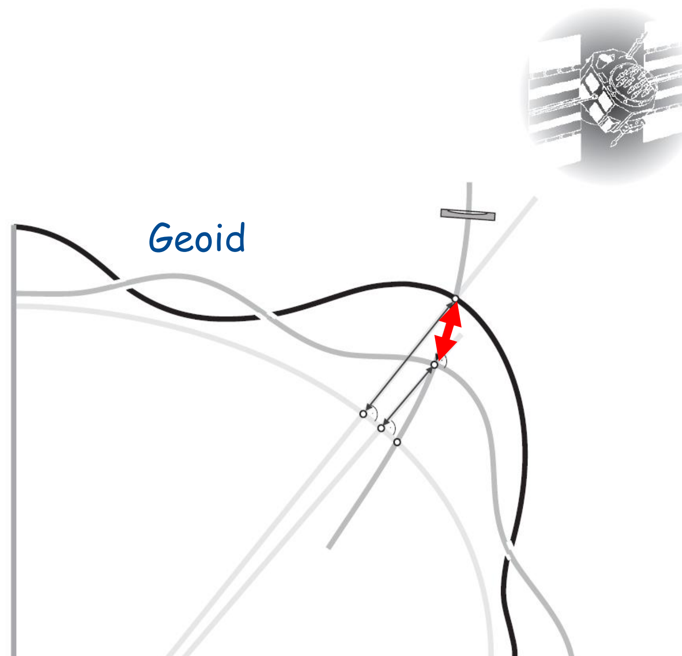
Ellipsoidal heights

- Geometrical reference surface
- GNSS-determined (i.e. from x,y,z)



Physical heights

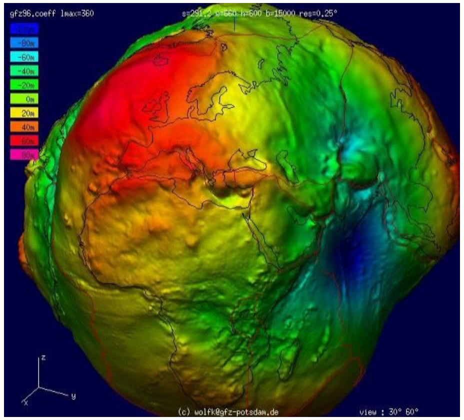
- Reference surface = Geoid
- Spirit levelling (+ gravity)



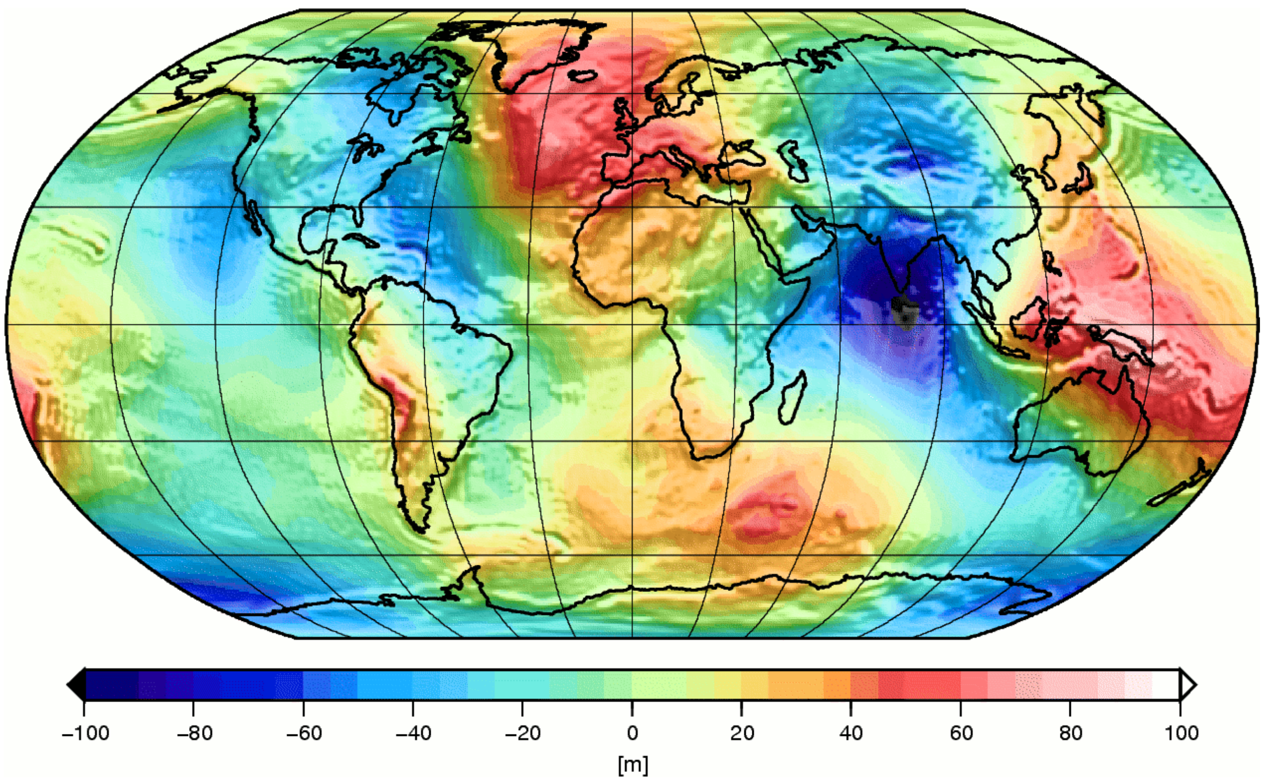
Ellipsoidal heights (GNSS) and physical heights:
Geoid undulations

