Coordinate Systems

Map Projections

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Coordinate Systems for Geoinformation

Earth-centered Cartesian coordinates

spherical/ellipsoidal coordinates 
latitude $\varphi$, longitude $\lambda$

projected coordinates

northing $y$
easting $x$
Coordinate Systems for Geoinformation

Earth-centered Cartesian coordinates

spherical/ellipsoidal coordinates
latitude $\varphi$, longitude $\lambda$

projected coordinates
northing $y$
easting $x$

$\begin{pmatrix} X_p \\ Y_p \\ Z_p \end{pmatrix} = \begin{pmatrix} r \cos \varphi_p \cos \lambda_p \\ r \cos \varphi_p \sin \lambda_p \\ r \sin \varphi_p \end{pmatrix}$

for spherical coordinates
Coordinate Systems for Geoinformation

Earth-centered Cartesian coordinates

spherical/ellipsoidal coordinates
latitude $\varphi$, longitude $\lambda$

projected coordinates

\[
\begin{bmatrix}
X_p \\
Y_p \\
Z_p
\end{bmatrix} = \begin{bmatrix}
r \cos \varphi_p \cos \lambda_p \\
r \cos \varphi_p \sin \lambda_p \\
r \sin \varphi_p
\end{bmatrix}
\]

for spherical coordinates
Relationship \((\varphi, \lambda) \leftrightarrow (x, y)\)

- Point on reference surface (e.g. sphere, ellipsoid) is mapped to plane with a **map projection**.
Relationship \((\varphi, \lambda) \leftrightarrow (x, y)\)

- Point on reference surface (e.g. sphere, ellipsoid) is mapped to plane with a map projection.

- Different map projections cause different distortions.
- Best choice of map projection depends on application.
Outline

• Plate carrée projection (＝ most basic cylindrical projection)
• What is a cylindrical projection?
• Mercator projection (＝ most important cylindrical projection)
• cylindrical equal-area projection
Plate Carrée Projection

Idea:

• Map areas between lines of constant latitude/longitude to squares of constant size (assuming $\Delta \varphi = \Delta \lambda$).
Plate Carrée Projection

Idea:

• Map areas between lines of constant latitude/longitude to squares of constant size (assuming $\Delta \varphi = \Delta \lambda$).
• Preserve lengths of equator and meridians.
Plate Carrée Projection

- mapping function:

  easting $x = \lambda[\text{deg}]$
  northing $y = \varphi[\text{deg}]$
Plate Carrée Projection

- mapping function:

\[
\begin{align*}
\text{easting } x &= \lambda [\text{deg}] \cdot \frac{\pi}{180^\circ} \cdot r \\
\text{northing } y &= \varphi [\text{deg}] \cdot \frac{\pi}{180^\circ} \cdot r
\end{align*}
\]

\( r = \text{Earth's radius} \)
Plate Carrée Projection

Disadvantages:

• shapes get “squeezed”, i.e., aspect ratios change
Plate Carrée Projection

Disadvantages:
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• angles change
Plate Carrée Projection

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• shapes get “squeezed”, i.e., aspect ratios change
• angles change
• areas change
Plate Carrée Projection

Disadvantages:
- shapes get “squeezed”, i.e., aspect ratios change
- angles change
- areas change

Advantage:
- lengths of equator and meridians are preserved
Coordinate Systems for Geoinformation

Earth-centered Cartesian coordinates

spherical/ellipsoidal coordinates

latitude \( \varphi \), longitude \( \lambda \)

projected coordinates

northing \( y \)
easting \( x \)

\[
\begin{bmatrix}
X_p \\
Y_p \\
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\end{bmatrix} = \begin{bmatrix}
r \cos \varphi_p \cos \lambda_p \\
r \cos \varphi_p \sin \lambda_p \\
r \sin \varphi_p \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_p \\
y_p \\
\end{bmatrix} = \begin{bmatrix}
r \lambda_p \\
r \varphi_p \\
\end{bmatrix}
\]

for spherical coordinates and plate carrée
Cylindrical Projections

- plate carrée \((x = r\lambda, y = r\varphi)\) is one example
Cylindrical Projections

• plate carrée \((x = r\lambda, y = r\phi)\) is one example

For all cylindrical projections in normal position:
• Lines of equal latitude are mapped to horizontal lines.
• Meridians are mapped to vertical lines.
• The equator is scaled with a constant factor (often with 1).

The distances between lines of equal latitude can be nonuniform.
Cylindrical Projections

• plate carrée \((x = r\lambda, y = r\varphi)\) is one example

For all cylindrical projections in normal position:

• Lines of equal latitude are mapped to horizontal lines.
• Meridians are mapped to vertical lines.
• The equator is scaled with a constant factor (often with 1).

The distances between lines of equal latitude can be nonuniform.

\[ x = r\lambda \]
\[ y = f(\varphi) \]
Cylindrical Projections

• For *transverse* cylindrical projections, consider one meridian as the Earth’s equator.

source: Spata (2010)
Mercator Projection

- arguably, the most important cylindrical projection

Gerhard Krämer (1512-1594)
Mercator Projection

Idea:

- Map areas between lines of constant latitude/longitude to rectangles that preserve aspect ratios.

\[
\frac{a}{b} = \frac{a'}{b'}
\]
Mercator Projection

Idea:

- Map areas between lines of constant latitude/longitude to rectangles that preserve aspect ratios.

- here shown for $\Delta \lambda = \Delta \varphi = 10^\circ$
Mercator Projection

Idea:

• Map areas between lines of constant latitude/longitude to rectangles that preserve aspect ratios.

• Here shown for $\Delta \lambda = \Delta \varphi = 10^\circ$

• For an exact construction, choose $\Delta \lambda, \Delta \varphi$ infinitely small.

\[ \frac{a}{b} = \frac{a'}{b'} \]
Mercator Projection

Advantage:
• Aspect ratios are preserved.
Mercator Projection

Advantage:
- Aspect ratios are preserved.
- This implies that angles are preserved!
- Thus, the Mercator projection is conformal.
Mercator Projection

Advantage:
- Aspect ratios are preserved.
- This implies that angles are preserved!
- Thus, the Mercator projection is conformal.

Disadvantages:
- Areas are distorted (esp. close to poles).
- Lengths of meridians are not preserved.

Mercator Projection – Maths
Requirement: same aspect ratios

\[
\frac{\left| \frac{\partial \tilde{x}}{\partial \phi} \right|}{\left| \frac{\partial \tilde{x}}{\partial \lambda} \right|} = \frac{\left| \frac{\partial \tilde{X}}{\partial \phi} \right|}{\left| \frac{\partial \tilde{X}}{\partial \lambda} \right|}
\]
Mercator Projection – Maths

Requirement: same aspect ratios

\[ \frac{|\frac{\partial \vec{x}}{\partial \varphi}|}{|\frac{\partial \vec{x}}{\partial \lambda}|} = \frac{|\frac{\partial \vec{X}}{\partial \varphi}|}{|\frac{\partial \vec{X}}{\partial \lambda}|} \]

\[ \iff \frac{\left(\frac{\partial \vec{x}}{\partial \varphi}\right)^2}{\left(\frac{\partial \vec{x}}{\partial \lambda}\right)^2} = \frac{\left(\frac{\partial \vec{X}}{\partial \varphi}\right)^2}{\left(\frac{\partial \vec{X}}{\partial \lambda}\right)^2} \]
Mercator Projection – Maths

\[
\left( \frac{\partial \vec{x}}{\partial \varphi} \right)^2 = \left( \begin{array}{c} 0 \\ f'(\varphi) \end{array} \right)^2 = (f'(\varphi))^2
\]

\[
\left( \frac{\partial \vec{x}}{\partial \lambda} \right)^2 = \left( \begin{array}{c} r \\ 0 \end{array} \right)^2 = r^2
\]