Geoid

(Geoid undulations = height of the Geoid above the Ellipsoid)



GFZ Potsdam



Coordinate transformations

(for given ellipsoid parameters)

Ellipsoidal coordinates → Cartesian coordinates

$$(\lambda, \phi, h) \rightarrow (x, y, z)$$

1. Compute N

$$N(\phi) = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}}$$

2. Compute x, y, z

$$\mathbf{x}(\phi, \lambda, h) = \mathbf{x}(\phi, \lambda, h) = \mathbf{x}(\phi, \lambda) + h\mathbf{n}(\phi, \lambda)$$

$$= \begin{pmatrix} (N(\phi) + h) \cos\lambda \cos\phi \\ (N(\phi) + h) \sin\lambda \cos\phi \\ \left(\frac{N(\phi)}{1+e'^2} + h\right) \sin\phi \end{pmatrix}$$

or

$$x = (N+h) \cos\lambda \cos\phi \qquad y = (N+h) \sin\lambda \cos\phi \qquad z = \left(\frac{N}{1 + e'^2} + h\right) \sin\phi$$

√

Cartesian coordinates → Ellipsoidal coordinates (I)

1. compute $\lambda = \arctan \frac{y}{x}$ $p = (x^2+y^2)^{1/2}$

$$\phi = \arctan\left(\frac{z}{p}\cdot\frac{N+h}{\frac{N}{1+e'^2}+h}\right) \qquad h = \frac{p}{\cos\phi} - N$$

2. Since N depends on φ, computing latitude and height…

…requires iterative solution.

E.g. start values:

$$h_{[0]} = 0$$
$$\phi_{[0]} = \arctan\left(\frac{z(1+e'^2)}{p}\right) \longrightarrow N(\phi) = \frac{a}{\sqrt{1-e^2\sin^2\phi}}$$

Cartesian coordinates → Ellipsoidal coordinates (II)

$$\phi = \arctan\left(\frac{z}{p} \cdot \frac{N+h}{\frac{N}{1+e'^2}+h}\right) \qquad h = \frac{p}{\cos\phi} - N$$

Terminate e.g. when $|h_{[k+1]} - h_{[k]}| < \varepsilon$
and $R|\varphi_{[k+1]} - \varphi_{[k]}| < \varepsilon$

(e.g. for ε=0.1 mm)

$$N(\phi) = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}}$$

Typically 4-5 iterations sufficient.

√