\[
\left( \frac{\partial \vec{X}}{\partial \varphi} \right)^2 = \left( \begin{array}{c} -r \sin \varphi \cos \lambda \\ -r \sin \varphi \sin \lambda \\ r \cos \varphi \end{array} \right)^2 = r^2
\]

\[
\left( \frac{\partial \vec{X}}{\partial \lambda} \right)^2 = \left( \begin{array}{c} -r \cos \varphi \sin \lambda \\ r \cos \varphi \cos \lambda \\ 0 \end{array} \right)^2 = r^2 \cos^2 \varphi
\]
Mercator Projection – Maths

Requirement:
same aspect ratios

\[ \left( \frac{\partial x}{\partial \phi} \right)^2 \frac{1}{\left( \frac{\partial x}{\partial \lambda} \right)^2} = \left( \frac{\partial X}{\partial \phi} \right)^2 \frac{1}{\left( \frac{\partial X}{\partial \lambda} \right)^2} \]
Mercator Projection – Maths

Requirement:
same aspect ratios

\[ \left| \frac{\partial \vec{x}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{x}}{\partial \lambda} \right| = \left| \frac{\partial \vec{X}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{X}}{\partial \lambda} \right| \]

\[ \Leftrightarrow \left( \frac{\partial \vec{x}}{\partial \varphi} \right)^2 = \frac{\left( \frac{\partial \vec{X}}{\partial \varphi} \right)^2}{\cos^2 \varphi} \]

\[ \Leftrightarrow \frac{\left( \frac{\partial \vec{x}}{\partial \lambda} \right)^2}{r^2} = \frac{\left( \frac{\partial \vec{X}}{\partial \lambda} \right)^2}{r^2 \cos^2 \varphi} \]
Mercator Projection – Maths

 Requirement:
 same aspect ratios

\[
\frac{|\frac{\partial \vec{x}}{\partial \varphi}|}{|\frac{\partial \vec{x}}{\partial \lambda}|} = \frac{|\frac{\partial \vec{X}}{\partial \varphi}|}{|\frac{\partial \vec{X}}{\partial \lambda}|}
\]

\[
\Leftrightarrow \left(\frac{\partial \vec{x}}{\partial \varphi}\right)^2 = \left(\frac{\partial \vec{x}}{\partial \lambda}\right)^2 \cos^2 \varphi
\]

\[
\Leftrightarrow \left(\frac{f'(\varphi)}{\frac{\partial \vec{X}}{\partial \lambda}}\right)^2 = \frac{r^2}{r^2 \cos^2 \varphi}
\]

\[
\Leftrightarrow f'(\varphi) = \frac{r}{\cos \varphi}
\]
Mercator Projection – Maths

Requirement:
same aspect ratios

\[
\left| \frac{\partial \vec{x}}{\partial \varphi} \right| = \left| \frac{\partial \vec{X}}{\partial \varphi} \right|
\]

\[
\left| \frac{\partial \vec{x}}{\partial \lambda} \right| = \left| \frac{\partial \vec{X}}{\partial \lambda} \right|
\]

\[
\left( \frac{\partial \vec{x}}{\partial \varphi} \right)^2 = \left( \frac{\partial \vec{X}}{\partial \varphi} \right)^2
\]

\[
\left( \frac{\partial \vec{x}}{\partial \lambda} \right)^2 = \left( \frac{\partial \vec{X}}{\partial \lambda} \right)^2
\]

\[
\left( f'(\varphi) \right)^2 = \frac{r^2}{r^2 \cos^2 \varphi}
\]

\[
f'(\varphi) = \frac{r}{\cos \varphi}
\]

\[
f(\varphi) = r \ln \tan\left( \frac{\pi}{4} + \frac{\varphi}{2} \right)
\]

\[f(0) = 0\]
Mercator Projection

Mapping function:

\[ x_p = r \lambda_p \]
\[ y_p = r \ln \tan \left( \frac{\pi}{4} + \frac{\varphi_p}{2} \right) \]

\( r = \text{Earth’s radius} \)
Mercator Projection

- often the transverse version is used; central meridian can be chosen arbitrarily.
Mercator Projection

- often the transverse version is used; central meridian can be chosen arbitrarily.

Gauß Krüger projection:

- Earth is subdivided into zones of width 3°.
- For different zones, different projections are used (central meridian = center of zone)
Mercator Projection

- often the transverse version is used; central meridian can be chosen arbitrarily.

Gauß Krüger projection:

- Earth is subdivided into zones of width 3°.
- For different zones, different projections are used (central meridian = center of zone)

- Usage: official surveys in Germany until 1991, then replaced by UTM

Mercator Projection

Universal Transverse Mercator (UTM):
• Similar to Gauß Krüger
• width of zone $6^\circ$
• scale $= 0.9996$

Mercator Projection

Mapping function:

\[ x_p = r \lambda_p \]
\[ y_p = r \ln \tan \left( \frac{\pi}{4} + \frac{\varphi_p}{2} \right) \]

\( r = \text{Earth's radius} \)
Mercator Projection

Mapping function:

\[ x_p = r \lambda_p \]
\[ y_p = r \ln \tan \left( \frac{\pi}{4} + \frac{\varphi_p}{2} \right) \]

\( r = \text{Earth’s radius} \)

Conclusion:

- Mercator projection is not appropriate for world maps.
- Frequently used for smaller zones
  - close to equator or
  - close to a central meridian (transverse version).
Cylindrical Equal-area Projection – Idea

Idea:

• Map areas between lines of constant latitude/longitude to rectangles of the same sizes.

• here shown for $\Delta \lambda = \Delta \varphi = 10^\circ$

• For an exact construction, choose $\Delta \lambda, \Delta \varphi$ infinitely small.

\[ ab = a'b' \]
Cylindrical Equal-area Projection – Maths
Requirement:
same sizes

\[ \left| \frac{\partial \hat{x}}{\partial \varphi} \right| \cdot \left| \frac{\partial \hat{x}}{\partial \lambda} \right| = \left| \frac{\partial \hat{X}}{\partial \varphi} \right| \cdot \left| \frac{\partial \hat{X}}{\partial \lambda} \right| \]
Cylindrical Equal-area Projection – Maths

Requirement:
same sizes

\[
\left| \frac{\partial \vec{x}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{x}}{\partial \lambda} \right| = \left| \frac{\partial \vec{X}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{X}}{\partial \lambda} \right|
\]

\[\Leftrightarrow \left( \frac{\partial \vec{x}}{\partial \varphi} \right)^2 \left( \frac{\partial \vec{x}}{\partial \lambda} \right)^2 = \left( \frac{\partial \vec{X}}{\partial \varphi} \right)^2 \left( \frac{\partial \vec{X}}{\partial \lambda} \right)^2\]
Cylindrical Equal-area Projection – Maths

Requirement:
same sizes
\[ \left| \frac{\partial \vec{x}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{x}}{\partial \lambda} \right| = \left| \frac{\partial \vec{X}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{X}}{\partial \lambda} \right| \]
\[ \iff \left( \frac{\partial \vec{x}}{\partial \varphi} \right)^2 \left( \frac{\partial \vec{x}}{\partial \lambda} \right)^2 = \left( \frac{\partial \vec{X}}{\partial \varphi} \right)^2 \left( \frac{\partial \vec{X}}{\partial \lambda} \right)^2 \]
\[ \iff (f'(\varphi))^2 r^2 = r^4 \cos^2 \varphi \]
Cylindrical Equal-area Projection – Maths

Requirement:

same sizes

\[
\left| \frac{\partial \vec{x}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{x}}{\partial \lambda} \right| = \left| \frac{\partial \vec{X}}{\partial \varphi} \right| \cdot \left| \frac{\partial \vec{X}}{\partial \lambda} \right|
\]

\[\Leftrightarrow \left( \frac{\partial \vec{x}}{\partial \varphi} \right)^2 \left( \frac{\partial \vec{x}}{\partial \lambda} \right)^2 = \left( \frac{\partial \vec{X}}{\partial \varphi} \right)^2 \left( \frac{\partial \vec{X}}{\partial \lambda} \right)^2\]

\[\Leftrightarrow (f'(\varphi))^2 r^2 = r^4 \cos^2 \varphi\]

\[\Leftrightarrow f'(\varphi) = r \cos \varphi\]
Cylindrical Equal-area Projection – Maths

Requirement:

same sizes

\[ \left| \frac{\partial \vec{x}}{\partial \phi} \right| \cdot \left| \frac{\partial \vec{x}}{\partial \lambda} \right| = \left| \frac{\partial \vec{X}}{\partial \phi} \right| \cdot \left| \frac{\partial \vec{X}}{\partial \lambda} \right| \]

\[ \Leftrightarrow \left( \frac{\partial \vec{x}}{\partial \phi} \right)^2 \left( \frac{\partial \vec{x}}{\partial \lambda} \right)^2 = \left( \frac{\partial \vec{X}}{\partial \phi} \right)^2 \left( \frac{\partial \vec{X}}{\partial \lambda} \right)^2 \]

\[ \Leftrightarrow (f'(\varphi))^2 r^2 = r^4 \cos^2 \varphi \]

\[ \Leftrightarrow f'(\varphi) = r \cos \varphi \]

\[ \Leftrightarrow f(\varphi) = r \sin \varphi \]

\[ f(0) = 0 \]
Cylindrical Equal-area Projection

Mapping function:

\[ x_p = r \lambda_p \]
\[ y_p = r \sin \varphi_p \]

\[ r = \text{Earth's radius} \]

Cylindrical equal-area projection leads to large distortions of aspect ratios and angles

More Map Projections

• For a cartographic visualization of the whole world, it is better to use a map projection that tries to compromise between area distortion and angle distortion.

• e.g. Robinson projection:


• No map projection preserves both areas and angles!
Coordinate Systems for Geoinformation

Earth-centered Cartesian coordinates

spherical/ellipsoidal coordinates

latitude $\phi$, longitude $\lambda$

projected coordinates

northing $y$

easting $x$

\[
\begin{align*}
\begin{pmatrix} X_p \\ Y_p \\ Z_p \end{pmatrix} &= \begin{pmatrix} r \cos \phi_p \cos \lambda_p \\ r \cos \phi_p \sin \lambda_p \\ r \sin \phi_p \end{pmatrix} \\
\begin{pmatrix} x_p \\ y_p \end{pmatrix} &= \begin{pmatrix} r \lambda_p \\ r \ln \tan\left(\frac{\pi}{4} + \frac{\phi_p}{2}\right) \end{pmatrix}
\end{align*}
\]

for spherical coordinates and Mercator projection
Coordinate Transformation

Earth-centered system: ETRS89
ellipsoid: GRS80
projection: UTM

Earth-centered system: WGS84
ellipsoid: WGS84
projection: Mercator
Coordinate Transformation

Aim: Combine both layers in GIS, using UTM

Earth-centered system: ETRS89
ellipsoid: GRS80
projection: UTM

Earth-centered system: WGS84
ellipsoid: WGS84
projection: Mercator
Coordinate Transformation

Earth-centered coords. (ETRS89)

Earth-centered coords. (WGS84)

Ellipsoidal coordinates

Projected coordinates

Ellipsoidal coordinates

Projected coordinates

Ellipsoidal coordinates

Projected coordinates

Earth-centered system: ETRS89

Ellipsoid: GRS80

Projection: UTM

Earth-centered system: WGS84

Ellipsoid: WGS84

Projection: Mercator